5 Equivalence Relations

5.1 Binary Relations

Functions	=	Operations
Relations	=	Properties

We will only consider *binary* relations.

Example A typical relation is the order relation $<$ on a set of numbers A. Let us take $A = \{0, 1, 2\}$. Thus $0 < 1$, $2 \not< 1$, etc. The relation $<$ is defined set theoretically in terms of its graph:

$$<= \{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 2\rangle\}$$

Example Consider the relation E on $A = \{0, 1, 2\}$ defined by

$$xEy \text{ iff } x - y \text{ is even}$$

Then

$$E = \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 0, 2\rangle, \langle 1, 3\rangle, \langle 2, 0\rangle, \langle 3, 1\rangle\}$$

Definition 5.1. Let A be a set. A *relation* on A is a subset of $A \times A$.

Notation 5.2. *If A is a set, R is a relation on A, and $\langle a, b\rangle \in A \times A$, it is customary to write*

$$aRb$$

rather than $\langle a, b\rangle \in R$.

Thus we may write $0 < 1$ rather than $\langle 0, 1\rangle \in<$ and $0 \in \{0\}$ rather than $\langle 0, \{0\}\rangle \in\in$.

Definition 5.3 (More general)**.** A *relation* is a set of ordered pairs.

Definition 5.4. If R is a relation, then we define the *domain, range,*, and *field* of R by

$$x \in \operatorname{dom} R \iff \exists y\, \langle x, y\rangle \in R \tag{1}$$
$$x \in \operatorname{ran} R \iff \exists y\, \langle y, x\rangle \in R \tag{2}$$
$$\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R \tag{3}$$

Remark 5.5. If R is a relation and $A = \operatorname{fld} R$, then R is a relation on A.

Example Let $R = \{\langle \ell, n\rangle : \ell$ is a letter, $n \in \mathbb{N}$, and the n-th letter of the English alphabet is $\ell\}$. Then

$$\operatorname{dom} R = \{a, b, c, \ldots, z\} \tag{4}$$
$$\operatorname{ran} R = \{1, 2, 3, \ldots, 26\} \tag{5}$$

The field of R is the union of both sets.

The most important kinds of relations are the following:

- functions $f : A \to B$;
- equivalence relations (next);
- order relations (following section)

A function f is a relation such that for every $\langle a, b\rangle \in f$, a determines b.

One can take unions of intersections of relations: $x(R_1 \cup R_2)y$ means "xR_1y or xR_2y" while $x(R_1 \cap R_2)$ means "xR_1y and xR_2y".

Exercise 5.6. Let D_n be the relation on $\mathbb{Z}$ defined by

$$xD_ny \text{ iff } x - y \text{ is a multiple of } n$$

(a) Is there a number m so that $D_4 \cap D_6 = D_m$?

(b) Is there a nubmer n so that $D_4 \cup D_6 = D_n$?

5.2 Equivalence Relations

Definition 5.7. Let R be a binary relation on a set A.

- R is *reflexive* iff xRx for all $x \in A$.
- R is *symmmetric* if $xRy \implies yRx$ for all $x, y \in A$.
- R is *transitive* if $xRy, yRz \implies xRz$ for all $x, y, z \in A$.
- R is an equivalence relation if R is reflexive, symmetric, and transitive.

Example The following relations are *not* equivalence relations.

- R_1: xR_1y iff x and y are *siblings* (field: people)
- R_2: xR_2y iff $x < y$ (field: $\mathbb{R}$)
- R_3: xR_3y iff $|x - y| < .001$ (field: $\mathbb{R}$)

Example
The following relations are equivalence relations.

- E_1: xE_1y if x and y have the same mother.(field: people)
- E_2: xE_2y if x and y have the same father.(field: people)
- $E_3 = E_1 \cap E_2$.(field: people)
- E_4 (field $M_3(\mathbb{R})$: AE_4B iff A and B have the same eigenvalues.
- D_n (field $\mathbb{Z}$) defined by xD_ny iff $x - y$ is a multiple of n; n fixed.
- E_5 (field $M_3(\mathbb{R})$: AE_5B iff there is an invertible matrix P with $A = PBP^{-1}$

- E_V (field $\mathbb{R}$): xE_Vy iff $x - y \in \mathbb{Q}$)

Proposition 5.8. *For any $n \in \mathbb{Z}$, the relation D_n is an equivalence relation on $\mathbb{Z}$.*

Proof.

- *Reflexivity:* For $x \in \mathbb{Z}$, we have $x - x = 0 = 0 \cdot n$, so xD_nx.
- *Symmetry:* Let $x, y \in \mathbb{Z}$ and suppose xD_ny. Then $x - y = nk$ for some $k \in \mathbb{N}$, and $y - x = n(-k)$, so yD_nx.
- *Transitivity:* Let $x, y, z \in \mathbb{Z}$ with xD_ny and yD_nz. Then $x - y = nk_1$ and $y - z = nk_2$ for some $k_1, k_2 \in \mathbb{Z}$. So $x - z = (x - y) + (y - z) = n(k_1 + k_2)$ and xD_nz.

□

Exercise 5.9. Show that E_1 is an equivalence relation.

Exercise 5.10. Show that if E_1, E_2 are equivalence relations on A, then $E_1 \cap E_2$ is an equivalence relation on A.

Our first four example all have the following form.

Definition 5.11. Let $f : A \to B$ be a function. The E_f is the relation on A defined by:

$$aE_fb \text{ iff } f(a) = f(b)$$

For E_1, E_2 let the function assign a person to his or her mother, or father, respectively. For E_3 use the ordered pair $\langle$father,mother$\rangle$. For E_4 assign to each matrix its set of eigenvalues.
Also D_{10}, D_2 can be put in this form: for D_{10} let $f(n) =$ the last digit of n. For D_2 let $f(n)$ be the *parity* of n (for n even this is 0, for n odd it is 1).
It is harder find functions f to represent the other relations D_n and E_5 or E_6 as E_f, but we will do this soon.

Exercise 5.12. If $f : A \to B$, show that the relation E_f is an equivalence relation on A.

Exercise 5.13. Define a relation E on $\mathbb{N} \times \mathbb{N}$ by

$\langle a, b\rangle E\langle c, d\rangle$ iff $a + d = c + b$. Give two proofs that E is an equivalence relation:

1. Show that $E = E_f$ where f is the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ defined by $f(m, n) = m - n$
2. Check directly that E is reflexive, symmetric and transitive, and do this *without using the notion of subtraction.*

Remark 5.14. Explanation: This exercise is the key step in the construction of $\mathbb{Z}$ from $\mathbb{N}$. The first proof we ask for is more direct than the second proof, but the first proof makes use of properties of $\mathbb{Z}$. So we actually will need the second proof!

5.3 Equivalence Classes and Partitions

Theorem 5.15. *Let E be an equivalence relation on the set A. Then there is a set B and a function $f : A \to B$ such that $E = E_f$.*

The proof requires some machinery.

Definition 5.16. Let R be an equivalence relation on the set A. Then for $x \in A$ we define the *R-equivalence class* of x (denoted $[x]_R$) by

$$[x]_R = \{y \in A | xRy\}$$

Proposition 5.17. *Suppose that R is an equivalence relation on the set A. Then the set*

$$\Pi = \{[x]_R : x \in A\}$$

is a partition of A.

ExampleConsider the relation D_2 on $\mathbb{Z}$. The class $[n]_{D_2}$ will be the class of all even integers if n is even, and all odd integers if n is odd.

Proof. We need to prove three things.

- Each set $[x]_R$ is nonempty.
- $\bigcup \Pi = A$;
- If $x, y \in A$ and $[x]_R, [y]_R$ are distinct, then they are disjoint.

In fact we will prove the following, which is more explicit.

- $x \in [x]_R$ for $x \in A$.
- If $x, y \in A$ and $[x]_R$ meets $[y]_R$, then $[x]_R = [y]_R$

If $x \in A$ then $x \in [x]_R$ by reflexivity and our first claim is proved. This implies that each of the sets $[x]_R$ is nonempty, and that their union is A.
Now we prove our second claim.
If $x, y \in A$ and $[x]_R$ meets $[y]_R$ then let $z \in [x]_R \cap [y]_R$. That is: xRz and yRz. Using symmetry and transitivity we get xRz, zRy, and then xRy. We show $[y]_R \subseteq [x]_R$: if $t \in [y]_R$, then yRt and by transitivity xRt, so $t \in [x]_R$. Similarly we may show $[x]_R \subseteq [y]_R$ and conclude $[x]_R = [y]_R$. $\square$

Example Let $f : Rr \to \mathbb{R}$ be the function $f(x) = x^2$. The relation E_f has equivalence classes defined by conditions $x^2 = C$ where $C \geq 0$. These have the form

$$\{\pm t\}$$

for $t \in \mathbb{R}$. One of these classes is the set $\{0\}$ and all the others are pairs.

Theorem 5.18. *Suppose that Π is a partition of the set A. Then the relation R_Π on A defined by*

$$xR_\Pi y \textit{ iff } [x]_R = [y]_R$$

is an equivalence relation.

Proof. Define $f : A \to \Pi$ by $f(x) = [x]_R$, and observe that $E_\Pi = E_f$. $\square$

5.4 Quotients

Definition 5.19. If R is an equivalence relation on the set A, then the *quotient* A/R denotes the associated partition into equivalence classes, and the *natural map* (or the "*canonical map*")

$$f : A \to A/R$$

is defined by

$$f(x) = [x]_R$$

Exercise 5.20. Let $f : A \to B$ be a surjection and E_f the corresponding equivalence relation on A. Define a function $\phi : B \to \mathcal{P}(A)$ by

$$\phi(b) = \{a \in A : f(a) = b\}$$

Then $\phi : B \leftrightarrow A/E_f$ is a bijection.

Example Let $f : \mathbb{N}^2 \to \mathbb{Z}$ be defined by

$$f(\langle m, n\rangle) = m - n$$

Then the associated function ϕ is a bijection between $\mathbb{N}^2/E_f$ and $\mathbb{Z}$. This observation will be used in the construction of $\mathbb{Z}$.

5.5 Summing Up

We have shown that each equivalence relation gives a partition and that each partition gives an equivalence relation. This has the practical consequence that equivalence relations and partitions can be treated as two different descriptions of the same idea. We make this more precise as follows.

Theorem 5.21. *Let A be a set, let $\mathcal{E}$ be the set of all equivalence relations on A, and let $\mathcal{P}$ be the set of all partitions of A. Define functions*

$$\pi : \mathcal{E} \to \mathcal{P},\ \epsilon : \mathcal{E} \to \mathcal{P}$$

by $\pi(E) = A/E$ for $E \in \mathcal{E}$, and $\epsilon(\Pi) = R_\Pi$ for $\Pi \in \mathcal{P}$. Then $\pi : \mathcal{E} \leftrightarrow \mathcal{P}$ and $\epsilon : \mathcal{E} \leftrightarrow \mathcal{P}$ are bijections, and each is the inverse of the other. In other words,

- $R_{A/E} = E$ *for* $E \in \mathcal{E}$;
- $A/R_\Pi = \Pi$ *for* $\Pi \in \mathcal{P}$.

The proofs are simply a matter of applying all the definitions carefully, so we leave them as exercises. But the result is important.

Example If A is finite, the number of equivalence relations on A is the same as the number of partitions of A. In practice, it is easier to count partitions than equivalence relations.

Exercise 5.22. Show that for a set A with 3 elements, there are 5 equivalence relations on A, and that for a set with 4 elements, there are 15 equivalence relations.

Example

- For $d = 0, 1, \ldots, 9$ let $L_d \subseteq \mathbb{N}$ be the set of natural numbers with last digit d. Then the associated equivalence relation is E_{10}: m and n have the same last digit if and only if $m - n$ is a multiple of 10.
- For $d = 0, 1, \ldots, 9$ let $F_d \subseteq \mathbb{N}$ be the set of natural numbers with first digit d. Then the simplest way to define this relation is simply to refer to the partition: two natural numbers are equivalent if they have the same first digit. Note that the equivalence class of 0 is $\{0\}$.

Appendix to §5. The Numerical Partition Function

Supplemental Notes: Not part of the regular course material.

One way to count the partitions of a set of n elements is to classify the partitions according to the number of elements in each "box".
For example, to count partitions of the set $\{1, 2, 3, 4\}$, we can list all the ways to write 4 as a sum of smaller numbers, then list the corresponding partitions, getting the following table.

Numerical Partitions	Partitions
4	$\{\{1, 2, 3, 4\}$
3+1	$\{\{2, 3, 4\}, \{1\}\}$, $\{\{1, 3, 4\}, \{2\}\}$, $\{\{1, 2, 4\}, \{3\}\}$, $\{\{1, 2, 3\}, \{4\}\}$
2+2	$\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$
2+1+1	$\{\{1, 2\}, \{3\}, \{4\}\}$, $\{\{1, 3\}, \{2\}, \{4\}\}$, $\{\{1, 4\}, \{2\}, \{3\}\}$, $\{\{2, 3\}, \{1\}, \{4\}\}$,
	$\{\{2, 4\}, \{1\}, \{3\}\}$, $\{\{3, 4\}, \{1\}, \{2\}\}$
1+1+1+1	$\{\{1\},\{2\},\{3\},\{4\}\}$

The five rows correspond to the five *numerical partitions* of the number 3. Note that we consider rearrangements such as $1 + 1 + 2$, $1 + 2 + 1$, and $2 + 1 + 1$ to all represent the same numerical partition. Evidently the number $p(n)$ of all numerical partitions of n is considerably less than the number of all partitions of n. A theorem of Hardy and Littlewood says that for large n, this number $p(n)$ is approximately

$$\frac{1}{\sqrt{48}} e^{\sqrt{\frac{2}{3}\pi n}}$$

Numerical partitions with odd, or distinct, parts

But what I want to discuss is a remarkable relationship between the set of numerical partitions in which only *odd* terms occur, and the set of numerical partitions in which all the terms are *distinct*. For example, when $n = 7$, there are five of each. We list the numerical partitions of 7 with odd entries on the left, and the numerical partitions of 7 with distinct entries on the right.

Odd	Distinct
1+1+1+1+1+1+1	1+2+4
1+1+1+1+3	3+4
1+1+5	2+5
1+3+3	1+6
7	7

Let us also write $p_O(n)$ and $p_D(n)$ for the number of partitions of each kind, "odd" or "distinct". Thus we have

$$p_O(7) = p_D(7) = 5$$

Theorem 5.23 (Euler)**.** *For all n, $p_O(n) = p_D(n)$.*

Proof. First, consider the infinite product

$$(1+x)(1+x^2)(1+x^3)\cdot\dots = \prod_{n=1}^{\infty}(1+x^n)$$

If one multiplies out, one gets an infinite series of the form:

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots$$

In fact the product of the first 7 terms already looks like this, and the later factors $(1+x^8)(1+x^9)\cdot\dots$ are not going to change the first few terms.
The coefficient of x^n in this infinite series is $p_D(n)$. For example, the term x^7 occurs in the following ways:

$$x^7 = x\cdot x^2\cdot x^4,\ x^7 = x^3\cdot x^4,\ x^7 = x^2\cdot x^5,\ x^7 = x\cdot x^6,\ x^7 = x^7$$

and these correspond to $7 = 1+2+4$, $7 = 3+4$, etc.
So we may write

$$\prod_{n=1}^{\infty}(1+x^n) = \sum_{n=1}^{\infty} p_D(n)$$

We have expanded the infinite product.

Next, consider the following expression.

$$\frac{1}{\prod_{1\infty}(1-x^{2n+1})}$$

In other words the denominator is the product $(1-x)(1-x^3)(1-x^5)\cdot\dots$ with odd exponents only. We expand this as an infinite series. To begin with, each expression

$$\frac{1}{1-x^{2n+1}}$$

expands to a geometric series, for example when $n = 5$ $(2n+1 = 11)$ this is the geometric series

$$1 + x^{11} + x^{22} + x^{33} + \cdots$$

with exponents the successive multiples of 11.

It is a little more complicated to multiply all of these together. When one does this, the coefficient of x^n comes out to be $p_O(n)$. This is because we are now writing the exponent n in x^n as a sum of *multiples of odd numbers:* if $k(2i+1)$ is a multiple of the odd number $2i+1$, then we can think of this as using the odd number $2i+1$ exactly k times.
So now we have two expansions.

$$\prod_{n=1}^{\infty}(1+x^n) = \sum_{n=1}^{\infty} p_D(n)x^n \tag{6}$$

$$\frac{1}{\prod_{n=1}^{\infty}(1-x^{2n+1})} = \sum_{n=1}^{\infty} p_O(n)x^n \tag{7}$$

We want to prove that $p_D(n) = p_O(n)$ for all n. This means that the expressions on the *right* in equations (1,2) are equal. So we will show that the equivalent expressions on the *left* in equations (1,2) are equal. We claim:

$$\prod_{n=1}^{\infty}(1+x^n) = \frac{1}{\prod_{n=1}^{\infty} 1-x^{2n+1}}$$

The key idea is to consider a third product:

$$f(x) = \prod_{n=1}^{\infty}(1-x^{2n})$$

taking even exponents this time: $(1-x^2)(1-x^4)\cdots\cdot$.
Since we can factor $1-x^{2n}$ as $(1-x^n)(1+x^n)$, $f(x)$ factors as

$$\prod_n(1-x^n)\prod_n(1+x^n)$$

So if we divide $f(x)$ by $\prod_n(1-x^n)$ we get the function associated with p_D.
On the other hand, if we divide $f(x)$ by $\prod_n(1-x^n)$, then all the terms in the numerator may be cancelled against terms in the denominator, leaving only odd powers in the denominator:

$$f(x) = \frac{1}{\prod_n(1-x^{2n+1})}$$

This is the function associated with p_O.
So both of these functions are equal to $\frac{f(x)}{\prod_n(1-x^n)}$ and thus the coefficients p_D and p_O are equal.
This proves Euler's Theorem. $\square$

The Bijection Problem

For each n, let $O(n)$ and $D(n)$ be the number of numerical partitions of n into odd parts, or into distinct parts, respectively—in other words, the sets which are counted by the functions p_O and p_D.
Since $O(n)$ and $D(n)$ have the same number of elements for each n, there must be bijections between them.

Problem 1. Is there a *natural* bijection betweeen $O(n)$ and $D(n)$?

This type of question is hard in general. When two sets are known to have the same size because of some manipulation of infinite series, there is no known method for converting that proof into an explicit bijection between the sets. One must start over.
In this particular case, a bijection is known. It is based on the following two facts.

- Every positive integer can be written in a unique way as a power of 2 times an odd number.
- Every positive integer can be written in a unique way as a sum of powers of 2.

In our chart above, each numerical partition of 7 with distinct terms can be converted to an expression with odd terms by applying our first point: for example, the expression "$3+4$" will be written as $1 \cdot 3 + 4 \cdot 1$ and reinterpreted as "one three and four ones", giving the corresponding expression $3+1+1+1+1$ as a sum of odd terms.
3*3+6*1+2*4 3+6, 2+4, 10 One more example: let $n = 25$, and let the numerical partition into distinct terms be

$$25 = 2 + 3 + 4 + 6 + 10$$

Writing $2 = 2 \cdot 1$, $3 = 1 \cdot 3$, $4 = 4 \cdot 1$, $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, this converts first to

$$1 + +1 + 3 + 1 + 1 + 1 + 1 + 3 + 3 + 5 + 5$$

and then may be written in increasing order as

$$1 + 1 + 1 + 1 + 1 + 1 + 3 + 3 + 3 + 5 + 5$$

(odd terms).
Can you give the general rule for converting numerical partitions with distinct entries into numerical partitions with odd entries? And can you give a rule for the inverse operation, reconstructing from each numerical partition with odd entries, the corresponding numerical partition with distinct entries? If so, you have shown that there is a natural bijection between the two sets.