

Chapter 1

Pairs of separably closed fields and exotic groups

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We look at simple groups associated primarily with the general theory of Moufang buildings, and analyze their relation to stability theory in the model theoretic sense. As the details become quite technical, a lengthy introduction surveys the developments at a less detailed level.

Beginning from the second section, the text first deals with some model theoretic algebra of fields, followed by an extended study of three associated families of simple groups coming from the theory of Tits buildings, the theory of Moufang polygons, and Timmesfeld's theory of exotic analogs of SL_2 .

The field theoretic part is fundamental (§ 2). The rest of the paper relates this to group theoretic constructions, with two sections surveying the consequences for the original Tits and Timmesfeld theory before concentrating on the more exotic groups associated with Moufang polygons.

A good deal of the group theoretical material is expository, aimed to make the relevant structural information meaningful to those coming from the direction of model theory.

1.1 Introduction

Our aim here is to construct some simple stable groups which are not algebraic (hence, “exotic”). These are not, strictly speaking, “new” groups, but instances of a phenomenon discovered by Tits long ago, in connection with the classification of buildings of spherical type [17]. He called them groups of “mixed type”. We became aware of this much later, while looking into the classification of *Moufang polygons* given in [18] and discussed below in § 1.5. Moufang polygons can be classified broadly speaking into algebraic (associated with algebraic groups), classical (in a historical sense), and *mixed*, reusing the term introduced by Tits to reflect both their similarities to the algebraic case, and the use of two fields rather than one in their construction; but in the case of Moufang polygons the meaning of the term becomes a bit broader.

So we have on the one hand the groups identified by Tits, which are analogs of algebraic groups in Lie rank at least 3, but with a coordinatization involving two fields $k \leq K$, and we have also various groups associated with Moufang polygons which are

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analogues of algebraic groups in Lie rank 2, but associated with a considerably more intricate collection of coordinatizing structures (including some of Tits' original type, constructed from a pair of fields). There is also a rank 1 analogue of $\mathrm{SL}_2(K)$ due to Timmesfeld, which we will consider as well.

A very natural program is then the following:

- (a) Construct some stable algebraic structures of the sorts used by Tits, Tits/Weiss, or Timmesfeld.
- (b) Deduce the existence of the corresponding stable simple groups.

This turns out to be more subtle than appears at first. So we aim not only to carry this through in some cases, but also to point out some issues that others might want to explore further.

In the Tits setting, things work out neatly but with more delicacy than one might expect. An ample supply of coordinatizing structures for Tits' purposes is afforded by Theorem 1.2.1, and in a generalized form, by Theorem 1.2.2. We cover some cases relevant to the Timmesfeld construction and an interesting case from the Tits/Weiss classification. However, while this is satisfactory as far as (a) is concerned, one is not quite done at this point.

One might expect that a general interpretability result would allow for the systematic treatment of step (b) above. This seems not to be the case (see Question 1.4.12). On the other hand, in the context of the groups of Tits' type associated to a pair of fields $k \leq K$, this is the case.

The problem in general is that when one moves beyond Tits' original setting, the groups are defined as those generated by a collection of subgroups. This is perhaps clearest in the rank 1 case (the Timmesfeld construction), which is given explicitly as a subgroup of $\mathrm{SL}_2(K)$ whose diagonal subgroup is generated by elements whose coordinates lie in an additive subgroup of K . The situation in rank 2 turns out to be much the same, but the notation involved is a good deal more complicated.

In fact, one may take a slightly different point of view on all of this, one that emerges most clearly in the rank 2 setting (Tits/Weiss). This becomes more technical. We describe this now, but the reader might prefer to look first at the more concrete rank 1 setting of § 1.4 where everything can be worked out in detail, from first principles, and only then return to a consideration of rank 2.

In any case, in the rank 2 setting, there are at least two groups naturally associated with a given Moufang polygon, and it becomes important to distinguish them, and to consider more generally the groups lying between them. The first group is the full automorphism group of the Moufang polygon. The second group, called the *little projective group*, is defined (by analogy with Chevalley groups) as the subgroup generated by the so-called *root groups*, which are the fundamental building blocks of the group from the point of view of either the Chevalley theory or the theory of Moufang polygons, and in the classical cases are copies of the additive group of the field. These

groups appear in the Moufang theory as subgroups of the automorphism group of the Moufang polygon, and then the group they generate is one of the main groups of interest within the automorphism group, and is certainly the smallest group of interest, for our purposes.

In most cases the latter group is simple, and is the socle of the full automorphism group (its unique minimal normal subgroup). Between this group and the full automorphism group there are some other groups which are interpretable in the coordinate system for the group, and whose commutator subgroup is our simple group. So if we start with a stable coordinate system then we can associate a stable group with a simple socle to it, but in passing to the commutator subgroup, while we gain simplicity, we may lose definability.

Accordingly, our exposition becomes more elaborate than we had expected, as we sort through these issues. To complicate matters, our sources for the three cases take varying points of view, from the explicit matrix theoretic point of view of Timmesfeld, to the style of Chevalley (and Steinberg) in terms of generators and relations in the Tits/Weiss setting, and (for the part that concerns us) much more directly in terms of the structure of algebraic groups in the Tits setting. So we have the choice of unifying our perspective or staying close to our sources as we go along. We try to unify the description, but at the same time we do need to quote specific material from each source.

The paper is aimed at model theorists with an interest in a variety of related topics. We have arranged it as follows:

In § 1.2, Theorems 1.2.1 and 1.2.2 give the supply of stable “coordinate systems” with which we work. This is self-contained and is closely related to well-known work on the model theory of separably closed fields. Here the first theorem serves as warm-up to the second, and provides enough information to deal with the groups of mixed type as originally considered by Tits. We describe such groups in § 1.3 and prove that we do indeed get stable simple groups of this type by passing to the context of separably closed fields and applying Theorem 1.2.1.

Now Theorem 1.2.2 is of interest because the algebraic systems considered are the natural parametrizing systems for the groups which interest us. At the same time, the groups themselves cannot always be defined in a first order way from these structures. A point of considerable technical interest is that in certain cases, enriching the original parametrizing structures to slightly richer structures *of the same kind* may make the group first order definable over the expanded coordinate system.

After considering Tits’ groups of mixed type in §1.3, we pass to the opposite extreme—rank 1—in the following section, working out the details of Timmesfeld’s construction and the consequences for the issues of definability and interpretability that concern us here. Everything can be seen very simply by repeating standard computations (either from the point of view of Chevalley theory, or from the point of view of elementary linear algebra in two dimensions). The unsettling phenomenon of a con-

flict between the desired simplicity and the desired interpretability, mentioned above, is encountered at this stage. One can say more precisely how the initial coordinate system should be expanded to make the simple group definable, but then the issue of stability has to be approached afresh, and the situation becomes much more complex. Much remains unclear, and perhaps someone will investigate this further.

The last three sections discuss the related groups of automorphisms of Moufang polygons at some length. Here we are in rank 2. At this point, the notation becomes noticeably more burdensome. Here we encounter everything that we have seen in the original Tits construction together with the complications that became visible in rank 1—but not much else, fortunately, other than some rather specific notation. At this point one needs to work rather concretely in the notation of root systems in order to sort out the details. Readers will probably find our presentation either excessively terse or excessively detailed, depending on their degree of familiarity with the notation used. The ultimate result, which is a theme throughout much of the latter part of Tits/Weiss—though not put in these terms—is that in rank 2 one has to deal with two separate instances of the rank 1 theory, and otherwise things are rather similar to the case of algebraic groups.

In more detail, the content of the last three sections runs as follows.

In § 1.5 we give an overview of what is done by Tits and Weiss in [18], and the notation used. Their goal is to give a classification theorem in terms of concrete coordinatizations by algebraic systems. This background material discusses what is common across all cases prior to the introduction of coordinates.

The next two sections then look into two particular cases of the classification of Moufang polygons as given by Tits and Weiss. The first concerns Moufang hexagons, where we encounter examples already noticed by Tits as rank 2 analogues of the algebraic group of exceptional type G_2 (so, $G_2(k, K)$ in his notation). The second, more subtle example, treated in the last section, concerns the Moufang quadrangles of so-called “indifferent type,” which are those most closely related to the Timmesfeld construction in rank 1. Our summary of the situation, above, focuses on this case: this is the setting which inherits the specific difficulties associated with the rank 1 case.

The classification of Moufang polygons involves further families which could be investigated model theoretically; they tend to involve structure incompatible with stability, but compatible, in principle, with model theoretic simplicity. The interested reader may want to look further in that direction, and in particular investigate the problem of building coordinate systems of the various types which are simple in the model theoretic sense.

We imagine that most readers will be interested either in looking into § 1.2 and taking much of the rest on faith (particularly from § 1.5 onward), or else taking § 1.2 on faith and looking into the following group theoretic issues (including the definability issues that arise). Either approach should be perfectly feasible. Most of what we have to say in the group theoretic part is intended to be expository, but it was not always

evident where to find clear statements in the literature of the facts most directly relevant to the model theoretic issues.

Up to this point, we have been very vague about the details, in order not to become lost in them. In the remainder of this introduction we give a more precise account of the main points (and the key definitions) concerning the original construction of Tits, the lower rank constructions of Timmesfeld and Tits/Weiss, and the role of the model theory of separably closed fields in the construction of stable coordinate systems of the appropriate types.

Remarks by the second author, added in revision

The story we take up here begins with [8] (see also [13]) and in a sense the present paper comments on that one.

Not long after this paper was submitted for publication (July 2024), we became aware of the paper [10], which uses the definability of centralizers as we do here, and distinctly more systematically, though not quite in the same cases. Accordingly, we had intended to return to this point, and perhaps adjust our exposition in light of it, but the first author died on January 22, 2025, and we did not get an opportunity to discuss that point further. Accordingly, the relevance of the cited paper will simply be noted in the text. But the reader who is interested in the details relating to definability would do well to consult that text.

It was a pleasure to work with Zoé on this paper, although (or possibly even because) we yelled at each other a bit in the process.

1.1.1 The Tits construction: $G(k, K)$

In [17, 10.3.2], Tits constructs analogs of (abstractly) simple algebraic groups over algebraically closed fields, in certain very special cases, defined from a suitable pair of fields (k, K) with $k \leq K$. The point of view taken is that of Chevalley, with a small twist. This relies on the description of these groups in terms of root systems and their Dynkin diagrams, which may be summarized very rapidly as follows. This is either a reminder, or a few points of reference for the discussion afterward.

We begin with the algebraic group $G(K)$, which in algebraic terms is a K -split simple algebraic group of adjoint type. The 1-dimensional subgroups are isomorphic to the additive or multiplicative group of K . A maximal torus T is a product of a certain number of copies of the multiplicative group of K ; that number is the Lie rank. The copies of the additive group of K invariant under the action of T are the *root groups* (with respect to T); these are permuted by the group $W = N(T)/T$; the action of W on the set of root groups can be identified with the action of a finite reflection group acting on real Euclidean space (a Coxeter group) and these are classified by the Dynkin diagrams of types A – G . The root groups then correspond to a finite set

of vectors invariant under the action of W (these vectors encode the homomorphisms from T to $K^\times$ which give the actions of T on the corresponding root groups).

From the Dynkin diagram, or the root system and the action of W , one can recover the construction of the group from the field K ; this is the description of $G(K)$ as a Chevalley group. We will see this concretely in the case of rank 2 in §§ 1.6, 1.7, where in the second case the construction is a generalization of the one described by Tits, and additional complications arise.

For our purposes it is important that the roots will always have either one or two root lengths. The setting for the Tits construction involves a simple split algebraic group of adjoint type over a field k associated with a root system in which, in fact, two root lengths occur. Furthermore we require the characteristic to be “exceptional” in a certain sense (in a familiar sense from the point of view of finite group theory, and explained by Tits in terms of *special isogenies*, [17, (5.7.3)]). The restriction on root lengths means concretely that the Dynkin diagram is of type B_n , C_n , F_4 , or G_2 , and the restriction on the characteristic then means that the characteristic is 2 unless we have type G_2 , in which case the characteristic will be 3.¹

In this setting, one fixes a second field K with

$$K^p \leq k \leq K.$$

With $G(k)$ the original algebraic group, one builds a group $G(k, K)$ containing $G(k)$, and contained in $G(K)$, much as one might construct $G(K)$ as a Chevalley group.

Namely, we consider a Borel subgroup $B = TU$ with T k -split, we extend the groups $T(k)$ and $U(k)$ to groups $T(k, K)$ and $U(k, K)$ in a manner to be described momentarily, and then we set $N(k, K) = N(k)T(k, K)$, so that $N(k, K)/T(k, K)$ is isomorphic to the usual Weyl group $W = N(k)/T(k)$. The group $G(k, K)$ is then defined as the group generated by $B(k, K) = T(k, K)U(k, K)$ and $N(k, K)$.

The group $U(k, K)$ is an exact analog of the maximal unipotent subgroup of a Borel subgroup from the point of view of Chevalley. Namely, $U(K)$ is generated by the root subgroups, which are copies of K_+ , subject to the *Chevalley commutator relations* determined by the root system. To get $U(k, K)$, one adjusts this construction by taking the root groups for long roots to be copies of the additive group of the smaller field k , and the root groups for the short roots to correspond to the larger field K . One may then check that the Chevalley commutator formula still makes sense (using the precise data in that formula, and the particular value of the characteristic).

¹Here the classification by Dynkin diagrams can be treated simply as a set of labels for the cases of interest, until we come down to the rank 2 case. Tits mainly deals with the case of F_4 in [17]; he is able to identify types B_n , C_n with groups he has treated from another point of view [17, (2), p. 204], and G_2 is mentioned in passing but lies outside the scope of that monograph.

At this point, one could reasonably proceed as follows: using the same modified notion of root group based on a pair of fields (k, K) , take the group inside $G(K)$ generated by all the long root groups over k and the short root groups over K . However, Tits proceeds in a different way, which connects up directly with his theory of *BN-pairs*. Before following him on this path, we discuss why one might do that.

1.1.1.1 BN-pairs and the Bruhat decomposition. In the first place, Tits' BN-pair theory gives a direct route toward connecting the new groups with the subject of his monograph [17]. In the second place, the data $B(k, K)$ and $N(k, K)$ are explicitly given direct analogs of the usual groups $B(K)$ and $N(K)$. On the other hand the group *generated by them* is potentially obscure; a priori it might very well be $G(K)$, for example. But the BN-pair theory implies a so-called *Bruhat decomposition*

$$G = \bigsqcup_w BwB$$

which is the double coset decomposition of $G = G(k, K)$ with respect to $B = B(k, K)$. (More properly, w is replaced by a representative in G , but the corresponding double coset is well-defined.) Comparing this decomposition to the corresponding decomposition of $G(K)$, we see that $B(K) \cap G(k, K)$ is $B(k, K)$, which is reassuring. And more generally, the Bruhat decomposition can be read as saying that $G(k, K)$ is built from $B(k, K)$ in exactly the way that $G(K)$ is built from $B(K)$.

1.1.1.2 The groups. The groups obtained in this manner are (in the Dynkin notation) the families $B_n(k, K)$, $C_n(k, K)$, and the exceptional groups $F_4(k, K)$, $G_2(k, K)$. The groups $C_2(k, K)$ are variations on the algebraic group $\mathrm{PSp}_4(K)$. Further variations are possible: these correspond to Moufang quadrangles of indifferent type in the sense of Tits and Weiss, discussed in § 1.1.2. Rather than taking a pair of fields k, K , we take a large field K of characteristic 2 and two additive subgroups K_0, L_0 , with

$$K^2 \leq L_0 \leq K_0 \leq K$$

where now L_0 is a vector space over K^2 and K_0 is a vector space over the field generated by L_0 . We then proceed to build a group $\mathrm{PSp}_4(L_0, K_0)$ in the manner of Tits, using L_0 to parametrize long root groups, K_0 for short root groups. This is not the description used by Tits and Weiss however; they build its associated Moufang polygon and then compute the subgroup of its automorphism group which is generated by the corresponding root subgroups (again, parametrized by L_0 and K_0 rather than k and K).

We have some unfinished business to attend to. On the one hand, we need to complete the definition of the groups $G(k, K)$. On the other hand, we should say a bit more as to how one actually obtains the BN-pair properties, or at least the Bruhat decomposition; this is the only way we will have of seeing that these groups are in fact new

groups. Tits refers a little vaguely to Chevalley for this point, in [17], though elsewhere he gave the argument explicitly (in the Chevalley context).

1.1.1.3 $G(k, K)$ (definition, concluded). We have described $U(k, K)$ as the subgroup of $U(K)$ generated by modified root subgroups. Tits defines the torus $T(k, K)$, as the subgroup of $T(K)$ whose elements act sensibly on the root groups: that is, the elements of $T(K)$ which leave the root groups of $U(k, K)$ invariant. In other words, these are the elements which act via multiplication by an element of k on the long root groups.

In particular the group $T(k, K)$ normalizes the group $U(k, K)$, and so we can define a “Borel subgroup” $B(k, K) = T(k, K)U(k, K)$. Here $T(k, K)$ is the largest such torus available inside $G(K)$. One could consider similar constructions in which the toral part varies. In the rank 2 case this last point is the subject of extended calculations in [18]; however the full automorphism group also contains elements inducing automorphisms of the coordinate system². In the cases of interest to us, these automorphisms are induced by certain field automorphisms, and they will not appear in an algebraic group.

It is reasonably clear that the “Borel subgroup” $B(k, K)$ is interpretable in the pair (K, k) ; more concretely, its underlying set is definable in $G(K)$ if we take K to be equipped with a predicate for the subfield k . It then follows from the Bruhat decomposition that the same applies to $G(k, K)$, and thus stability of the coordinate system will give rise to stability of the group; the converse also holds. Indeed (k, K) is interpretable in $U(k, K)$ (see Theorem 1.6.4 for G_2 , and Theorem 1.7.2 for some more exotic cases in rank 2).

1.1.1.4 B and N . Now we come back to the point that the groups $B(k, K)$ and $N(k, K)$ give a Bruhat decomposition for $G(k, K)$, indicating how this goes in the setting of Chevalley groups, and how it relates to the theory of BN-pairs. For brevity we will now write G, B, N, U , and T for the various groups involved in the definition of $G(k, K)$. So G is generated by B and N , by definition, and the desired Bruhat decomposition is

$$G = \bigsqcup_w BwB.$$

As Tits mentions in [17, 10.3.2], a key ingredient is the fact that the nilpotent group U can be written as the product of its root subgroups, *taken in any order*. Another ingredient is the fact that the Bruhat decomposition holds in rank one (in $SL_2(k)$, $SL_2(K)$, or the projective versions of these groups). To this one adds some observations about the operation of the reflections corresponding to a simple root on the set of positive roots, and the fact that two opposite root groups generate a rank one subgroup.

²Here that system is (k, K) with natural structure.

We run over some of the more formal aspects of this argument, taking as our initial goal the Bruhat decomposition. As G is generated by N and B , and $W = N/T$ with T contained in B , the double coset decomposition exhibited is well-defined and contained in G . In order to show that this subset exhausts G , it suffices to show that it is closed under multiplication by (representatives for) W and under multiplication by B , the latter point being evident. Also, as W is generated by reflections w_α corresponding to simple roots α , it suffices to check that sets of the form $w_\alpha BwB$ are contained again in the double cosets exhibited. What is claimed, in fact, is the following:

$$w_\alpha BwB \subseteq Bw_\alpha wB \cup BwB.$$

This is one of the fundamental axioms in the theory of BN-pairs, in fact, so the question is how to verify it.

This can be further reduced by similar formal manipulations, since $B = TU$ and W normalizes T , to a consideration of $w_\alpha Uw$, and then even further by consideration of the structure of U . Namely, U may be written as U^*U_α where U_α is the root group corresponding to α , and where U^* is the product of the remaining root groups, which is itself invariant under w_α . One reduces quickly to a consideration of $w_\alpha U_\alpha w$. Then either w or $w_\alpha w$ carries U_α into another root group contained in U . In the first case $w_\alpha U_\alpha w \subseteq w_\alpha wU$ and one finds that $w_\alpha BwB = Bw_\alpha wB$. In the second case one applies the same reasoning to $w_\alpha w$ in place of w , but one also uses the Bruhat decomposition for the rank one group generated by U_α and $U_{-\alpha}$.

The last details are found in the proofs of [12, Lemma 25, § 3; (b) p. 34] or [16, (16), p. 323].

1.1.2 Tits-Weiss and Timmesfeld: subtleties

So far, everything proceeds according to plan. Now complications arise as we encounter some variations corresponding to Lie ranks 1 or 2, where the underlying algebraic systems are of a more general type.

For us, the most interesting case concerns Moufang quadrangles of “indifferent” type, similar to the buildings associated with Tits’ groups $C_2(k, K)$, but more general. Most of the complexity of this case, as far as the model theory is concerned, can be traced back to the rank 1 groups associated with simple roots in this setting, which turn out to be the groups Timmesfeld calls $SL_2(L_0)$ and $SL_2(K_0)$ (we are in characteristic 2, so we do not need to distinguish SL_2 and PSL_2).

Interesting comments about the history and the differing emphases of the various approaches taken to this subject by [15, 17, 18] are found in the review of [15] in the AMS Bulletin by Richard Weiss [19]. The following has considerable relevance here:

In a spherical building, groups of rank one appear as groups generated by pairs of “opposite” root groups, In the classification of Moufang

buildings, in fact, these subgroups are avoided to the maximal extent possible. The philosophy of abstract root groups is just the opposite—groups of rank one are enshrined in the hypothesis themselves and play a central role in the whole theory.

We will approach the rank 2 case via the rank 1 case, in order to encounter the model theoretic issues in their simplest “pure” state. This means in particular that we will be crossing over between two rather different points of view.

We are again in characteristic 2 with an imperfect field K , and we begin in rank 1. In the Timmesfeld setting—or rather, the special case of interest to us here—we will have an *additive subgroup* L of K containing K^2 and invariant under multiplication by K^2 . Timmesfeld’s description of his group involves generation by two “root subgroups” parametrized by L , but as we will check later, we can give a description similar to the one given by Tits above.

We begin with a *single* root group $U(L)$ (where L is not necessarily a subfield) which we may take to be the upper unitriangular matrices with coefficients in the additive group L . If we followed Tits’ construction we would also define a torus $T(L)$ at this point. In fact we will take the root group $U(L)$ and its opposite, and the group they generate, and then *compute* the torus $T(L)$ generated as a subgroup of $T(K)$. This turns out to be parametrized by the multiplicative subgroup of K which is *generated* by the nonzero elements of L . This is the point at which nondefinability enters into the picture.

On other hand, after this detour we could start afresh and define $T(L)$ as the particular group of diagonal matrices just mentioned, then define $B(L) = T(L)U(L)$, and let $\mathrm{SL}_2(L)$ be the group generated by $B(L)$ and a suitable Weyl group element. The usual Weyl group element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

will do (and we can omit the minus sign, as the characteristic is 2). This preserves the connection with the Tits construction; but we will in fact take Timmesfeld’s definition as our point of departure.

As there is only one pair of roots, the field K does not play much of a role here, and it could be replaced by the subfield k generated by L .

On the other hand, the torus $T(L)$ is not the strict analog of the one considered by Tits. The direct analog of Tits’ $T(k, K)$ in this context would be the subgroup of the diagonal group $T(K)$ which normalizes U . But this is $T(K)$, since L is a vector space over K^2 . So that torus would depend on the choice of K .

Notice that it is the small torus $T(L)$ which is a maximal torus in the simple group $\mathrm{SL}_2(L)$. But in general it is the larger torus $T(K)$, defined in the manner of Tits, which is definable from the coordinate system, so here we have a definable group $T(K) \mathrm{SL}_2(L)$

with simple socle, stable if (K, L) is, and the commutator subgroup of this group is simple, but not necessarily stable.

All of this can be checked by direct computations—computations which we will make, and which are the usual computations made over a field in the context of Chevalley groups. In particular one verifies the Bruhat decomposition in this context, and that leads to a proof of the BN-pair axioms also in rank 2 (carried out in a different way in [18]).

Turning to this rank 2 case, let us denote by $G_0(L_0, K_0)$ the group associated by Tits and Weiss to the coordinate system $(K; L_0, K_0)$. Namely, one first defines $U(L_0, K_0)$ by strict analogy with the case of Chevalley groups, as in the algebraic group $\mathrm{PSp}_4(K)$, with L_0 and K_0 parametrizing the long and short root groups respectively, using the Chevalley commutator relation to define the group law.

In an algebraic group setting one may then take the opposite group and the group they generate. In the setting of Moufang polygons, one may define the corresponding Moufang polygon (with some effort) and then consider the subgroup of the full automorphism group which is generated by its root subgroups (in the sense of Moufang polygons). From this point of view one also computes the torus (with considerable effort in this setting). This gives a simple group which is not necessarily first order definable, because the torus itself is not necessarily definable, and in fact rank one groups of type $\mathrm{SL}_2(L_0)$ and $\mathrm{SL}_2(K_0)$ are involved. The analysis of Tits and Weiss determines both the minimal torus (splitting the normalizer of the group U as $T \cdot U$ in the corresponding simple group) and the maximal torus (giving a similar splitting, but in the full automorphism group of the Moufang polygon)³.

The result is that inside the automorphism group of the Moufang polygon, and above the group generated by root subgroups, we have a family of groups, corresponding to a family of “tori” (in a very broad sense, allowing actions by field automorphisms on the coordinate system).

The smallest of these groups is simple but not necessarily definable over the coordinate system (in the first order sense), while the largest is rather too large for any of our purposes; but in between one can find a definable group whose commutator subgroup is the corresponding simple group (i.e., the associated torus is abelian). Here, definability refers to definability in the structure $(K; L_0, K_0)$.

In particular when the coordinate system is stable, the closest we come, in general, to building a stable simple group is to build a stable group with simple commutator subgroup.

³Tits and Weiss give in [18, § 37] a complete description of the automorphism group of the polygon for the various Moufang examples, which involves an “algebraic part” and a subgroup coming from automorphisms of the field K ; here, by *full automorphism group* we will mean the “algebraic part”.

On the other hand, as yet we have *no negative results* in the more challenging cases. In particular we do not know whether some of the simple groups which are not interpretable in the associated algebraic systems might themselves be stable, for other reasons.

This last is not intrinsically a group theoretic question, since the simple group of interest is definable from a coordinatizing structure expanding (L_0, K_0) by the torus T of the group and its action on the root subgroups (and conversely, this structure can be recovered from the group, if one is careful about the formulation). The torus can be made a little more concrete as it is a product of 1-dimensional tori which can be taken separately and come from rank 1 subgroups of Timmesfeld's type.

So stability of the simple group is equivalent to stability of the structure (L_0, K_0) together with the two 1-dimensional tori associated with the rank 1 subgroups corresponding to simple roots, and their actions on all the root groups. In this sense, one can set aside the simple group and work with an expanded language of fields instead.

1.1.3 Some model theory of fields

A few introductory remarks about the model theory of fields are also in order, just to set the scene properly. From our perspective, what was intriguing was the central role of imperfect fields in all of these constructions, and the known fact that separably closed fields have stable theories. This is what suggested the current line of investigation, and, in particular, our interest in the case of Moufang quadrangles of indifferent type.

The question as to whether every stable field is in fact separably closed is of long standing (see for example [9]). This question has been placed in a broader framework by Shelah and others, and occurs now in a number of formulations generally all going by the name of *Shelah's conjecture* for (e.g.) dependent fields [5]. This broader question is being actively pursued at present and leads into very different issues outside stability theory. But certainly in the present state of knowledge the only definite source of constructions of stable simple groups in which fields can be interpreted will pass through the theory of separably closed fields. If one enlarges the scope to simple unstable theories, then some other constructions from the theory of Moufang polygons would come into play, involving automorphisms and various semilinear or quadratic forms.

We turn now to the details, beginning with the model theoretic algebra that produces a good supply of stable structures suitable for use as coordinatizing structures, with three cases: Timmesfeld's rank one groups $SL_2(L)$, and the two families of rank two groups $G_2(k, K)$ and the indifferent type for PSp_4 . With that in hand we will take up the three sorts of groups of interest, starting with Tits' theory over pairs of fields, where matters are simplest in some fundamental sense (though with the usual apparatus of algebraic groups, root systems, and also BN-pairs all in the mix). Then

we pass to the rank 1 case as a relatively transparent context where real problems of definability arise, before coming finally to the most interesting case, Moufang polygons of indifferent type, where the groups to be constructed are stable, with simple socle equal to the commutator subgroup, and nonalgebraic.

1.1.4 Main results of the paper

As explained before, our aim was to study examples of “exotic groups,” preferably simple ones, from a model-theoretic point of view. We concentrated on three cases: $\mathrm{SL}_2(L)$, $G_2(k, K)$ and groups obtained as automorphism groups of the Moufang polygons of [18] coordinatized by an *indifferent set*, and in particular the groups generated in that setting by the *root groups* associated with the Moufang polygon (and a fixed apartment).

The first point is that stable coordinatizing systems exist in all three cases (§ 1.2). This is proved by fixing an imperfect separably closed field K of the appropriate characteristic, and studying the model theory of various enrichments of K : by subfields of K , or by K^2 -vector spaces lying between K^2 and K . The field theoretic results are valid, in part, in arbitrary characteristic p . The main result of § 1.2 is:

Theorem 1.2.2. *Let K be a separably closed field in characteristic p , and*

$$K^P = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq K_{m+1} = K$$

a chain of subfields of K containing K^P . Furthermore, for $1 \leq i \leq m$ let R_i be an additive subgroup of K_{i+1} which contains K_i and is a vector space over K_i , and which satisfies, in addition, the following two conditions:

- (1) $K_i = \{a \in K \mid aR_i = R_i\}$,
- (2) *Any subset of R_i which is linearly independent over K_i is p -independent over K_i .*

Then the structure $(K, K_1, \dots, K_m, R_1, \dots, R_m)$ is stable, and the complete theory is given by the properties stated together with simple numerical invariants: the dimensions of both R_i and K_{i+1} over K_i , as finite values or the formal symbol ∞ .

One also obtains a variation of this result by slightly modifying the vector spaces R_i (Theorem 1.2.6).

These results will be applied in characteristic 3 to a pair of fields and in characteristic 2 to two fields and two additive subgroups meeting the additional requirements.

Let us first start with two results on the groups $G_0(k, K)$ (“à la Tits”).

Theorem 1.3.3. *Suppose that $G(k)$ is of adjoint type (centerless) and split over k . Then for $K \neq \mathbb{F}_2, \mathbb{F}_3$, the group $G_0(k, K)$ is simple.*

For the Tits groups, stability of the group is equivalent to stability of the coordinatizing pair of fields, and we have

Theorem 1.3.5. *Suppose $G(K)$ is simple of type of type B_n , C_n , F_4 , or G_2 and (K, k) is a pair of fields with*

$$K^p \leq k \leq K$$

and p the appropriate characteristic (3 for type G_2 , and 2 otherwise).

Then the following hold:

- (1) *If the pair of fields (K, k) is a stable structure, then the groups $G_0(k, K)$ and $G(k, K)$ are stable.*
- (2) *If K is separably closed then $G_0(k, K)$ and $G(k, K)$ are stable groups.*

(The converse of item (1) is proved separately for type G_2 and PSp_4 , see Theorem 1.6.4 for G_2 , and Theorem 1.7.2 for $G = \mathrm{PSp}_4$.)

In particular for the case of groups of type G_2 we have a class of automorphism groups of Moufang hexagons which are both stable and simple, and nonalgebraic.

We turn now to the rank one case, Timmesfeld's exotic simple groups of type $\mathrm{SL}_2(L)$, defined in Definition 1.4.1 in terms of generalized root groups. Here we have the following standard facts.

Theorem 1.4.2. *Let K be an imperfect field of characteristic 2 and L an additive subgroup satisfying*

$$K^2 \leq L \leq K,$$

where L is a vector space over K^2 . Let $T(L) \leq \mathrm{SL}_2(K)$ be the subgroup of $\mathrm{SL}_2(K)$ with coordinates in the multiplicative subgroup of K generated by L^ . Let $B = T(L)L$ and $N = T(L)\langle w \rangle$.*

Then we have the Bruhat decomposition

$$\mathrm{SL}_2(L) = B \cup BwB.$$

In particular, L is the group of upper unitriangular matrices in $\mathrm{SL}_2(L)$, and $T(L)$ is the diagonal subgroup.

Furthermore, $\mathrm{SL}_2(L)$ is simple.

The definability theoretic properties of the group $\mathrm{SL}_2(L)$ are more subtle and lead us to consider a slight, more generally, groups of the form $T \mathrm{SL}_2(L)$ where T is a subgroup of the diagonal matrices in a larger group $\mathrm{SL}_2(K)$ over a field. We take T to contain the diagonal matrices of $\mathrm{SL}_2(L)$.

Corollary 1.4.9. *Given a (slightly generalized) Timmesfeld group $T \mathrm{SL}_2(L)$, with additive group L and torus T , there is a structure $(\tilde{K}, L, \tilde{T})$ with $\tilde{K}$ a field and $\tilde{T}$ a subgroup of $\tilde{K}$ such that the following are equivalent:*

- (1) *The group $T \mathrm{SL}_2(L)$ is stable.*

- (2) The structure $(\tilde{K}, L, \bar{T})$ with the field structure on $\tilde{K}$ and the additive and multiplicative subgroups L and $\bar{T}$ is stable.

In particular, when T is the subgroup of $\mathrm{SL}_2(L)$ consisting of diagonal matrices (and $T \mathrm{SL}_2(L)$ is $\mathrm{SL}_2(L)$), the corresponding group $\bar{T}$ may be taken to be the subgroup of $\tilde{K}^\times$ generated by the nonzero elements of L .

This is the point at which one realizes that $\mathrm{SL}_2(L)$ is likely to be undefinable in first order terms relative to its natural coordinatization by $(\tilde{K}, L)$, and examples falling under Theorem 1.2.2 confirm this.

Turning to rank 2 and automorphism groups of Moufang polygons, we first consider hexagonal systems of type 1/F, which turn out to coincide with $G_2(k, K) = G_{20}(k, K)$.

Theorem 1.6.3. Suppose (K, k) is a pair of fields of characteristic 3 with

$$K^3 \leq k \leq K$$

Then the following hold:

- (1) The group $G_2(k, K)$ is stable if and only if the pair of fields (K, k) is a stable structure.
- (2) If K is separably closed then $G_2(k, K)$ is a stable simple group.

Theorem 1.6.4. Let (K, k) be a pair of fields in characteristic 3 with

$$K^3 \leq k \leq K$$

and let $U = U(k, K)$ in the sense of $G_2(k, K)$. Then each of U and (K, k) is definable in the other.

This immediately gives

Theorem 1.6.6. The group $G_2(k, K)$ is stable (model-theoretically simple, NTP_2 , NSOP_1 , ...) if and only if the pair of fields (K, k) is stable (resp. model-theoretically simple, ...).

Now we turn to our real interest: the rank two case, and specifically automorphism groups of certain Moufang hexagons (§ 6) and Moufang quadrangles (§ 4). Here we deal with weak indifferent sets in the sense of Definition 1.5.2.

Theorem 1.7.2. Let $(K; L_0, K_0)$ be a weak indifferent set and let U be the group $U(L_0, K_0)$ in the sense of $\mathrm{PSP}_4(L_0, K_0)$. Then each of U and $(K_0, L_0, +, *)$ is definable in the other, where

$$a * b = a^2 b$$

on K_0 .

Theorem 1.7.4. *Let $(K; L_0, K_0)$ be a weak indifferent set, $T(K)$ a maximal torus of $\mathrm{PSp}_4(K)$, and T a subgroup of $T(K)$ normalizing the group $\mathrm{PSp}_4(L_0, K_0)$ and containing $T(K) \cap \mathrm{PSp}_4(L_0, K_0)$.*

Let $\mathcal{M}$ be the structure

$$(K_0; L_0, +, T, \mu)$$

consisting of the group K_0 with the subset L_0 , the abstract group T with its multiplication, and the following additional structure:

- (1) *the map $\mu : K_0 \times K_0 \rightarrow K_0$ defined by $\mu(a, b) = a^2b$;*
- (2) *actions of T on K_0 and on L_0 which correspond to the actions of T on two root subgroups U_α, U_β with α, β the two simple roots, where α is short and β is long.*

Then the group $G = T \mathrm{PSp}_4(L_0, K_0)$ is interdefinable with $\mathcal{M}$.

In particular, G is stable if and only if $\mathcal{M}$ is stable.

1.2 Stable pairs of fields and related structures

Results

For our applications, we need to work with *pairs* of fields, or with more general structures (but again, in pairs), with one contained in the other. But what can be done with a pair of nested fields can also be done, in the same way, with more than two nested fields, and also with the more general coordinatizing systems called *indifferent sets*. Our first result in this line will be the following, which we will need in characteristics 2 and 3, and with $m = 1$, so that we have two distinct fields at our disposal. And we will see that in some particular cases even when we begin with such a pair, it may be useful to pass to a related triple.

Theorem 1.2.1. *Let K be a separably closed field of characteristic $p > 0$, and let*

$$K^p = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq K_{m+1} = K$$

be a chain of subfields of K containing K^p , viewed as a structure with predicates for the fields. Then the theory of this structure is stable.

Furthermore, this theory is axiomatized by the stated properties together with a specification of the dimensions $[K_{i+1} : K_i]$ (as finite values or the formal symbol ∞).

The method of proof will pass through an elimination of quantifiers in an appropriate language—the language customarily used for quantifier elimination in separably closed fields, reviewed below, together with the appropriate unary predicates.

This result already supports the Tits constructions, including some in rank 2, notably in the case of G_2 , which was first described in [17, § 10.3, p. 205 (Remark)].

But as we have explained, we need a more varied supply of coordinatizing structures, involving some additive subgroups as well as subfields—in characteristic 2. The following will be sufficient for our current purposes, though as previously discussed, the question of stability of the associated simple groups would require even more elaborate coordinatizing structures, at this greater level of generality.

The relevant value of m in the next theorem will usually be 2, as we will be working mainly with the two additive groups R_1 and R_2 .

Theorem 1.2.2. *Let K be a separably closed field in characteristic p , and*

$$K^P = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq K_{m+1} = K$$

a chain of subfields of K containing K^P . Furthermore, for $1 \leq i \leq m$ let R_i be an additive subgroup of K_{i+1} which contains K_i and is a vector space over K_i , and which satisfies, in addition, the following two conditions:

- (1) $K_i = \{a \in K \mid aR_i = R_i\}$,
- (2) Any subset of R_i which is linearly independent over K_i is p -independent over K_i .

Then the structure $(K, K_1, \dots, K_m, R_1, \dots, R_m)$ is stable, and its complete theory is given by the properties stated, together with simple numerical invariants: the dimensions of both R_i and K_{i+1} over K_i , as finite values or the formal symbol ∞ .

Algebraic preliminaries

Definition 1.2.3. Let $F \supset E$ be fields of characteristic $p > 0$.

- (1) A subset B of F is p -independent in F if $[F^P[C] : F^P] = p^{|C|}$ for every finite subset C of B ; otherwise, it is said to be p -dependent. A maximal p -independent subset B of F is called a p -basis of F , and one then has $F^P[B] = F$.
- (2) A subset B of F is p -independent over E in F if $[EF^P[C] : EF^P] = p^{|C|}$ whenever C is a finite subset of B . Note that if $E \supset F^P$, we could equally say: B is p -independent in $E^{1/p}$.
- (3) The *degree of imperfection* of the field E is $e \in \mathbb{N} \cup \{\infty\}$ such that $[E : E^P] = p^e$. Equivalently, it is the cardinality of a p -basis if E has a finite p -basis, and the symbol ∞ otherwise.

Notation 1.2.4. Let K be a field of characteristic $p > 0$.

- (1) For each $n > 0$, we fix an enumeration $m_{i,n}(x_1, \dots, x_n)$, $0 \leq i < p^n$, of the p -monomials in $x_1, \dots, x_n$, i.e., of all monomials in $x_1, \dots, x_n$ where the exponents are between 0 and $p - 1$. Without loss of generality, $m_{0,n}(x_1, \dots, x_n) = 1$ for each n .

(2) The λ -functions $\lambda_{i,n}$ on K are defined in the following way:

- If $a_1, \dots, a_n$ are p -independent, and b is p -dependent on $a_1, \dots, a_n$, then the values of the $\lambda_{i,n}$ are uniquely defined by the condition

$$b = \sum_{i=0}^{p^n-1} \lambda_{i,n}(a_1, \dots, a_n; b)^p m_{i,n}(a_1, \dots, a_n).$$

- otherwise, $\lambda_{i,n}(a_1, \dots, a_n; b) = 0$

- (3) Let $\mathcal{L}$ be the language of fields $\{+, -, \cdot, ^{-1}, 0, 1\}$, and let the language $\mathcal{L}_\lambda$ be $\mathcal{L} \cup \{\lambda_{i,n} \mid n \in \mathbb{N}, 0 \leq i < p^n\}$. Observe that the inverse of the Frobenius map is $\mathcal{L}_\lambda$ -quantifier-free definable on K^p : if $b \notin K^p$, then $\lambda_{0,1}(b; x^p) = x$.
- (4) Let B be a p -independent subset of K . For each n and $i < p^n$, we denote by $\lambda_{i,n}^B : B^n \times K \rightarrow K$ the corresponding restriction of $\lambda_{i,n}$. If $a \in K$, we will say that the λ^B -functions are *well-defined at a* when $a \in K^p[B]$. Similarly, the iterates of the λ^B are said to (all) be well-defined at a if $a \in K^{p^n}[B]$ for all $n > 0$.
- (5) Suppose we have a nested sequence of fields

$$K_1 \leq \dots \leq K_m \leq K_{m+1} = K.$$

We define the language

$$\mathcal{L}^m = \mathcal{L}_\lambda \cup \{K_1, \dots, K_m\} \cup \{\lambda_{i,n}^{K_j} \mid n \in \mathbb{N}, 0 \leq i < p^n, j = 1, \dots, m\},$$

where the K_j are unary predicates for the subfields K_j , and the function symbols λ^{K_j} are interpreted as the usual $\lambda_{i,n}$ functions on the field K_j , and 0 outside. If B_j is a p -basis of K_j , then λ^{K_j, B_j} denotes the λ^{K_j} -functions restricted to $B_j^n \times K_j$ (all n).

We now collect some useful results, mostly classical (and trivial if the degree of imperfection of K is finite). We will give most of the proofs, though briefly. More detailed proofs can be found at various points in [1], [2] or [11].

Remark 1.2.5.

(1) Let E be a subfield of K . Then the following are equivalent:

- K is a separable extension of E
- E is closed under the λ -functions of K
- the elements of any (or, some) p -basis of E stay p -independent in K .

In this case, the λ -functions of E and of K agree on E .

- (2) Let $B \subset K$ be p -independent. Assume that the iterates of the λ^B -functions are well-defined at the element a of K , and let A_0 denote the set of these iterates. Then $\mathbb{F}_p(B, A_0)$ is closed under the λ^B -functions. Hence $\mathbb{F}_p(B, A_0)$ has p -basis B , K is a separable extension of $\mathbb{F}_p(B, A_0)$, and $\mathbb{F}_p(B, A_0)$ is closed under the λ -functions of K .
- (3) Let E be a subfield of K closed under the λ -functions of K . Assume that B is a p -basis of K such that $E \cap B$ is a p -basis of E . Let $C \subset K$ be closed under the λ^B -functions. Then $E(C)$ is closed under the λ -functions of K .
Note that in general it is not true that if A_1 and A_2 are $\mathcal{L}_\lambda$ -substructures of K , then so is the field $A_1 A_2$. For example, take a_1, a_2, a_3, a_4 p -independent, and consider $A_1 = \mathbb{F}_p(a_1, a_2)$, $A_2 = \mathbb{F}_p(a_3, a_1 a_2 + a_4^p)$.
- (4) Let E be a subfield of K closed under the λ -functions of K , and let $a \in K$. If A is the closure of $E(a)$ under the λ -functions of K , then A is countably generated over E .
- (5) The λ -functions of K extend uniquely to the separable closure K^s of K .
- (6) Suppose the subfield E of K is an $\mathcal{L}^m$ -substructure of K , and let $a \in K$. Then the $\mathcal{L}^m$ -substructure A of K generated by $E(a)$ is countably generated over E .
- (7) Let $K^p \leq L \leq K$. Let B_1 be a p -basis of L over K^p , and B_2 a p -basis of K over L . Then $B_1 \cup B_2^p$ is a p -basis of L and

$$L = \bigcap_{n \in \mathbb{N}} K^{p^n} [B_1, B_2^p].$$

- (8) Let K and L be separably closed fields, and $E \leq K, L$ an $\mathcal{L}_\lambda$ -substructure. Suppose that K and L are both saturated of the same cardinality κ with $\kappa > |E| + \aleph_0$. Let B be a p -basis of K such that $E \cap B$ is a p -basis of E , and let B' be a p -basis of L containing $E \cap B$. If $f : B \setminus E \rightarrow B' \setminus E$ is a bijection, then $f \cup \text{id}_E$ extends to an isomorphism $K \rightarrow L$.

Proof.

- (1) See [1] or a similar (general) text.

(2) If c is a p -independent n -tuple in K and a, b are two elements of $K^p[c]$, then $\lambda_{i,n}(c; a+b)$ and $\lambda_{i,n}(c; ab)$ belong to the ring generated over $\mathbb{F}_p[c]$ by the elements

$$\lambda_{i,n}(c; a), \lambda_{i,n}(c; b) \text{ for } 0 \leq i < p^n.$$

Moreover, $a^{-1} = a^{-p}(a^{p-1}) \in K^p[a]$; this gives the first assertion, and the second follows by (1).

(3) By (2), $E(C)$ is closed under the λ^B -functions, and the result follows by (1).

(4) Let A be as above, and extend a p -basis of E to a p -basis B of K . Let A_0 be the set of λ^B -iterates of a . As this set of functions is countable, the set A_0 is countable, and involves only countably many elements of B . That is, there is a countable subset B_0 of B such that all iterates of the λ^{B_0} -functions are well-defined at a .

Now by (3), $E(B_0 A_0)$ is closed under the λ -functions of K , and contains A .

(5) We know that for $a \in K^s$ and for each m , we have $a \in K[a^{p^m}]$, so that there are polynomials $f_m \in K[X]$, depending only on the minimal polynomial of a over K , such that $a = f_m(a^{p^m})$ for all m . Given a p -basis B of K the polynomials f_m determine uniquely the values of the iterates of the $\lambda_{i,n}^B(a)$.

(6) For each $j = 1, \dots, m+1$, select a p -basis B_j of K_j such that $B_j \cap E_j$ is a p -basis of E_j . Then, using (4), we build an increasing sequence A_i of subfields of K , where A_0 is the $\lambda^{B_{m+1}}$ -closure of $\{a\}$, and for $i > 0$, with $i \equiv j \pmod{m+1}$, A_i is the λ^{K_j, B_j} -closure of $A_{i-1} \cap K_j$. Since each A_i is countable by (4), so is $A' = \bigcup_{n \in \omega} A_n$, and by (4), since E and A' are closed under the functions λ^{K_j, B_j} , so is $E(A')$.

(7) As B_2 is a p -basis of K over L , B_2^p is a p -basis of K^p over L^p , and therefore $L = L^p[B_1, B_2^p]$ and B_1, B_2^p is a p -basis of L .

Observe now that

$$L = \bigcap_n L^{p^n}[B_1, B_2^p] \leq \bigcap_n K^{p^n}[B_1, B_2^p] \leq K^p[B_1, B_2^p] = L.$$

(8) This is a straightforward back-and-forth argument, using the stability of the theory of separably closed fields of infinite degree of imperfection.

By (5), $f \cup \text{id}_E$ extends to some $\hat{f}$ defined on $E(B)^s$. Assume now that we have an isomorphism

$$g : E_1(B)^s \rightarrow E'_1(B')^s$$

extending $\hat{f}$, with E_1, E'_1 $\mathcal{L}_\lambda$ -substructures of K, L respectively, such that $|E_1| < \kappa$ and $E_1 \cap B$ a p -basis of E_1 .

Let $a \in K \setminus E_1$. By (4), the λ^B -closure A of $E_1(a)$ is countably generated over E_1 , and adding countably many elements of B if necessary, we may assume that $A \cap B$ is a p -basis of A ; let A_1 be countable, closed under the λ^B , and such that $A = E_1(A_1)$.

By saturation of L , there is some $A'_1 \in L$ which realizes $g(\text{tp}(A_1/E_1(A_1 \cap B)^s))$. Then A_1 is independent from B over $A_1 \cap B$ and is separable over $E_1[A_1 \cap B]$, while the type $\text{tp}(A_1/E_1(A_1 \cap B)^s)$ is stationary. It follows that A'_1 realizes $g(\text{tp}(A_1/E_1(B)^s))$. This proves one direction, and the other is symmetric. ■

Proofs

We now give the proofs of Theorems 1.2.1 and 1.2.2. We restate Theorem 1.2.1 in a more explicit form as follows:

Theorem 1.2.1. *Let K be a separably closed field of characteristic $p > 0$, and let*

$$K^P = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq K_{m+1} = K$$

be a chain of subfields of K containing K^P , viewed as a structure with predicates for the fields. Then the theory of this structure is stable.

Furthermore, this theory is axiomatized by the stated properties together with a specification of the dimensions $[K_{i+1} : K_i]$ (as finite values or the formal symbol ∞), and admits elimination of quantifiers in the associated language $\mathcal{L}^m$, with the predicates K_j and the functions $\lambda_{i,n}^{K_j}$ interpreted naturally.

Proof. Since all K_i contain K^P and are contained in K , the sequence

$$K_0 \leq K_1 \leq \cdots \leq K_{m+1} = K$$

is a series of purely inseparable extensions.

Let T_K be theory stating that the sequence of fields has the stated properties, and which, in addition, specifies the degrees $[K_{j+1} : K_j]$. We show first that this theory is complete and allows quantifier elimination.

Let $\mathcal{E} = (E, E_1, \dots, E_m)$ be an $\mathcal{L}^m$ -substructure of $(K, K_1, \dots, K_m)$. Then $K^P \cap E = E^P \subset E_1$, each extension K/E , K_j/E_j is separable, and for $j > 0$, K_j and E_{j+1} are linearly disjoint over E_j , since E_{j+1}/E_j is purely inseparable.

We may assume that $(K, K_1, \dots, K_m)$ is saturated of cardinality $\kappa > |E|$, and we fix another model $(L, L_1, \dots, L_m)$ of T_K containing $\mathcal{E}$ which is also saturated of cardinality κ .

Let B_1 be a p -basis of E_1/E^P , B_2 a p -basis of E_2/E_1 , ..., B_{m+1} a p -basis of E/E_m ; extend B_1 to a p -basis C_1 of K_1/K^P , B_2 to a p -basis C_2 of K_2/K_1 (this is possible because E_1 and K^P are linearly disjoint over E^P , so that $K^P \leq K^P E_1 \leq K_1$ are purely inseparable extensions), ..., B_{m+1} to a p -basis C_{m+1} of K/K_m (again, use that $K/K_m E$ purely inseparable).

Then $\tilde{C} := C_1 \cup \cdots \cup C_{m+1}$ is a p -basis of K , and for each $i = 1, \dots, m$, $\tilde{C}_i := \bigcup_{j=i+1}^{m+1} C_j^P \cup \bigcup_{j=1}^i C_j$ is a p -basis of K_i . Do the same with $L, L_1, \dots, L_m$ to obtain corresponding p -bases $D_1, \dots, D_{m+1}$, and observe that necessarily, either $|C_i| = |D_i|$ is finite, or $|C_i| = |D_i| = \kappa$, by saturation of K and L .

Thus, if f is a bijection between $\bigcup_{j=1}^{m+1} (C_j \setminus B_j)$ and $\bigcup_{j=1}^{m+1} (D_j \setminus B_j)$ which sends each $C_j \setminus B_j$ to $D_j \setminus B_j$, then $\text{id}_E \cup f$ extends to an isomorphism of fields $K \rightarrow L$, which is the identity on E , and sends each K_j to L_j : use (8) and (7) in Remark 1.2.5. This shows that T_K is complete and eliminates quantifiers in $\mathcal{L}^m$.

Now we show stability of T_K . Let $a \in K$, and let A be the $\mathcal{L}^m$ -substructure of K generated by (Ea) . With A_i defined as in the proof of Remark 1.2.5(6), we may assume that the p -bases C_i are chosen to contain a p -basis of A_i over A_{i-1} extending B_i . By Remark 1.2.5(6) (and (4)), there is a countable $\mathcal{L}^m$ -substructure A' of A containing a and such that $EA' = A$ and $C \cap A'$ is a p -basis of A' . By elimination of quantifiers, $\text{tp}_{T_K}(a/E)$ is entirely determined by $\text{qftp}_{\mathcal{L}^m}(A/E)$, and because $A = EA'$ and by Remark 1.2.5(2), there are at most $|E|^{\aleph_0}$ such types. Thus the theory is stable. ■

The proof of Theorem 1.2.2 is very similar. Again, we reformulate it in more precise terms:

Theorem 1.2.2 (in detail). *Let K be a separably closed field in characteristic p , and*

$$K^p = K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_m \leq K_{m+1} = K$$

a chain of subfields of K containing K^p . Furthermore, for $1 \leq i \leq m$ let R_i be an additive subgroup of K_{i+1} which contains K_i and is a vector space over K_i , and which satisfies, in addition, the following two conditions:

- (1) $K_i = \{a \in K \mid aR_i = R_i\}$,
- (2) *Any subset of R_i which is linearly independent over K_i is p -independent over K_i .*

Then the structure $(K, K_1, \dots, K_m, R_1, \dots, R_m)$ is stable, and its complete theory is given by the properties stated together with simple numerical invariants: the dimensions of both R_i and K_{i+1} over K_i , as finite values or the formal symbol ∞ . This theory admits elimination of quantifiers in the associated language $\mathcal{L}_R^m$, with the predicates K_i, R_i and the functions $\lambda_{i,n}^{K_j}$ interpreted naturally.

Proof. Let

$$\mathcal{E} = (E, E_1, \dots, E_m, F_1, \dots, F_m) \subset (K, K_1, \dots, K_m, R_1, \dots, R_m)$$

be a substructure. As in the proof of Theorem 1.2.1, the sequence

$$E^p \leq E_1 \leq \cdots \leq E_m \leq E$$

is purely inseparable, and each K_i/E_i is separable.

As usual, we suppose that the $\mathcal{L}_R^m$ -structure K is saturated, of cardinality $\kappa > |E| + \aleph_0$, and that $(L, L_1, \dots, L_m, S_1, \dots, S_m)$ is another such model containing $\mathcal{E}$. By saturation, any of the invariants which are not finite take on the value κ in both of the $\mathcal{L}^m$ -structures K and L .

The only change in what follows, relative to the proof of Theorem 1.2.1, will lie in the initial choice of p -bases B_i, C_i and D_i , so as to respect the additional structure.

Let B_1 be a p -basis of E_1 over E^P and extend it to a p -basis C_1 of K_1 over K^P . For $i \geq 1$, let B_i be a p -basis of E_{i+1} over E_i , such that $\{1\} \cup (B_i \cap F_i)$ is an E_i -basis of the E_i -vector space F_i . Extend B_i to a p -basis C_i of K_{i+1} over K_i in such a way that $\{1\} \cup (C_i \cap R_i)$ is a K_i -basis of the K_i -vector space R_i ; this is possible because $B_i \cap F_i = B_i \cap R_i$ is a p -basis of the purely inseparable extension $E_i[F_i]$ of E_i , so that $K_i[F_i] \leq K_i[R_i]$ are also purely inseparable extensions of K_i . Choose p -bases D_i within L similarly.

As in Theorem 1.2.1, if $f_i : C_i \setminus B_i \rightarrow D_i \setminus B_i$ is a bijection for $i = 1, \dots, m$, then $\text{id}_E \cup f_1 \cup \dots \cup f_m$ extends to an $\mathcal{L}_\lambda$ -isomorphism $g : K \rightarrow L$, which is an $\mathcal{L}^m$ -isomorphism, and sends R_i to S_i for $i = 1, \dots, m$. This gives completeness of the theory and also quantifier elimination for this language because of the λ -functions and conditions (1) and (2) on (K_i, R_i) .

The proof that the theory is stable goes much as before.

Let E and K be as above, with $E = E^s$, and let $a \in K$. By Remark 1.2.5(4)(6), we know that there is some countable $A_0 \subset K$ containing a , closed under the $\mathcal{L}_R^m$ -functions, containing a p -basis of A_0 , and such that EA_0 is closed under the $\mathcal{L}_R^m$ -functions. By stability of $(K, K_1, \dots, K_m)$, there is some countable substructure E_0 of E , which is separably closed, and such that $\text{tp}_{\mathcal{L}^m}(A_0/E)$ does not fork over E_0 , and enlarging A_0 we may assume that A_0 contains E_0 as an $\mathcal{L}_R$ -substructure. There are $2^{\aleph_0}$ possibilities for $\text{qftp}_{\mathcal{L}_R^m}(A_0/E_0)$, and $|E|^{\aleph_0}$ -many $\mathcal{L}^m$ -formulas saying that $\text{tp}_{\mathcal{L}^m}(A_0/E)$ does not fork over E_0 , so that there are at most $|E|^{\aleph_0}$ types over E . Thus the theory is stable. ■

As an easy corollary, we obtain

Theorem 1.2.6. *Let $K^2 = K_0 \leq K_1 \leq R_1 \leq \dots \leq K_m \leq R_m \leq K$ satisfy the hypotheses of Theorem 1.2.2, and let $S_1, \dots, S_m$ be additive subgroups of K , with S_i a finite-dimensional K_i -vector space contained in K_{i+1} . Then the $\mathcal{L}_R^{2m}$ -structure*

$$\mathcal{K}' = (K, K_1, K_1[R_1 + S_1], \dots, K_m, K_m[R_m + S_m], R_1 + S_1, \dots, R_m + S_m)$$

is stable.

Proof. As the S_i are finite dimensional over K_i , both $R_i + S_i$ and $K_i[R_i + S_i]$ are definable (with parameters) in the $\mathcal{L}_R^{2m}$ -structure $\mathcal{K}$. ■

Lemma 1.2.7. *Let K be a field in characteristic p , and let*

$$K^p = R_0 \leq R_1 \leq R_2 \leq \dots \leq R_m \leq K$$

be an increasing chain of additive subgroups of K . Suppose that for all i with $1 \leq i \leq m$, we have $R_{i-1} \cdot R_i = R_i$. Consider the structure

$$\mathcal{M} = (R_m; R_0, R_1, \dots, R_{m-1}, +, \mu)$$

where the R_i are given as subgroups of R_m and

$$\mu : R_m \times R_m \rightarrow R_m$$

is the function $\mu(a, b) = a^p b$.

Then there are $\mathcal{M}$ -definable fields $K_0, K_1, \dots, K_m, \tilde{K}$ such that

$$\tilde{K}^p = K_1 \leq R_1 \leq K_2 \leq \dots \leq K_m \leq R_m \leq K \leq \tilde{K}$$

and each R_i is a vector space over K_i .

Proof. The element 1 in R_m is clearly definable, hence the p -th power map $F : R_m \rightarrow R_m$ is definable. The restriction of multiplication to R_m is a partial binary operation $a \circ b$ defined by setting $a \circ b = c$ whenever

$$\mu(a, F(b)) = F(c).$$

Define K_m as the multiplicative stabilizer of R_m under $\circ$:

$$\{a \in R_m \mid a \circ R_m = R_m\}.$$

This is a definable subfield of R_m which contains R_{m-1} .

Note that the structure on $\mathcal{M}$ induces the corresponding structure on all R_i and hence we have definable subfields $K_i \leq R_i$ such that R_i is a vector space over K_i , where in addition K_i contains R_{i-1} if $i > 1$, and K_1 contains K^p . Let $\tilde{K} = K_1^{1/p}$. Then $K \leq \tilde{K}$. ■

From a model theoretic point of view, the reduced structure just on the R_i is more convenient for interpretability results as the additional structure may then be treated as coming for free. Normally it would seem prudent, model theoretically, not to add undefinable structure to a given coordinate system. In practice, doing so can be either highly undesirable, or extremely convenient, depending on the circumstances. In the context of Theorem 1.2.2, adding undefinable fields is harmless, and also, at times, extremely convenient. We will see instances of the latter eventually (notably in the setting of rank 1 groups).

We conclude this section with some related questions, which concern the choice of the vector spaces in Theorem 1.2.2. That result gives us a good understanding of the most extreme case. Beyond that case, there may well be other natural theories of similar kinds.

Problems 1.2.8.

- (1) For $p > 2$ and $K^p \leq K_1 < K$, choose $(a_i)_{i \in \mathbb{N}}$ p -independent elements of K over K_1 , and consider $R_1 = \sum_{i \in \mathbb{N}} K_1[a_i]$. Is $\text{Th}(K, K_1, R_1)$ stable? (The case $p = 2$ is covered by Theorem 1.2.2).

Note that the union $A = \bigcup_i \{K_1[a_i] \setminus K_1 \mid i \in \mathbb{N}\}$ is definable in the specified language via the formula $R_1(x) \wedge R_1(x^2) \wedge \neg K_1(x)$. This tends to suggest a level of complexity that may be incompatible with stability. In the structure as we have defined it, modulo K_1 all elements of R_1 have a finite “support” in the set A (and in any model, elements with arbitrary finite supports will occur). Whether this translates concretely into definable complexity remains unclear.

- (2) Let $K^p \leq K_1 \leq K$ be separably closed fields of infinite degree of imperfection, with $[K_1 : K^p] = [K : K_1] = \infty$. Let $\{a_i, b_i \mid i \in \mathbb{N}\}$ be a subset of K consisting of elements p -independent over K_1 , and set

$$R_1 = \sum_{i \in \mathbb{N}} K_1[a_i, b_i].$$

Is $\text{Th}(K, K_1, R_1)$ stable?

Note that a κ -saturated model will not be of the same form, even if $p = 2$.

1.3 Groups of mixed type $G(k, K)$ according to Tits [17]; or a variation

In the present section we will discuss the groups of mixed type over pairs of fields in the spirit of [17] (with some slight variation), continuing on from the broad discussion in the introduction, § 1.1.1. In this context, by applying Theorem 1.2.1, we can identify some simple stable groups which are not algebraic but which one might reasonably call “algebraic over two intimately connected fields.” In Tits’ monograph the focus was on rank at least 3 as far as classification is concerned, but the constructions make sense in rank 2, and in particular the case of G_2 was covered in [17, § 10.3, p. 205 (Remark)].

Definition 1.3.1. Let $G(K)$ be a Chevalley group associated with a root system with roots of two lengths: that is, type B_n, C_n, F_4 , or G_2 .

Fix a pair of fields (k, K) satisfying

$$K^p \leq k \leq K$$

where $p = 3$ if the type is G_2 , and $p = 2$ otherwise.

For α in the root system, define $U_\alpha(k, K)$ to be $U_\alpha(K)$ if α is short and $U_\alpha(k)$ if α is long.

Let $G_0(k, K)$ be the subgroup of $G(K)$ generated by the root subgroups $U_\alpha(k, K)$. The G_0 notation indicates that we follow Tits' construction of $G(k, K)$, but not exactly. The question is what part of the torus to take from $G(K)$ and as we will see in Lemma 1.7.1, there is some latitude in this respect in the case of $C_2(k, K)$, and more generally $\mathrm{PSp}_4(L_0, K_0)$.

Remark 1.3.2. The group $G_0(k, K)$ has a BN-pair

$$B_0(k, K) = T_0(k, K)U(k, K), \quad N_0(k, K),$$

where $U(k, K)$ is generated by root subgroups $U_\alpha(k, K)$ for α positive, and $T_0(k, K)$ is generated by the corresponding root tori; the latter can be defined as the groups obtained by intersecting the rank 1 groups $\langle U_\alpha(k, K), U_{-\alpha}(k, K) \rangle$ with $T(K)$, or more directly as the groups $h_\alpha[U_\alpha(k, K)^*]$ in the notation of [12, Lemma 19]. Then $N_0(k, K)$ may be defined as $N_{G(k)}(T(k))T_0(k, K)$ (which normalizes $T_0(k, K)$ and has as quotient the Weyl group of $G(K)$). That it constitutes a BN-pair can be proved with the classical arguments, using the fact that $U(k, K)$ can be written as a product of the root groups taken in any order, and that the result holds for subgroups of the type of $\mathrm{SL}_2(k)$ and $\mathrm{SL}_2(K)$ (treated more generally in § 1.1.1).

Theorem 1.3.3. *Suppose that $G(k)$ is of adjoint type (centerless) and split over k . Then for $K \neq \mathbb{F}_2, \mathbb{F}_3$, the group $G_0(k, K)$ is simple.*

Proof. We use the Tits simplicity criterion for groups with a BN-pair, as can be found in § 29 of [7]; see in particular Theorem 29.5.

Since our groups have BN-pairs, it suffices to check the following points, with $B = B_0(k, K)$ and $G = G_0(k, K)$.

- (a) B is solvable and centerless.
- (b) The set of generators of W corresponding to the simple roots does not decompose into a union of disjoint, nontrivial, commuting subsets.
- (c) B contains no nontrivial normal subgroup of the full group G .
- (d) G is perfect.

Of these four points, the first is clear, and the second is a basic fact about the classification of the associated root systems. In terms of the usual Dynkin diagram representation it means the diagram is connected. (In the rank two case with which we will be principally concerned, it means that the two simple roots are nonorthogonal—so that the corresponding generators of the Weyl group do not commute.)

The third point may be argued as follows: The group B has a conjugate B^w for which $B \cap B^w = T$, so any normal subgroup X of the full group contained in B would be contained in T . Then $[X, U] \leq X \cap U = 1$ and X centralizes U , forcing $X = 1$ as the torus acts faithfully on U . This last point depends on the fact that the group has no center.

The proof that the group is perfect reduces to the condition $U_\alpha \leq [U_\alpha, T]$ for the roots α , since the root groups generate the full group. This computation can take place in the rank 1 group $\langle U_\alpha, U_{-\alpha} \rangle$, which is SL_2 or PSL_2 over one of the fields k or K . Here we may work concretely with U_α the group of upper triangular unipotent matrices in $\mathrm{SL}_2(I + A$ with A strictly upper triangular) and T the group of diagonal matrices.

Writing $x(a)$ for

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and $h(t)$ for the diagonal matrix with entries (t, t^{-1}) , we have the commutator law

$$[h(t), x(a)] = x(a(1 - t^{-2}))$$

Now we have only to choose t so that t is nonzero and $t^2 \neq 1$ to get the general element of U_α as a commutator. ■

We gave the final computation explicitly as it will serve again in the more general setting of Timmesfeld's rank one groups, below.

Remark 1.3.4. There are no exceptions over $\mathbb{F}_3$, in fact, though for what one might call accidental reasons. Over $\mathbb{F}_3$, our definitions only allow one group, the Chevalley group $G_2(\mathbb{F}_3)$, and it is simple, for reasons like the ones we give, but more delicate [12, Lemma 32].

Of these, types G_2 and C_2 recur below in the context of Moufang polygons (Moufang hexagons and Moufang quadrangles, respectively). Type C_2 is a particular case of the class of Moufang quadrangles said to be of *indifferent type*. As we will see, in a fairly precise sense, the class of groups associated to Moufang polygons of indifferent type is related to the narrower class of groups $C_{2,0}(k, K)$ in exactly the way that Timmesfeld's groups $\mathrm{SL}_2(L)$ are related to the usual groups $\mathrm{SL}_2(k)$ over fields.

From our point of view the interest of these groups lies in the following:

Theorem 1.3.5. *Suppose $G(K)$ is simple of type of type B_n, C_n, F_4 , or G_2 and (K, k) is a pair of fields with*

$$K^p \leq k \leq K$$

and p the appropriate characteristic (3 for type G_2 , and 2 otherwise).

Then the following hold:

- (1) *If the pair of fields (K, k) is a stable structure, then the groups $G_0(k, K)$ and $G(k, K)$ are stable.*
- (2) *If K is separably closed then $G_0(k, K)$ and $G(k, K)$ are stable groups.*

Here $G(k, K)$ denotes the groups as originally defined by Tits in [17], where a suitable torus $T(k, K)$ acting on $U(k, K)$ is defined explicitly.

We will look into definability issues in a sharper form for the case of G_2 in § 1.6. Clearly (1) relates to a couple of claims about interpretability and (2) then follows via Theorem 1.2.1. But we give a proof of this form, in general.

It is also important to note that we set out with the initial expectation that something very similar would also occur in the analogous cases (at greater generality) in ranks 1 and 2, particularly in view of Theorem 1.2.2, but this is not necessarily the case: as already explained in the introduction, things become more subtle in rank 1, and then in rank 2 they remain equally subtle (but become no worse).

Proof. In view of Theorem 1.2.1 it suffices to prove the first point. For that we will use some coarse definability arguments. One should perhaps prove a bi-interpretability result characterizing definability exactly but it is not necessary for our purposes.

To show that the group is definable from the coordinate system (in first order terms) we work inside the algebraic group $G(K)$, which is certainly definable. It suffices to show that the underlying sets of $G_0(k, K)$ and $G(k, K)$ are also definable, in the coordinate system (K, k) , as the group multiplication is inherited.

In view of the Bruhat decompositions

$$G_0(k, K) = \bigsqcup_w B_0(k, K)wB_0(k, K) \quad \text{and} \quad G(k, K) = \bigsqcup_w B(k, K)wB(k, K)$$

with w varying over a finite set of representatives, it suffices to show that $B_0(k, K)$ and $B(k, K)$ are definable.

Relative to the extended coordinate system (K, k) the root groups are definable (parametrized by one of the fields). The group $U(k, K)$ is the product (in any order) of its root subgroups, so it is definable.

The root tori that generate $T_0(k, K)$ are root tori of $G(K)$ or $G(k)$, hence definable in the coordinate system. So $T_0(k, K)$ is definable and $B_0(k, K)$ is definable. The torus $T(k, K)$ is a definable subgroup of $T(K)$ (in the pair of fields (K, k)), so $B(k, K)$ is definable. ■

Remark 1.3.6. It turns out that the condition given in Theorem 1.3.5 (1) is also necessary: one interprets the pair of fields (K, k) in $G_0(k, K)$ and in $G(k, K)$, using the commutator relations. We will give the precise computations in two cases, see Theorem 1.6.4 for G_2 , and Theorem 1.7.2 for $G = \mathrm{PSp}_4$. (In fact, in the case of $\mathrm{PSp}_4(k, K)$ we even prove outright definability.)

Theorem 1.3.5 sets out the model for what we try to do in this paper. This turns out to be more demanding than we initially expected. Theorem 1.2.2 prepares the ground by making an ample supply of some coordinate systems needed to generalize Tits' construction in rank 2, but the definability issues are more severe as well. Namely, when one defines a group as “the group generated by” something, and the coordinate system

defines the generators, then the algebraist may be reasonably happy with that (particularly if a Bruhat decomposition results, and one can tell from that what group one has), but the model theorist needs to worry about the definability of the constituents of the Bruhat decomposition as well. One might reasonably object that if we had followed Tits we would also have defined not only the subgroup $U(k, K)$ but the torus $T(k, K)$ and the group $N(k, K)$ as well, from the coordinate system, and the issue would disappear. We will see next why this is clearly not the case when we take up Timmesfeld's construction in rank 1, and then we will see why the difficulties that appear in rank 1 reappear in rank 2. The only reason they do not appear in higher ranks is that the coordinate systems that appear in higher rank are of a particularly simple type, and in particular the only rank 1 groups that occur in that construction are $\mathrm{SL}_2(k)$, $\mathrm{SL}_2(K)$, and $\mathrm{PSL}_2(K)$.

1.4 The rank 1 case according to Timmesfeld [15]

Timmesfeld presents a very general theory of groups generated by abstract root groups which includes the automorphism groups of most Moufang buildings, and starts off in rank 1 in what amounts to the study of split BN-pairs of rank 1 from another point of view. In particular, even the more exotic rank 1 groups arising as groups generated by pairs of opposite root groups in the context of Moufang buildings are captured by his theory. We are interested in the ones which arise in the specific case of Moufang quadrangles of indifferent type, which we will come to in the next section. In that case, we arrive at the particular rank 1 groups with which Timmesfeld begins his discussion in [15], namely his Example 1.5, as specialized further in [15, Example 1.6 (2), p. 6].

In the presentation below, we begin with the explicit definition, but work out in detail the standard calculations in the manner of Chevalley or [12], in their minimalist form (2×2 matrices). These calculations are identical to the usual calculations in $\mathrm{SL}_2(K)$, but we must pay close attention to where the entries of the matrices lie—and, in particular, which diagonal matrices are actually obtained in Timmesfeld's setting, and whether or not that set is first order definable from the initial data.

Definition 1.4.1 ($\mathrm{SL}_2(L)$ according to Timmesfeld). We begin with an imperfect field K of characteristic 2 and an *additive subgroup* L satisfying

$$K^2 \leq L \leq K,$$

where L is a vector space over K^2 . We then define the group $\mathrm{SL}_2(L)$ to be the subgroup of $\mathrm{SL}_2(K)$ generated by upper and lower unitriangular matrices in $\mathrm{SL}_2(K)$ with coefficients in L .

That is, we have the “root groups” A, A^{op} consisting of the elementary matrices

$$a(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad b(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

respectively, with $t \in L$. And we consider the group $\text{SL}_2(L) = \langle A, A^{\text{op}} \rangle$.

There is a good deal to be said about the group $\text{SL}_2(L)$. Our main concern is with a criterion for stability, which naturally leads us to consider related definability issues, notably the definability of the subgroup of diagonal matrices. This last issue turns out to recur substantially, afterward, in our discussion of rank 2 groups, as some of them contain Timmesfeld’s groups. And for that matter, it is implicit in our treatment of Tits’ construction (where we avoided beginning with a description of the torus), though in that construction the rank 1 tori involved were just the multiplicative groups of the two fields k, K . Here things become more delicate.

We begin with the Bruhat decomposition. As a point of notation, we will denote by L^* the set of non-zero elements of the additive group L . We make elementary calculations but keep track particularly of the diagonal matrices that appear.

Theorem 1.4.2. *Let K be an imperfect field of characteristic 2 and L an additive subgroup satisfying*

$$K^2 \leq L \leq K,$$

where L is a vector space over K^2 . Let $T(L) \leq \text{SL}_2(K)$ be the diagonal subgroup of $\text{SL}_2(K)$ with coordinates in the multiplicative subgroup of K generated by L^ . Let $B = T(L)A$ and $N = T(L)\langle w \rangle$, with A and w as above.*

Then we have the Bruhat decomposition

$$\text{SL}_2(L) = B \cup BwB.$$

In particular, A is the group of upper unitriangular matrices in $\text{SL}_2(L)$, and $T(L)$ is the diagonal subgroup.

Furthermore, $\text{SL}_2(L)$ is simple.

Proof. Given any $a \neq 1$ in A , there is a unique $b \in A^{\text{op}}$ such that $A^b = (A^{\text{op}})^a$, and we write $b = f(a)$; then $f(a(t)) = b(-t^{-1})$. (Even though we are in characteristic 2, we use the minus sign since the computation works in any characteristic). With $a_0 = a(1)$, we find that

$$w := a_0 f(a_0) a_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is an element of the Weyl group of $\text{SL}_2(K)$, and that the elements $a(t)f(a(t))a(t)w$ are diagonal matrices in $\text{SL}_2(L)$ of the form $\text{Diag}(t, t^{-1})$, for $t \in L^*$.

It follows that the subgroup of diagonal elements of $\mathrm{SL}_2(L)$ contains all elements of the group $T(L)$. From the formula $a(t)f(a(t))a(t) = \mathrm{Diag}(t, t^{-1})w$, we deduce that

$$b(-t^{-1}) = a(-t)\mathrm{Diag}(t, t^{-1})wa(-t),$$

so that $A^{\mathrm{op}} \leq \langle A, w, T(L) \rangle$, and in fact,

$$A^{\mathrm{op}} \subseteq AT(L)wA \cup \{1\}.$$

Now we check that these calculations give

$$\mathrm{SL}_2(L) = B \cup BwB$$

by formal manipulations, as in the case of fields.

On the one hand, we know that both $T(L)$ and the element w lie in $\mathrm{SL}_2(L)$, so the inclusion from right to left holds. In the opposite direction it suffices to check that the right hand side is closed under multiplication by A and A^{op} , which is obvious for A . Hence for $A^{\mathrm{op}} = wAw$ it suffices to check closure under multiplication by w , which reduces to the following relations:

$$\begin{aligned} wBwB &= wT(L)AwB = T(L)wAwB = T(L)A^{\mathrm{op}}B \\ &\subseteq T(L)(BwA \cup \{1\})B \subseteq B \cup BwB. \end{aligned}$$

It now follows that $T(L)$ is the full diagonal subgroup of $\mathrm{SL}_2(L)$ and that A is the full subgroup of upper unitriangular matrices of $\mathrm{SL}_2(L)$, since both points are clear in the context of the subgroup B , and the other double coset BwB is disjoint from B .

For the simplicity of the group we use the BN-pair and follow the line of [16, (16), p. 323]. We first show that the group $\mathrm{SL}_2(L)$ is perfect for $|L| > 2$. It suffices to show that A is contained in the commutator subgroup, since then the conjugate A^{op} is also contained in the commutator subgroup, and these two groups generate $\mathrm{SL}_2(L)$.

We claim in fact that $[A, T(L)] = A$. We have

$$[\mathrm{Diag}(t^{-1}, t), a(s)] = a(s(1 - t^2))$$

which for t fixed and not equal to 0 or 1 represents a general element of A . The claim follows.

Now consider a normal subgroup X of $\mathrm{SL}_2(L)$.

If X is contained in B then X is contained in the conjugate B^w and hence in the intersection, which is the group $T(L)$ of diagonal matrices in $\mathrm{SL}_2(L)$. We then have $[X, A] \subseteq X \cap A = 1$, so X is in $C_{T(L)}(A) = 1$, that is, X is trivial.

So suppose now X is not contained in B . Then the group XB contains B properly, and is a union of B double cosets, so by the Bruhat decomposition $XB = \mathrm{SL}_2(L)$; hence the quotient $\mathrm{SL}_2(L)/X$ is isomorphic to a quotient of B , and in particular is

solvable. On the other hand as $\mathrm{SL}_2(L)$ is perfect the quotient is also perfect, and a perfect solvable group is trivial. So in this case $X = \mathrm{SL}_2(L)$.

Thus $\mathrm{SL}_2(L)$ is simple. For a statement from a broader point of view see [15, I (2.10)]. ■

One should notice at this point that the torus $T(L)$ is likely to be undefinable in any natural language (at least, a priori; this is an interesting question in itself). Accordingly, even if the structure (K, L) is stable we run the risk that the group $\mathrm{SL}_2(L)$ is not. But there is a closely related group which is definable from the coordinate system, and has $\mathrm{SL}_2(L)$ as its commutator subgroup: namely, the normalizer of $\mathrm{SL}_2(L)$ in $\mathrm{SL}_2(K)$. So we examine this.

1.4.3. The normalizer of $\mathrm{SL}_2(L)$

For the present we fix the notation K, L as in Timmesfeld's setting and consider $\mathrm{SL}_2(L)$ within $\mathrm{SL}_2(K)$.

Remark 1.4.4. The full diagonal subgroup $T = T(K)$ of $\mathrm{SL}_2(K)$ normalizes $\mathrm{SL}_2(L)$, and the group $T \mathrm{SL}_2(L)$ has the Bruhat decomposition

$$T \mathrm{SL}_2(L) = \hat{B} \cup \hat{B}w\hat{B}$$

with $\hat{B} = T(K)A$.

The point here is that diagonal matrices $\mathrm{Diag}(t, t^{-1})$ act on A and on A^{op} by multiplication by $t^{\pm 2}$, so $T(K)$ leaves A and A^{op} invariant. Then the Bruhat decomposition for $\mathrm{SL}_2(L)$ gives the Bruhat decomposition for $T(K) \mathrm{SL}_2(K)$.

The interest of this group is that it is definable over (K, L) in view of the Bruhat decomposition, and its commutator subgroup is $\mathrm{SL}_2(L)$ since $T(K)$ is abelian. Thus we have a definable stable group with simple commutator subgroup associated to any stable coordinate system (K, L) ; this depends intrinsically on K as well as L , though it would be very natural to take for K the field generated by L to get a more canonical construction (in similar settings in rank 2, this is actually part of the standard approach).

We note that in our definition of Tits' groups we preferred to follow Timmesfeld, and rather than defining a torus in advance, we let it be computed in the group generated by root subgroups. As the coordinate system used was a pair of fields, the rank 1 subgroups $\mathrm{SL}_2(k)$ and $\mathrm{SL}_2(K)$ appearing there were not problematic. But we will need to keep these extra complications—and the need in some cases to sacrifice simplicity for definability—firmly in mind going forward.

Lemma 1.4.5. *In Timmesfeld's setting, the normalizer in $\mathrm{SL}_2(K)$ of the subgroup $\mathrm{SL}_2(L)$ is the group $T(K) \mathrm{SL}_2(L)$.*

Proof. We work first in $\mathrm{GL}_2(K)$. Let $\hat{T}(K)$ denote the full subgroup of diagonal matrices. This also normalizes $\mathrm{SL}_2(L)$. It suffices to check that the normalizer in $\mathrm{GL}_2(K)$ of $\mathrm{SL}_2(L)$ is $\hat{T}(K) \mathrm{SL}_2(L)$. We have noticed that the normalizer contains this group.

Let n belong to the normalizer of $\mathrm{SL}_2(L)$ in $\mathrm{GL}_2(K)$. If n normalizes A then it lies in the Borel subgroup $\hat{T}(K)A(K)$ of $\mathrm{GL}_2(K)$ (where $A(K)$ is the full set of upper triangular unipotent matrices). Hence after multiplying by an element of $\hat{T}(K)$ we may suppose $n \in A(K)$, and write $n = a(t)$ for $t \in K$. In that case consider $L_1 = \langle L, t \rangle$. Since $a(L_1)$ and w lie in the normalizer of $\mathrm{SL}_2(L)$, the group $\mathrm{SL}_2(L_1)$ is also contained in the normalizer of $\mathrm{SL}_2(L)$. But $\mathrm{SL}_2(L_1)$ is a simple group (Theorem 1.4.2), so we find these two groups are equal and $n \in \mathrm{SL}_2(L)$.

If n normalizes A^{op} then wn normalizes A and we conclude similarly.

So suppose $A^n \neq A, A^{\mathrm{op}}$. As the torus $\hat{T}(K)$ acts transitively on the root groups of $\mathrm{SL}_2(K)$ (which correspond to the points of the projective line other than $0, \infty$) we may adjust by $\hat{T}(K)$ and suppose that A is conjugated into a root group of the form $A(K)^b$ where $b \in \mathrm{SL}_2(L)$. But then adjusting by this element of $\hat{T}(K)$ we may again take n to normalize A , and conclude as before. ■

Thus the family of groups normalizing $\mathrm{SL}_2(L)$ in $\mathrm{SL}_2(K)$ is parametrized by the family of groups T_1 lying between $T(L)$ and $T(K)$. We would like to take T_1 to be definable in (K, L) , ideally, but we would be perfectly happy as long as (K, L, T_1) is stable. Here T_1 may be viewed as an abstract multiplicative group with an action on L (corresponding to the action on A in $\mathrm{SL}_2(L)$), or as the image of that action in $\mathrm{Aut}(L)$, or more concretely as the multiplicative subgroup of K whose action on L is given by multiplication. Note that in the second interpretation the action of $\mathrm{Diag}(t^{-1}, t)$ is multiplication by t^2 and in the third interpretation the multiplicative subgroup is actually the corresponding subgroup of K^2 .

An attractive choice for the intermediate torus T_1 is the multiplicative group of the field K_L generated by L^* . This will often not be definable in (K, L) , but we can work equally well with (K_L, L) . And there are good chances that $T(L)$ will be equal to $T(K_L)$ in concrete cases; this leads to interesting questions.

Again: the choice of $T = T(L)$ gives a simple group; the choice of $T = T(K)$ gives a group definable in the original structure (K, L) with $\mathrm{SL}_2(L)$ as commutator subgroup; and the choice $T_1 = K_L^\times$ gives a group which in general is not definable in (K, L) , but is definable in (K_L, L) ; and if Theorem 1.2.2 applies to (K, L) , it will also apply to (K_L, L) . And as always, what we encounter here in rank 1 will recur in much the same form in rank 2.

We formalize the foregoing discussion further as follows:

Theorem 1.4.6. *Let K be an imperfect field K of characteristic 2 and L an additive subgroup satisfying*

$$K^2 \leq L \leq K,$$

where L is a vector space over K^2 . Let T be a group lying between the group $T(L)$ and the group $T(K)$. Let $\bar{T}$ be

$$\{a \in K \mid \text{Multiplication by } a \text{ is induced by some element of } T \text{ acting on } A\}.$$

Let $G = T \text{SL}_2(L)$.

Then the following hold:

- (1) The group G is definable in the structure $(L, \bar{T}, \cdot, \sigma)$, where $\cdot$ is the multiplication map on $L \times \bar{T}$, and σ is the squaring map.
- (2) Conversely, this structure is definable in G .

Proof.

1. One builds the group B from $\bar{T}$, A , and the action. One then builds the group G as

$$B \cup BwB = TA \cup TAwA$$

since w normalizes T . On the right side elements are uniquely represented either by pairs in $T \times A$ or by triples in $T \times A \times A$ (since $A^{\text{op}} \cap B = 1$). Multiplication on this set is then determined by multiplication in B and multiplication by w on the right. This is trivial for the map from TA to TAw and in the case of $TAwAw$ it reduces to the expression of $a(s)^w$ in terms of the Bruhat decomposition, given in the proof of Theorem 1.4.2 as

$$b(-t) = a(-t^{-1})\text{Diag}(t^{-1}, t)wa(-t^{-1}).$$

We may set aside the minus signs as superfluous. We need the operation of multiplicative inversion on L , which comes from squaring followed by the action of $\bar{T}$ on L , and the coordinate of $\text{Diag}(t^{-1}, t)$ in $\bar{T}$, which is t^2 .

2. A and T are, respectively, the centralizers in G of any of their nontrivial elements. So G gives A and T and the action of T on A . This gives $\bar{T}$ as a subset of A .

The element w allows us to define the function f used in the proof of Theorem 1.4.2 to compute the map from $a(t)$ in A to $\text{Diag}(t, t^{-1})$ in T . Thus we have the map from $a(t)$ to multiplication by t^{-2} on L . This then gives both the set $\bar{T}$ as a subset of L , and its action on L by multiplication. That is, $\bar{T}$ is the image of $a(1)$ under T , the image of $a(t)$ under the corresponding element $\text{Diag}(t, t^{-1})$ of T is $a(t^{-1})$, and the squaring map is given by $a(t^{-1}) \mapsto \text{Diag}(t^{-1}, t) \mapsto t^2$. ■

Corollary 1.4.7. *A group of the form $T \text{SL}_2(L)$ in Timmesfeld's setting is stable if and only if the coordinatizing structure*

$$(L, \bar{T}, \cdot, \sigma)$$

is stable.

Now let us give a coordinatization that looks more normal from the algebraic point of view.

Theorem 1.4.8. *Let K be an imperfect field of characteristic 2, L an additive subgroup of K , and $\bar{T}$ a multiplicative subgroup of K^2 which contains L^2 . Suppose that $\bar{T} \cdot L \subseteq L$. Then the structure*

$$\mathcal{L} = (L, \bar{T}, \cdot, \sigma)$$

in which $\cdot$ gives the multiplication on $\bar{T}$ and σ gives the squaring map from L to $\bar{T}$, is bi-interpretable with a structure

$$\mathcal{K} = (K_1, L, \bar{T})$$

where K_1 is a field satisfying Timmesfeld's conditions:

$$K_1^2 \leq L \leq K_1$$

and $\bar{T} \subseteq K_1^2$.

Proof. It suffices to recover a suitable field K_1 definably from $\mathcal{L}$.

We have the multiplication on $\bar{T}$ and the squaring map to T . We may view both as extended to $\bar{T} \cup \{0\}$. Then the restriction $*$ of multiplication from K to L is given, as a partial function, by $a * b = c$ iff $a^2 \cdot b^2 = c^2$.

Let K_0 be the multiplicative stabilizer of L in L :

$$K_0 = \{a \in L \mid aL \leq L\}.$$

This set is definable in $\mathcal{L}$. Furthermore, it is a field containing T and contained, as a set, in L .

The addition on K_0 is inherited from L . The multiplication is inherited from $*$, and is total. So K_0 is a definable field.

Let K_1 be $K_0^{1/2}$ with its field structure, taken as an isomorphic copy of K_0 with an embedding $L \rightarrow \tilde{K}$ corresponding to the squaring map to K_0 . We then have the structure

$$T \leq K_1^2 \leq L \leq K_1$$

with the field structure on K_1 inducing the remaining structure. ■

Corollary 1.4.9. *In the (slightly generalized) Timmesfeld setting, the following are equivalent:*

- (1) $T \text{SL}_2(L)$ is stable.
- (2) The structure $(L, \bar{T}, \cdot, \sigma)$ with $\bar{T} \subseteq L$, $\cdot$ the multiplication on $\bar{T}$, and σ the squaring map from L to $\bar{T}$, is stable.
- (3) The structure $(\tilde{K}, L, \bar{T})$, with $\tilde{K}$ as above is stable.

In the last clause, note also that the group $\mathrm{SL}_2(L)$ in the sense of K is also $\mathrm{SL}_2(L)$ in the sense of $\tilde{K}$.

One can do something quite similar in the rank 2 indifferent case, in principle; namely there will be two rank 1 groups of Timmesfeld type—associated to the two simple roots—and the condition is that both are stable (i.e., both exist within a *single* stable structure).

Let us come back now to the case of $\mathrm{SL}_2(L)$, and consider the problem of stability. This raises interesting questions of model theoretic algebra. We are considering structures

$$(K, L, T)$$

where $T = T(L)$ is the subgroup of $K^\times$ generated by L^* , given as an additional element of structure. By proper choice of K , the problem of stability for $\mathrm{SL}_2(L)$ becomes the problem of stability for structures of this kind. If T is definable in (K, L) there is no difficulty (Theorem 1.2.2). If T happens to be the multiplicative group of the field K_L generated by L , then we may take the ambient field to be K_L , apply Theorem 1.2.2 to that, and in this way force $T(L)$ to be definable. It is not yet clear how often that is the case.

So the questions are of two sorts: when is $T(L)$ in fact the multiplicative group of a field, and in general, when is the expanded structure stable?

Lemma 1.4.10. *Let K be an imperfect field of characteristic 2, and L an additive subgroup of K with*

$$K^2 \leq L \leq K$$

and L a vector space over K^2 . Suppose in addition that L contains a subfield K_1 of codimension 1 (as a vector space over K^2). Then $T(L)$ is the multiplicative group of the field generated by L , and every element of $T(L)$ is the product of two elements of L^ .*

Proof. We write $L = K_1 \oplus K^2 u$ for some $u \in L$, with K_1 a field.

Then L generates the field $K_1(u) = K_1 \oplus K_1 u = K_1 \cdot L \subseteq L \cdot L$. The claim follows. ■

Note therefore that we can always make $T(L)$ definable, in this setting, by including the field K_1 in the coordinate system. In the context of Theorem 1.2.2, the theorem will continue to apply.

In particular, we have the following:

Corollary 1.4.11. *Let K be an imperfect field of characteristic 2, with $[K : K^2] \geq 4$. Let a, b be 2-independent elements of K , and consider $L = K^2 + aK^2 + bK^2$. Then every nonzero element of $K^2[a, b]$ is the product of 2 elements of L^* , and therefore $T(L) = K^2[a, b]^\times$ is definable in (K, L) .*

Proof. As $K_1 = K^2(a)$ is a subfield of L of codimension 1, Lemma 1.4.10 applies. ■

Issues of stability in the groups $\mathrm{SL}_2(L)$ have led us to consider issues of definability in the underlying coordinate systems. It is clear that the field K_L generated by L plays a special role here. One has in general the question of definability of K_L in some particular coordinate system, but when working in the context of separably closed fields, which is the only concrete case known currently, we have observed that this field should be added to the coordinate system and one should consider the issue of definability of $T(L)$ in the extended coordinate system, and in particular the question as to whether $T(L)$ always coincides with the multiplicative group of K_L , a question which reduces to the case of $T(L)$ finite dimensional over K^2 .

Question 1.4.12. Let K be an imperfect field of characteristic 2, and L an additive subgroup of K containing K^2 which is a vector space over K^2 .

- (1) Is $T(L)$ the multiplicative group of K_L ? Does this hold at least if K is separably closed?
- (2) If this is not the case, and the field K is separably closed, is it possible for $T(L)$ to be definable in (K, K_L) nonetheless?
- (3) Can $\mathrm{SL}_2(L)$ be stable when (K, K_L) is not stable?

The first question, restricted to the case of K separably closed, is the main question at present. In the event of a negative solution, the second question should be taken as the natural refinement. Finally, in a situation in which $T(L)$ is not definable in any structure covered by Theorem 1.2.2, the third question remains. This is not strictly a group theoretic question but a question about extending Theorem 1.2.2 to include certain multiplicative subgroups as well as additive subgroups, which seems very difficult.

Since question (1) reduces to the finite dimensional case and one can in principle make detailed computations in that case, it would be of interest to take up the minimal open cases, in which L has dimension 4 over K^2 , or more generally, where L contains a subfield of codimension 2. This seems accessible.

1.5 The rank 2 case: Automorphism groups of Moufang polygons

Our own introduction to this subject came via the elegant work of Tits and Weiss in [18] concerning certain rank 2 groups (or rather, the geometries on which they act). So now we come, finally, to what was our point of departure. In practice we will focus on two of the cases which they consider, where the results of Theorem 1.2.2, or the special case Theorem 1.2.1, are directly applicable. In one case the group considered is the group $G_2(k, K)$ already considered by Tits (though we give it a slightly different definition, one should bear in mind). In the other case it is a substantial generalization

of the Tits group of type C_2 in which the pair of fields used by Tits is replaced by a pair of suitably chosen abelian subgroups of fields in the manner of Timmesfeld.

In this section we will review the point of view of [18], though as we find the groups easier to work with as subgroups of algebraic groups, we will adopt Tits' point of view for the more concrete discussions afterward. So we now discuss the way in which these groups were originally identified, within the scope of a broad classification project (a project initially proposed in [17] in a remark toward the end of the monograph).

The subject of [17] is the theory of buildings, the geometries on which simple algebraic groups, classical groups, and some other groups act naturally; a classification is given in dimension at least 3, which can be taken as a classification of the corresponding groups. These geometries generalize projective geometry, and just as high dimensional projective geometries satisfy the Desargues condition and can then be classified, all the higher dimensional buildings satisfy a related Moufang condition, and are thus called Moufang buildings. Tits proposed the problem of classifying all Moufang buildings in dimension 2 or higher; or more specifically, classifying them in dimension 2 specifically and then reducing the higher dimensional classification to that one. The project is carried through in [18], with some surprises along the way.

In rank 2 the Moufang buildings are called *Moufang polygons*. They are combinatorial point-line geometries which are naturally represented as bipartite graphs where the parts are the points and lines, and the edge relation is incidence. One may also interpret the same graph with the points taken as lines and the lines taken as points, which would be treated as a dual geometry. Accordingly the *automorphisms* are taken to leave the points and lines invariant, and any graph automorphism which switches the parts would be called an *anti-automorphism* in the geometric terminology. Tits and Weiss consider in great detail the structure of the geometric automorphism group $\text{Aut}(\Gamma)$ of a Moufang polygon Γ and in particular a certain subgroup $G^\dagger$ which is almost always simple and which includes the usual Chevalley groups along with many other groups with a very similar structure. In particular, the theory begins with a definition of *root subgroups* directly in terms of the action of the automorphism group on the graph, and $G^\dagger$ is by definition the subgroup generated by a certain family of root groups (those associated with the vertices of an "apartment", which is a cycle of minimal length in the graph).

As in the case of Chevalley groups, one may define a "maximal unipotent" subgroup U generated by half of the root groups (taking a path which covers half of the cycle), which turns out to be a nilpotent group with the root groups as generators and with a generalized Chevalley commutator formula as a set of defining relations. We will consider some cases in which these commutator relations are the ones associated with certain rank two Chevalley groups.

Namely, we consider the Moufang hexagons, which correspond to type G_2 , and more specifically to the groups $G_2(k, K)$, and then the richer family of Moufang quadrangles of *indifferent type*, which correspond to type C_2 , and are realized in PSp_4 (or

Sp_4 , since we work in characteristic 2). In general, the polygon is called an n -gon if the shortest cycle length is $2n$: geometrically an n -gon has n points and n lines and forms a cycle of length $2n$ in the incidence graph. In particular the group U is the (noncommuting) product of n root groups in a Moufang n -gon.

The main result of [18] is a classification theorem for Moufang polygons. Accordingly the various things known about Chevalley groups must not only be generalized, but proved in detail from first principles in a combinatorial setting. This complicates matters relative to the theory of Chevalley groups or algebraic groups, where the main facts are proved algebraically and may even be taken as belonging in part to the initial definition of the group (as in [12]).

But in addition to this, [18] contains detailed studies of the automorphism groups in all of the cases identified in the classification theorem, including that of the (mostly) simple group $G^\dagger$, as well as the full automorphism group and the quotient $\mathrm{Aut}(\Gamma)/G^\dagger$, which can be viewed as a group of automorphisms of U . One of the main results of this analysis is the BN-pair structure for all of the groups between $G^\dagger$ and $\mathrm{Aut}(\Gamma)$. As we have seen in the case of Timmesfeld's groups $\mathrm{SL}_2(L)$, we have reasons to consider larger groups than $G^\dagger$ from the point of view of definability—though we will set aside the portion of $\mathrm{Aut}(\Gamma)$ which corresponds to nontrivial automorphisms of the coordinate system, which is not useful from the point of view of first order definability, and which does not appear in the corresponding algebraic group (when there is one).

Remark 1.5.1. A very general lemma of [18, (7.5)] states that a Moufang polygon is uniquely determined by the associated automorphism group U and its sequence of root subgroups $U_1, \dots, U_n$. In particular the Chevalley commutator formula in U determines the group $G^\dagger$.

We now describe the groups corresponding to the coordinate systems of indifferent type, which generalize Timmesfeld's systems (K, L) .

Definition 1.5.2. A *weak indifferent set* is a triple (K, K_0, L_0) , where K is a field of characteristic 2, and K_0, L_0 are additive subgroups of K for which

$$K^2 \leq L_0 \leq K_0 \leq K,$$

L_0 is a vector space over K_0^2 , and K_0 is a vector space over the field generated by L_0 .

If a weak indifferent set satisfies the additional constraint that the field K is generated by the set K_0 then it is called an *indifferent set*.

It is customary to use indifferent sets in the strong sense in the literature, and we are introducing the terminology *weak indifferent set* here to emphasize the variation. The distinction is not very significant from an algebraic perspective as there would be no harm in replacing the large field K in a weak indifferent set by the field generated by K_0 . However, from a model theoretic point of view, the notion of weak indifferent

set is axiomatizable, and the notion of indifferent set is not, so there is some advantage to allowing the broader notion into the formalism. It does not create any new examples of groups, however.

It is tempting to call a weak indifferent set an indifferent *pair* (even though it is a triple) because the groups L_0, K_0 play the roles previously played by the pair of fields k, K in mixed type groups.

Definition 1.5.3. Let (K, K_0, L_0) be a weak indifferent set. Then

$$\mathrm{PSp}_4(L_0, K_0)$$

is the subgroup of $\mathrm{PSp}_4(K)$ generated by the subgroups $U_\alpha(K_0)$ for α a short root, and by $U_\alpha(L_0)$ for α a long root. We call these groups the *root subgroups* of $\mathrm{PSp}_4(L_0, K_0)$ (which will require a little justification).

Remark 1.5.4. The group $\mathrm{PSp}_4(L_0, K_0)$ is defined by analogy with the group of mixed type $\mathrm{PSp}_4(k, K)$, replacing the pair of fields k, K by an indifferent set. As such it should more properly be denoted

$$\mathrm{PSp}_{4,0}(L_0, K_0)$$

and we may make use of that heavier notation if the point requires emphasis.

We have also identified a suitable torus in the verification of the BN-pair condition and the Bruhat property, and so we could also have followed the route taken by Tits in defining $G(k, K)$. But in any case it is important to us (and to [18]) that this group is generated by its root subgroups.

There is some pathology in this construction, inherited from the rank 1 case, which will require close attention to the torus that appears in $\mathrm{PSp}_4(L_0, K_0)$, and to other tori that normalize this group.

The definition of weak indifferent pair ensures that this group has more or less the same properties as $G_0(k, K)$ where $G = \mathrm{PSp}_4$ and $k = L_0, K = K_0$ are fields. We recall the relevant properties now.

First, the Chevalley commutator formula continues to make sense in our context. For positive roots α, β , writing U_α, U_β for the root groups relative to L_0 or K_0 (as specified), this formula gives the decomposition of elements of $[U_\alpha, U_\beta]$ as products of elements lying in the root groups of the ambient group $\mathrm{PSp}_4(K)$. The point is that these particular elements do in fact lie in the corresponding root subgroups of $\mathrm{PSp}_4(L_0, K_0)$. However, this only holds because in the special characteristics we consider, some terms in the general Chevalley commutator formula vanish, and the corresponding entries do not occur. So this is actually what makes everything work, in algebraic terms.

At the same time, the rank 1 groups $L_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$ become $\mathrm{SL}_2(L_0)$ or $\mathrm{SL}_2(K_0)$ in the sense of Timmesfeld.

One gets the BN-pair property, the corresponding Bruhat decomposition, and simplicity as previously. The computations we made in rank 1 close the gap between the

usual $\mathrm{SL}_2(K)$ and the Timmesfeld variations, and the rest of the argument for the BN-pair is formal, modulo the rank 1 case.

Notice also that $\mathrm{PSp}_4(L_0, K_0)$ lies between $\mathrm{PSp}_4(K^2)$ and $\mathrm{PSp}_4(K)$.

Lemma 1.5.5. *The groups $G_2(k, K)$ and $\mathrm{PSp}_4(L_0, K_0)$ are simple (for $K, K_0 \neq \mathbb{F}_2$).*

Proof. We use the Tits simplicity criterion for groups with a BN-pair, as can be found in § 29 of [7], see in particular Theorem 29.5. Since our groups have BN-pairs, it suffices to check the following points: (a) B is solvable and centerless; (b) the set of generators of W corresponding to the simple roots does not decompose into a union of disjoint, nontrivial, commuting subsets; (c) B contains no nontrivial normal subgroup of the full group G ; and (d) G is perfect.

Of these four points, the first two are clear since there are only two simple roots and the corresponding reflections do not commute (W is a dihedral group of order greater than 4). The other two points were noticed in the proof of the rank 1 case (Theorem 1.4.2), and the proofs given there continue to work. We repeat the main points. The group B has a conjugate B^w for which $B \cap B^w = T$, so any normal subgroup of the full group contained in B would be contained in T , after which it follows easily that it centralizes U , hence lies in U , hence is trivial. The proof that the group is perfect reduces to the condition $A \leq [A, T]$ for the root subgroups A , which is already shown in the rank 1 case. ■

Lemma 1.5.6. *The groups $G_2(k, K)$ and $\mathrm{PSp}_4(L_0, K_0)$ are the groups $G^\dagger$ of [18] corresponding to the Moufang hexagons of type $(1/F)$ and the Moufang quadrangles of indifferent type in the sense of [18].*

Proof. We suppose the field $K \neq \mathbb{F}_2$. By [3, Thm. 6.1], if G is the universal Steinberg group with the same presentation as $G^\dagger$ then $G/Z(G)$ is simple.

Since $G^\dagger$ and the groups of type G_2 or indifferent type are generated by root groups satisfying the same relations, both are homomorphic images of G . Furthermore both groups are simple by Theorem 1.3.3, Remark 1.3.4, Lemma 1.6.2 and [18, (37.3)]. So the kernel in both cases is $Z(G)$ and the two quotients are isomorphic. ■

1.6 Some Moufang hexagons

1.6.1. The hexagonal case

We return to a discussion of the groups of type $G_2(k, K)$, which are among the tamer examples of slightly exotic automorphism groups of Moufang polygons. There are also the groups $\mathrm{PSp}_4(k, K)$, which fall under the indifferent case treated in the next section, but we will not single them out for attention. So $G_2(k, K)$ will serve as our model for the discussion when the definability issues important to model theory are not

severe, and Theorem 1.2.1 (rather than the more general Theorem 1.2.2) is adequate for our purposes.

Before going into details, we remind the reader that our definition of $G_2(k, K)$ is not literally the same as Tits'. On the other hand, as we are in characteristic 3 it becomes quite clear that the torus Tits introduces in his definition is part of our group as well, and the two coincide. In fact there is only one possible torus in this case (as opposed to the situation encountered above in rank 1 and characteristic 2).

Lemma 1.6.2. *Working in $G_2(K)$, the normalizer of the group $U(k, K)$ in the torus $T(K)$ is the torus*

$$T(k, K) := T(K) \cap G_2(k, K).$$

Proof. The torus of $G_2(k, K)$ contains the root tori $H_\alpha(k)$ for α long and $H_\alpha(K)$ for α short. The torus $T(K)$ is the product of the two root tori for the simple roots α and β with α short and β long. So it suffices to check that the normalizer of U_β in $H_\beta(K)$ is $H_\beta(k)$.

This is a rank 1 computation which would not work in characteristic 2, but we are in characteristic 3. The action of the root group element $h_\alpha(t)$ (corresponding to $\text{Diag}(t, t^{-1})$ in our rank 1 computations) is via multiplication by t^{-2} . We also have $K^3 \subseteq k$ so for any t in $H_\beta(K)$ normalizing $U_\beta(k)$, both t^{-2} and t^3 lie in k , and thus $t \in k$. ■

One can make similar computations in other cases—in fact, whenever there are two long roots—but working with rank 2 subgroups. On the other hand, this does not apply to $\text{PSp}_4(k, K)$ and we will return to that point.

At this point we may use the notation $G_2(k, K)$ with a clear conscience to refer to the group either as defined by Tits or as defined here.

Now we recall Theorem 1.3.5, specialized to our context.

Theorem 1.6.3. *Suppose (K, k) is a pair of fields of characteristic 3 with*

$$K^3 \leq k \leq K$$

Then the following hold:

- (1) *The group $G_2(k, K)$ is stable if and only if the pair of fields (K, k) is a stable structure.*
- (2) *If K is separably closed then $G_2(k, K)$ is a stable simple group.*

This goes beyond Theorem 1.3.5, as part (1) has been strengthened to “if and only if.” So it remains to discuss the interpretability of the coordinate system (K, k) in the group $G = G_2(k, K)$. We will give a more precise statement in Theorem 1.6.4 below.

The interpretability of the coordinate system in the group G comes from its interpretability in the Borel subgroup B ; in fact most of it would come from the rank 1

case where B is just $K^\times$ acting on K_+ , which is simply the field disguised as a group. However when two fields k, K are present one also has to embed k into K and for this one uses commutation relations in U .

The question then arises as to whether the commutation relations in U are already enough to recover the coordinate system. This is obviously false in rank 1 since U is just an abelian group in that case. However it is to be expected (or hoped) in rank 2, and we will make this analysis in the two cases of interest here. In G_2 life is greatly simplified (relative to the indifferent case) by the fact that all the root groups are parametrized by fields.

Theorem 1.6.4. *Let (K, k) be a pair of fields in characteristic 3 with*

$$K^3 \leq k \leq K$$

and let $U = U(k, K)$ in the sense of $G_2(k, K)$. Then each of U and (K, k) is definable in the other.

One major question is the extent to which the root subgroups of U are definable (without additional parameters). As homomorphisms from U to $Z(U)$ produce automorphisms of U that move the root groups, this is not literally the case. These definability issues are more straightforward in $B = TU$ since after defining T (from a suitable parameter) one then can define the set of root subgroups without introducing additional parameters, and without looking at the commutation formulas.

Proof. Definability of $U(k, K)$ from the coordinate system is clear in all of the cases we consider, since the original definition of $U(k, K)$ is actually given in first order terms relative to the coordinate system, and there is really nothing to prove. The one point to notice is that the definition begins by specifying commutator relations relating the root groups, and only afterward is it shown that the group U is itself a product of the root groups, taken once each, in any order, so that the group law can be written out on U relative to coordinates in the root groups.

In the particular case of G_2 , we have (positive) root groups $U_1, U_2, \dots, U_6$, with the roots alternately short and long, parametrized by K or k respectively. We form a group generated by the six root groups with the following three *commutator relations*:

$$\begin{aligned} [x_1(a), x_5(b)] &= x_3(-ab); & [x_2(t), x_6(u)] &= x_4(tu); & ([1,5]; [2,6]) \\ [x_1(a), x_6(t)] &= x_2(-ta^3)x_3(ta^2)x_4(t^2a^3)x_5(-ta), & & ([1,6]) \end{aligned}$$

where $a, b \in K$ and $t, u \in k$, and we also require that all other pairs of root groups U_i, U_j commute. (As the characteristic is 3, some terms of the usual Chevalley commutator formula vanish in our case.) Now writing $U = \prod_i U_i$ (in any order) one can write out the group law equally explicitly on U .

Now we want to define the coordinate system from the group U . We will fix some additional parameters shortly. First we work in U purely algebraically.

The commutation formula shows that

$$Z(U) = U_3 \times U_4; \quad Z_2(U) = \langle U_2, Z(U), U_5 \rangle.$$

We set $\bar{U} = U/Z_2(U)$, $\bar{V} = Z_2(U)/Z(U)$, and $Z = Z(U)$ and we have

$$Z = U_3 \times U_4; \quad \bar{V} = \bar{U}_2 \times \bar{U}_5 \simeq U_2 \times U_5; \quad \bar{U} = \bar{U}_1 \times \bar{U}_6 \simeq U_1 \times U_6.$$

Furthermore commutation induces bilinear maps

$$\gamma : \bar{U} \times \bar{U} \rightarrow \bar{V} \quad \gamma' : \bar{V} \times \bar{U} \rightarrow Z.$$

These last maps and groups are definable in U without parameters; but the factorizations mentioned are not.

Now fix the parameters $u_1 = x_1(1)$ and $u_6 = x_6(1)$ in U_1 and U_6 respectively, and thus their images $\bar{u}_1$ and $\bar{u}_6$ in $\bar{U}$, and consider the linear maps

$$\lambda_i : \bar{V} \rightarrow Z$$

given by $\gamma'(x, \bar{u}_i)$ for $i = 1$ or 6 . The image of λ_1 is U_3 and the kernel is $\bar{U}_2$ (use the law [1,5]). The image of λ_2 is U_4 and its kernel is $\bar{U}_5$ (use the law [2,6]). So $\bar{U}_2, U_3, U_4, \bar{U}_5$ are definable. Furthermore λ_1 gives an isomorphism λ_{53} of $\bar{U}_5$ with U_3 , and λ_2 gives an isomorphism λ_{24} of $\bar{U}_2$ with U_4 , given by

$$\lambda_{53}(\bar{x}_5(a)) = x_3(a); \quad \lambda_{24}(\bar{x}_2(t)) = x_4(t).$$

We treat these maps as canonical definable identifications of $\bar{U}_2$ with U_4 , and of $\bar{U}_5$ with U_3 .

Now we consider the linear maps $\pi_i : \bar{U} \rightarrow \bar{V}$ for $i = 1$ or 6 defined by $\gamma(\bar{u}_1, x)$ and $\gamma(x, \bar{u}_6)$. The kernel of π_i is $\bar{U}_i$ (by [1,6]) so these groups are definable. Since $\bar{U}$ and $\bar{V}$ now split definably as $\bar{U}_1 \times \bar{U}_6$ and $\bar{U}_2 \times \bar{U}_5$, the map γ induces maps

$$\mu_j : \bar{U}_1 \times \bar{U}_6 \rightarrow \bar{U}_j$$

with $j = 2$ or 5 , given by

$$\mu_2(\bar{x}_1(a), \bar{x}_6(t)) = \bar{x}_2(-ta^3); \quad \mu_5(\bar{x}_1(a), \bar{x}_6(t)) = \bar{x}_5(-ta).$$

So with $a = -1$ this gives a definable isomorphism of $\bar{U}_6$ and $\bar{U}_2$ respecting coordinates, as well as an embedding of $\bar{U}_6$ into $\bar{U}_5$ respecting coordinates. Furthermore with $t = -1$ we get a definable isomorphism of $\bar{U}_1$ with $\bar{U}_5$ respecting coordinates, and a definable embedding of $\bar{U}_1$ into $\bar{U}_2$ corresponding to the cubing map.

To sum up: after fixing two parameters, we have found isomorphic copies of the root groups U_i which are definable subquotients of U . We have definable identifications of the root groups with even index, respecting coordinates, and also of those with odd index, again respecting coordinates. So it is not excessive now to change notation and to refer to those with even index as k and to those with odd index as K , and to take the map from k to K as an inclusion, letting $\xi : K \rightarrow k$ be the map in the reverse direction, which in coordinates would correspond to cubing.

Modulo these identifications we can simplify the notation (while recalling that the field multiplication is not yet defined in U) and write $m_i(a, t)$ for $\mu_i(\bar{x}_1(a), \bar{x}_6(t))$ ($a \in K, t \in k$), and obtain

$$m_2(a, t) = -a^3t, \quad m_5(a, t) = -at, \quad \xi(a) = a^3.$$

with $a \in K$ and $t \in k$. From these maps we can define multiplication on K by the condition $m_2(a, \xi(b)) = -\xi(ab)$. We now have K, k with the field structure on K and an inclusion map from k to K .

This concludes the proof. ■

Corollary 1.6.5. *Let (K, k) be a pair of fields in characteristic 3 with*

$$K^3 \leq k \leq K.$$

Let U be the group $U(k, K)$ in the sense of $G_2(k, K)$ and let U_i be the root groups occurring in U , with the usual numbering. Then the following structures are bi-interpretable:

- (1) (K, k) ;
- (2) $(U; U_1, U_6, u_1, u_6)$;
- (3) $(U; (U_i)_{1 \leq i \leq 6}, u_1, u_6)$.

Proof. This means that we can not only interpret each in the other, but if we apply both interpretations to recover a copy of one of the structures in itself, the result is definably isomorphic to the original structure, without parameters. As usual, if we start with (K, k) we get an explicit coordinatization of the third structure, and then we can simply trace the interpretation in coordinates.

The other direction is a bit less clear, in terms of what we have done so far. We recovered (K, k) from considerably less information than is given in (2), but for bi-interpretability we need the specified root groups, and not just U together with the parameters that allow us to define isomorphic copies of the root groups as subquotients. The point is that the root groups, represented explicitly as subgroups of U , are certainly definable from the coordinate system (since U is defined as their product), so for bi-definability we must have the actual root groups, and not just isomorphic subquotients. ■

As a further corollary we obtain

Theorem 1.6.6. *The group $G_2(k, K)$ is stable (model-theoretically simple, NTP_2 , $NSOP_1$, ...) if and only if the pair of fields (K, k) is stable (resp., model-theoretically simple, ...).*

1.7 Moufang polygons of indifferent type

Now we come to the case of particular interest, associated with Moufang polygons of indifferent type, or, alternatively, mixing the terminology of Tits and Timmesfeld, the groups $P\mathrm{Sp}_4(L_0, K_0)$ as in Definition 1.5.3.

There are some complications at the outset at this level of generality. The rank 1 subgroups of this group have the form $\mathrm{SL}_2(L_0)$, $\mathrm{SL}_2(K_0)$, sitting inside $\mathrm{SL}_2(K)$, but there is a subtlety. One expects to see either $\mathrm{SL}_2(K)$ or $\mathrm{PSL}_2(K)$ here, depending on the root chosen, but we are in characteristic 2 so we are free to call the group $\mathrm{SL}_2(K)$, and then recognize the subgroup generated by either L_0 or K_0 as one of Timmesfeld's groups. Hence we don't really need to concern ourselves with this point.

A more substantial concern is the nature of the torus in $P\mathrm{Sp}_4(L_0, K_0)$, and more generally, the possible need to enlarge that torus to a larger group T so that the group $T P\mathrm{Sp}_4(L_0, K_0)$ has better definability properties. The torus in $P\mathrm{Sp}_4(L_0, K_0)$ is generated by rank 1 root tori coming from $\mathrm{SL}_2(K_0)$ and $\mathrm{SL}_2(L_0)$, in each case parametrized by the subgroup of K generated by the nonzero elements of K_0 or L_0 , respectively (or by the values under the root map giving the multiplicative action on two root groups, for a pair of simple roots).

We elaborate briefly. It is clear that these rank 1 tori appear in $P\mathrm{Sp}_4(L_0, K_0)$. It is also clear that $P\mathrm{Sp}_4$ is generated by the two rank 1 subgroups associated to simple roots, since they generate U and U^{op} . It follows that if we let T be the torus generated by the two rank 1 tori and we write out the Bruhat decomposition using TU as the group B , then the resulting union of double cosets is in fact the full group, and in particular the torus of $P\mathrm{Sp}_4(L_0, K_0)$ lies in B .

The question that needs to be addressed is, how much of the full torus $T(K)$ of $P\mathrm{Sp}_4(K)$ normalizes the group $P\mathrm{Sp}_4(L_0, K_0)$.

Lemma 1.7.1. *Given a weak indifferent set (K, L_0, K_0) , the subgroup of the full torus $T(K)$ in $P\mathrm{Sp}_4(K)$ which normalizes $P\mathrm{Sp}_4(L_0, K_0)$ is generated by $H_\alpha(K)$ and $H_\beta(K_1)$, where K_1 is the subfield of K which stabilizes K_0 multiplicatively. Here α, β are simple roots with α short, and H_α, H_β are the corresponding rank 1 tori in $T(K)$.*

Proof. We compute via [12, Lemma 19].

The Cartan matrix of this root system is

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

where the first root is short and the second is long.

This means that the short root torus $H_\alpha(K)$ operates by squaring (or its inverse) on both root subgroups (for simple roots), and the long root torus $H_\beta(K)$ operates by squaring on $U_\beta(K)$ and multiplication (after inversion) on the other root subgroup.

As $K^2 \leq L_0 \leq K_0$, the only constraint on the torus is that the element $h_\beta(t)$ will occur only if $tK_0 = K_0$. ■

Notice that K_1 contains the field generated by L_0 , which we know a priori given the structure of the torus in $\mathrm{PSP}_4(L_0, K_0)$ —but it is reassuring that it also follows from the definition of K_1 and the hypotheses.

We mentioned earlier that $\mathrm{PSP}_4(L_0, K_0)$ (see Definition 1.5.3 and Lemma 1.5.5) is simple. For definability purposes we may want to consider the group $T\mathrm{PSP}_4(L_0, K_0)$ for some torus normalizing $\mathrm{PSP}_4(L_0, K_0)$ with better definability properties. The goal as always is to obtain simple groups with stable theory of the specified type, or, failing that, at least to find a group $T\mathrm{PSP}_4(L_0, K_0)$ with stable theory, and be content to have a simple commutator subgroup.

Accordingly, it remains only to discuss definability issues in groups of the form $T\mathrm{PSP}_4(L_0, K_0)$. We can more or less combine Theorems 1.3.5 and 1.4.6 to get a definability theorem for $T\mathrm{PSP}_4(L_0, K_0)$, namely Theorem 1.7.4 below. But since we also intend to look more narrowly at definability in $U(k, K)$, we begin with that.

Theorem 1.7.2. *Let $(K; L_0, K_0)$ be a weak indifferent set and let U be the group $U(L_0, K_0)$ in the sense of $\mathrm{PSP}_4(L_0, K_0)$. Then each of U and $(K_0, L_0, +, *)$ is definable in the other, where*

$$a * b = a^2 b$$

on K_0 .

Proof. To go from the coordinate system to the group, recall that we can find a definable field $\tilde{K}$ so that

$$(\tilde{K}^2 \leq L_0 \leq K_0 \leq \tilde{K})$$

(Theorem 1.4.8).

As usual U is constructed explicitly from root groups U_1, U_2, U_3, U_4 which are copies of L_0 or K_0 , most of which are taken to commute, while U_1, U_4 satisfy

$$[x_1(t), x_4(a)] = x_2(t^2 a) x_3(ta), \quad ([1, 4])$$

where t runs over K_0 and a runs over L_0 , and we use the vector space structure on L_0 and K_0 .

We illustrate the structure of this formula as follows, for reference.

$$\begin{array}{ccc}
 U_1 & U_2 \cdot U_3 & U_4 \\
 \hline
 & [x_t(t), x_4(a)] & \\
 x_1(t) & x_2(t^2 a) \cdot x_3(at) & x_4(a) \\
 t \in K_0 & t^2 \cdot a \in L_0, at \in K_0 & a \in L_0
 \end{array}$$

This is clearly definable in our coordinate system, if we include $\tilde{K}$, to get the action of L_0 on K_0 .

In the converse direction, working in the group U , there is more to be done.

Let $Z = Z(U)$, a definable subgroup. From the commutator formula and the fact that all other pairs of root groups in U commute, we find that $Z = U_2 U_3 \simeq U_2 \times U_3$.

Claim 1. The subset $U_2 \cup U_3$ of U is definable.

We may describe $U_2 \cup U_3$ as the set of elements $x_2(a)x_3(t)$ with $a \in L_0, t \in K_0$ and at least one coordinate equal to 0. For $t \in K_0$ and $a \in L_0$ both nonzero, we have

$$[x_1(at^{-1}), x_4(t^2 a^{-1})] = x_2(a)x_3(t).$$

Thus elements of Z which are not in $U_2 \cup U_3$ are therefore commutators. On the other hand, by inspection of the commutator formula [1,4], we see that the only elements of $U_2 \cup U_3$ which are commutators are those of the form $[x_1(t), x_4(a)]$ with at least one of t or a equal to 0; such an element must be the identity. This proves the claim.

We will now fix some additional parameters $u_i = x_i(1) \in U_i$ for $i = 1, 2, 3, 4$. We let $\bar{U}_1, \bar{U}_4$ denote the images of U_1, U_4 in $\bar{U} = U/Z$, and note that $\bar{U} = \bar{U}_1 \times \bar{U}_4$.

Claim 2. Relative to the specified parameters, the groups $\bar{U}_1, U_2, U_3, \bar{U}_4$ are definable.

By the commutator formula, the centralizer in U of u_1 is $U_1 Z$, and the image of this group in $\bar{U}$ is $\bar{U}_1$. So $\bar{U}_1$ is definable, and similarly $\bar{U}_4$ is definable.

On the other hand, U_2 is the intersection of $u_2 \cdot (U_2 \cup U_3)$ with $U_2 \cup U_3$ and so the group U_2 is definable. The group U_3 is definable similarly.

This proves the claim.

Now the commutator induces a bilinear map

$$\bar{U}_1 \times \bar{U}_4 \rightarrow Z = U_2 \times U_3$$

and in view of Claim 2, we can interpret the commutation law as giving two definable functions of two variables from $\bar{U}_1 \times \bar{U}_4$ to U_2 and U_3 , respectively. These two functions, expressed in coordinates, are the maps

$$\begin{aligned}
 \mu_2 : (\bar{x}_1(t), \bar{x}_4(a)) &\mapsto x_2(t^2 a) \in U_2, \quad (\text{here } t^2 a \in L_0); \\
 \mu_3 : (\bar{x}_1(t), \bar{x}_4(a)) &\mapsto x_3(ta) \in U_3 \quad (\text{here } ta \in K_0).
 \end{aligned}$$

With $t = 1$ the map μ_2 gives an isomorphism $\bar{U}_4 \rightarrow U_2$ respecting coordinates.

With $a = 1$ the map μ_3 gives an isomorphism $\bar{U}_1 \rightarrow U_3$ respecting coordinates.

With these definable isomorphisms available, we identify $\bar{U}_1$ with U_3 and with the additive group of K_0 . Likewise, we identify $\bar{U}_4$ with U_2 and the additive group of L_0 . Now K_0, L_0 are interpreted in U and the maps μ_2, μ_3 become definable maps m_2, m_3 from $K_0 \times L_0$ to L_0 and K_0 respectively, satisfying the following conditions:

$$\begin{aligned} m_2(t, a) &= t^2 a; \\ m_3(t, a) &= ta. \end{aligned}$$

With $t = 1$ the map m_3 defines the embedding of L_0 into K_0 , so we now view L_0 as a subset of K_0 , and both maps m_2 and m_3 now have image in K_0 (identified with U_3). In other words, the definable structure present is

$$(K_0; L_0, +, m_2, m_3).$$

With $a = 1$, m_2 defines the squaring map on K_0 . Now for $a, b \in K_0$, we have $a^2 \in L_0$, and $m_3(b, a^2) = a^2 b$. ■

Corollary 1.7.3. *Let $(K; L_0, K_0)$ be a weak indifferent set and let U be the group $U(k, K)$ in the sense of $\text{PSP}_4(k, K)$, with root subgroups U_i ($1 \leq i \leq 4$) numbered as usual. Then the following structures are bi-interpretable:*

- (1) $(K_0, L_0; +, *)$ where $a * b = a^2 b$ for $a, b \in K_0$.
- (2) $(U; +, (U_i)_{1 \leq i \leq 4}, u_1, \dots, u_4)$ with U_i the root groups and $u_i = x_i(1) \in U_i$.

Proof. As in the proof of Corollary 1.6.5. ■

Theorem 1.7.4. *Let $(K; L_0, K_0)$ be a weak indifferent set, $T(K)$ a maximal torus of $\text{PSP}_4(K)$, and T a subgroup of $T(K)$ normalizing the group $\text{PSP}_4(L_0, K_0)$ and containing $T(K) \cap \text{PSP}_4(L_0, K_0)$.*

Let $\mathcal{M}$ be the structure

$$(K_0; L_0, T, +, \mu)$$

consisting of the group K_0 with the subset L_0 , the abstract group T with its multiplication, and the following additional structure:

- (1) *the map $\mu : K_0 \times K_0 \rightarrow K_0$ defined by $\mu(a, b) = a^2 b$;*
- (2) *actions of T on K_0 and on L_0 which correspond to the actions of T on two root subgroups U_α, U_β with α, β the two simple roots, where α is short and β is long.*

Then the group $G = T \text{PSP}_4(L_0, K_0)$ is interdefinable with $\mathcal{M}$.

In particular, G is stable if and only if $\mathcal{M}$ is stable.

Note that by Lemma 1.2.7 we may also include fields $\tilde{K}, \tilde{L}$ with

$$\tilde{K}^2 \leq L_0 \leq \tilde{L} \leq K_0 \leq \tilde{K},$$

where $\tilde{K}$ contains K , but to do so we need to choose fields which are definable in the given structure.

Typically T is a product of “root tori” (intersections with root tori in $T(K)$), and this can be simplified further to just take the action of a given root torus on the corresponding root group. But we prefer to give the statement in its general form.

Proof. We repeat the proof of Theorem 1.3.5 with minor adjustments.

1. First we show that G is definable from the coordinatizing structure $\mathcal{M}$. We will take a definable field $\tilde{K}$ with

$$\tilde{K}^2 \leq L_0 \leq K_0 \leq K \leq \tilde{K}$$

and work in the larger group $G(\tilde{K})$ as the “ambient” group.

We build the group $U(k, K)$ as a subgroup of the group $\mathrm{PSp}_4(\tilde{K})$ using the coordinate system $(\tilde{K}; L_0, K_0)$. We use the action of T on L_0 and K_0 to define an action of T on the root groups $U_\alpha(K_0)$ and $U_\beta(L_0)$ for α, β simple, α short, β long, which determines the action on U by the commutator formula; that is, the two given root groups generate $U(k, K)$ in a bounded way. One can then find a unique embedding of T into $T(\tilde{K})$ which respects that action. So now B is a definable subset (in an appropriate language) in $\mathrm{PSp}_4(\tilde{K})$. The Bruhat decomposition therefore gives G as a definable subset of $\mathrm{PSp}_4(\tilde{K})$, and the multiplication law is inherited, so the group G is definable.

2. For the converse, we first show that U and T are definable in G using parameters. We exploit the definability of centralizers as follows⁴. Working in $\mathbb{G}(K^{alg})$ for $\mathbb{G}$ an algebraic group, we know that the descending chain condition on centralizers holds, i.e., that given $X \subset \mathbb{G}(K^{alg})$, there is some finite $X_0 \subset X$ such that $C_{\mathbb{G}(K^{alg})}(X) = C_{\mathbb{G}(K^{alg})}(X_0)$. But this property is preserved by going to subgroups: i.e., if $X \subset G$, then $C_G(X)$ is definable. In particular, as the torus $T_0(k, K)$ is self-centralizing in the group $\mathrm{PSp}_4(k, K)$, the group T is self-centralizing in G , hence definable. Similarly, the centralizer Z of $U(k, K)$ in G is the center of $U(k, K)$. So T and Z are definable in G .

Claim. Each root subgroup U_α is equal to $CC(U_\alpha)$, hence definable.

Proof of the claim. First, we can suppose the root subgroup is in the center of U , because root groups of the same length are conjugate (see Appendix I.15 in [12], or Lemma C, section 10.4 in [6] in the classical case). Then its centralizer contains U , and contains the rank 1 component $L_\beta = \langle U_\beta, U_{-\beta} \rangle$, with $\pm\beta$ the unique root orthogonal to α so

⁴Cf. [10], as mentioned in the note on p. 5.

that $L_\beta \leq C_G(U_\alpha)$. Hence the double centralizer of U_α is contained in the center of U and commutes with L_β . Looking at Lemma 19(c) in [12], we see that the action of a typical element $h_\beta(t)$ of the torus of L_β acts on a root subgroup U_γ as multiplication by $t^{\langle \beta, \gamma \rangle}$, where $\langle \beta, \gamma \rangle$ denotes the inner product of the roots β, γ in $\mathbb{R}^2$. In our case α is the only positive root which is orthogonal to β , which proves our assertion. ■

Thus the group U is definable as the product of the U_α with α positive.

If $L_\alpha = \langle U_{\pm\alpha} \rangle$, we can also define the group TL_α . Indeed, we have the group $B_\alpha = TU_\alpha$, and then TL_α is definable via its Bruhat decomposition. If $k = L_0$ or K_0 is the parametrizing group for U_α , this group gives us (k, T) with T represented via its action on k .

As we have the group U , we have the remaining structure on the coordinate system, by Theorem 1.7.2. ■

1.8 Concluding Remarks

While the paper is certainly not self-contained, we have developed some parts of the group theoretic analysis in large detail, as it casts considerable light on the definability issues and other important matters. Much of this is of the sort found in Steinberg's notes [12] or, from a radically different point of view, in the book of Tits and Weiss [18].

Once again, Theorem 1.2.2 (as well as Theorem 1.2.6) applies to give a number of stable simple groups acting as automorphism groups of Moufang polygons of indifferent type, or in some cases stable groups with simple commutator subgroup acting as automorphism groups of the same kind. In particular, in some cases adding a non-definable field to the coordinatizing structure will provide additional examples which are in fact simple.

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