MEMORIAL TRIBUTE

Richard L. Wheeden—In Memoriam 1940–2020

Sagun Chanillo, Bruno Franchi, Carlos E. Kenig, and Eric T. Sawyer

1. Introduction

Sagun Chanillo

Richard Wheeden, or Dick Wheeden as he was known to his many friends, collaborators, and colleagues, passed away in a tragic accident while walking near his home in St. Michaels, Maryland, on April 9, 2020. He was 79 years old. With the passing away of Benjamin "Ben" Muckenhoupt a few days later on April 13, 2020, at the age of 86, the curtain fell down on a golden period for analysis at Rutgers University. These two important harmonic analysts along with Richard Gundy, whose works were intertwined, had brought the Rutgers analysis group to international focus and contributed immensely to the development of harmonic analysis. Some of these contributions are discussed by Bruno Franchi, Carlos Kenig, and Eric Sawyer in subsequent sections of this memorial article.

Dick was born in Baltimore and maintained a lifelong attachment to the waters of the Chesapeake Bay. After retirement from Rutgers in December 2016, he moved to St. Michaels, a small town on the Chesapeake Bay, where he even owned a boat. As a student, Dick attended local area schools, enrolling in the Polytechnic Institute, an engineering high school in Baltimore, beginning in February 1955. There he was part of the demanding A-course, taking classes in mechanical drawing, machine shop, mathematics, and chemistry. The mathematics courses included solid and analytic geometry and calculus. In addition he represented his high school in swimming and gymnastics. In 1958, he enrolled in the pre-med program at Johns Hopkins University, but switched to mathematics in his senior year, graduating in the spring of 1961.

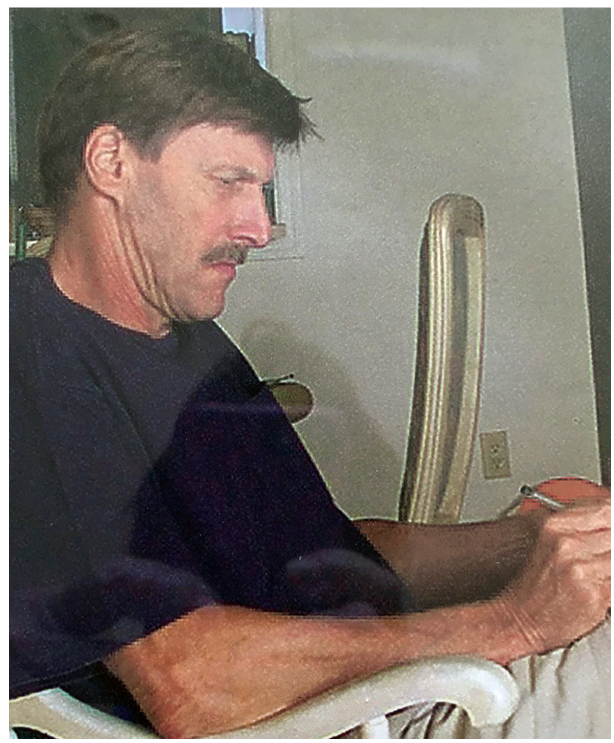

Figure 1. Dick Wheeden working.

Johns Hopkins was well known for algebraic geometry, centered around W.-L. Chow who had attracted luminaries like S. S. Abhyankar and J. Igusa, and in 1961 the Fields medallist K. Kodaira. Remarkably enough, in the fall of

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1961 Wheeden decided to go to the University of Chicago, a powerhouse of analysis. In the spring of 1962, after obtaining his masters degree, he embarked on his PhD under the supervision of A. Zygmund. He married the former Sharon McGlasson in fall 1962 and completed his PhD in 1965 on the subject of hypersingular integrals, of which we will say a bit more later. Dick was totally devoted to Zygmund and one of the important byproducts of this period is that later Dick transcribed the notes from Zygmund's course on real analysis into a book [1]. Shortly before retiring, Dick revised this book and produced a second edition. Totally dedicated to the viewpoint of Zygmund, one of the frequent refrains one heard Dick make during the revision was "Zygmund would not have approved of this proof." Thus a certain proof would be out as not fitting into the style of Zygmund.

1965, the year that Dick finished his PhD, was a momentous year for harmonic analysis. The 1950s had seen a significant breakthrough with real-variable methods used to study classical problems in harmonic analysis by A. P. Calderón, A. Zygmund, and E. Stein. These results immediately led to opening up the subject to several variables with powerful consequences and applications to PDEs. Calderón then applied these ideas in 1958 to the study of uniqueness in the Cauchy problem. The volume of the *American Journal of Mathematics* that contains this paper contains two other landmark papers, the paper of John Nash on regularity theory of elliptic PDEs and Harishchandra's paper on spherical functions. In 1965, a conference was given in honor of the 65th birthday of Zygmund, and the papers that were presented (mainly on singular integrals and applications) appear in a volume of the AMS series Proceedings of Symposia in Pure Mathematics, vol. 10 (1967). These papers reflect the state of the art at the time. The last paper in this volume is by Dick Wheeden and is based in part on his thesis on hypersingular integrals. 1965 was the dawn of a new era when new stars on the horizon were just beginning to emerge. This would herald a second era of vigorous activity in harmonic analysis and allied areas with D. Burkholder, R. Coifman, C. Fefferman, R. Gundy, R. Hunt, B. Muckenhoupt, and R. Wheeden, among several others, leading the way.

Fortuitously, Wheeden decided to spend the year immediately after obtaining his PhD (1965–1966) at the University of Chicago as an instructor. This coincided with Richard Hunt arriving there as an instructor. In that period the two of them embarked on their study of potential theory on Lipschitz domains, which Carlos Kenig describes admirably in his contribution here.

In 1966–1967, Wheeden moved to the IAS at Princeton. At the end of that year, he had offers from Yale and Princeton but these were not tenure-track assistant professorships. Rutgers made a tenure-track offer and, since Wheeden now had a son, the security of the Rutgers offer trumped the other offers. Ben Muckenhoupt and Richard Gundy were already at Rutgers. Muckenhoupt had a lifelong interest in classical orthogonal polynomials. Here, like for Jacobi polynomials, the measure that makes these polynomials orthogonal is a polynomial to various powers, sometimes fractional powers. Questions that arise for classical Fourier series get turned into similar questions for functions expanded by means of these orthogonal polynomials, but now the measure involves a polynomial density instead of standard Lebesgue measure as in Fourier series. Thus one is naturally led to the study of weighted norm inequalities for all the classical operators of analysis arising in the study of Fourier series and singular integrals like the Hardy-Littlewood maximal function, singular integral operators like the Hilbert transform, and so on. Muckenhoupt had already investigated boundedness properties on weighted L^p spaces for the Hardy-Littlewood maximal function in a seminal paper. Now together with Wheeden (newly arrived at Rutgers) and R. Hunt, he turned his attention to the other important operator in harmonic analysis, the Hilbert transform. The deep result of R. Hunt, B. Muckenhoupt, and R. Wheeden which completely solves the weighted norm inequality problem for the Hilbert transform (the prototype for all singular integrals) dates to this early period of Wheeden at Rutgers. This important result on the Hilbert transform and other results on weighted norm inequalities for other classical operators which Muckenhoupt and Wheeden established in a very vigorous collaboration is discussed in the contribution by Eric Sawyer, while Bruno Franchi's contribution lists the many applications that Wheeden discovered of this weighted theory to elliptic PDE. Weighted inequalities also have their role in the potential theory for Lipschitz domains, as in Carlos Kenig's contribution.

Wheeden remained at Rutgers from 1967 until his retirement in December 2016, except for two sabbaticals in the early 1970s, one to Argentina to the University of Buenos Aires, where he arrived with his family in the midst of a military coup, and another to Purdue University to visit Richard Hunt.

I first met Dick in spring 1978. I was a graduate student at Purdue and my advisor was Richard Hunt. Dick came to Purdue and gave a series of lectures on the proof of the Calderón commutator theorem. He took time to talk to me and then encouraged me to attend an AMS summer school in harmonic analysis at Williams College in Williamstown, MA, that summer. Two years later I was a postdoc at Rutgers. The seminars in analysis at Rutgers were extremely lively with Gundy, Muckenhoupt, and Wheeden. They were mostly held on Fridays and when no external speaker was available, people reported on somebody's paper. Dick's graduate topic courses, many of which I attended, were characterized by careful presentations of all details. The blackboard would be covered in equations written in an orderly way in Dick's neat handwriting. In addition, both Sharon and Dick took very good care of me. They were both tremendously warm. In all my associations with Dick, I seldom have heard him say anything negative about someone. He was always self-deprecating with a dry sense of humor at times. Both Wheeden and Gundy were natural athletes who ran and took part in the annual May day race between the Rutgers and Princeton math departments, now called the Fred Almgren memorial race. This race is run alternately in opposite directions between Rutgers and Princeton along the towpath of the Delaware and Raritan canal, a distance of 27 miles. Teams participate using a relay system and occasionally some dauntless individual like J.-E. Fornaess runs the entire course singlehandedly. Muckenhoupt viewed these activities with a jaundiced eye. Then in 1985, Muckenhoupt took up running with a gusto that surpassed all. He would even run in bitter cold only in his running shorts.

Dick and Ben Muckenhoupt had very distinctive styles of working. Nothing illustrates this better than the year 1984–1985, when I was visiting the IAS. In that year I was involved with a project with each of them. With Dick, I was working on eigenvalue counting estimates for the time-independent Schrödinger operator. Once a week, Dick would come to the IAS early in the morning, spend the entire day with me and leave after tea time. He liked to discuss his ideas and listen to what I had to say, and we would work together for the better part of a day. Sections written by Dick when the paper was ready would be written out on lined paper in a beautiful hand with annotations. Ben and I were working on endpoint estimates on L^p spaces for what are called Bochner-Riesz operators. Communications with him would be terse and on the phone and would occasionally involve visits to Rutgers for a brief meeting. Ideas or germs of computations from Ben would be written out on paper that had been used before, and written in a scrawl.

Richard Wheeden maintained a lifelong interest in the fine properties of functions, that is, their smoothness and differentiability properties as measured in some scale of spaces like Sobolev spaces. This stems from his work on hypersingular integrals. A typical hypersingular integral is an operator of the type

$$Tf(x) = \int_{\mathbf{R}^n} (f(x-y) - f(x)) \frac{\Omega(y)}{|y|^{n+\alpha}}\, dy, 0 < \alpha < 2,$$

where $\Omega(y)$ is a homogeneous function of degree zero, smooth on the sphere S^{n-1}, and having a cancelation property when $1 \leq \alpha < 2$ given by

$$\int_{S^{n-1}} y_j' \Omega(y') d\sigma(y') = 0 \qquad \forall j = 1, 2, \dots, n, y' = (y_1', \dots, y_j', \dots, y_n').$$

$d\sigma$ is the surface measure on the sphere and the integral defining $Tf(x)$ is to be taken in a suitable principal value sense. For the operator to exist in some $L^p(R^n)$ sense, one notices that the difference in the numerator brings the smoothness of the function into play and this can be used to ameliorate the singular kernel $\frac{1}{|y|^{n+\alpha}}$. Thus the smoothness of the function assists in obtaining various L^p bounds for Tf and in particular the operator Tf can be used to study the smoothness of f and characterize it. E. Stein had studied the case $\Omega(y') \equiv 1$ earlier in a paper in the *Bulletin* of the AMS in 1961. The first systematic treatment of such operators is due to Wheeden [2].

Another integral that measures the differentiability properties of a function is the Marcinkiewicz integral [3]. It is a form of what is nowadays called a Littlewood-Paley-Stein square function. This integral and its boundedness properties on L^p spaces were studied by both Marcinkiewicz and his advisor A. Zygmund. It is defined in its global version by

$$Mf(x) = \left(\int_{\mathbf{R}^n} \frac{|f(x+y) + f(x-y) - 2f(x)|^2}{|y|^{n+2}}\, dy \right)^{1/2},$$

and the local version is given by

$$Mf(x) = \left(\int_{|y|<\delta} \frac{|f(x+y) + f(x-y) - 2f(x)|^2}{|y|^{n+2}}\, dy \right)^{1/2}.$$

One of the early papers of Wheeden on this subject dates to 1969 with an article in the *Studia Mathematica*. It is quite remarkable that a second difference in the integrand allows one to study differentiability, which involves a first difference quotient. This delicate control by symmetric second differences over first differences can be seen by means of a desymmetrization argument found in Chapter 8 of E. Stein's seminal textbook on singular integrals [4].

When I arrived at Rutgers in 1980, Wheeden suggested a problem to me to derive precise quantitative inequalities (known as the Burkholder-Gundy good λ inequalities stemming from martingale theory) that would yield the desymmetrization of the Marcinkiewicz integral, and the equivalence of the differentiability of a function with the finiteness of the local Marcinkiewicz integral defined above and the equivalence of the $L^p(\mathbf{R}^n)$ norm of the Marcinkiewicz integral with the $L^p(\mathbf{R}^n)$ norm of the gradient of the function f. These are involved results; the details are to be found in [5]. Other facets of Dick Wheeden's work that combines his interest in smoothness of functions and PDEs are discussed in the contribution by Bruno Franchi.

Dick Wheeden had a profound influence on me mathematically and otherwise. He was encouraging to many young people and always supported the underdog. His encouragement not only extended to his own students and postdocs, but also to the students of other colleagues. My own student Guozhen Lu benefitted immensely from Dick's advice and by working on problems with him. Dick's advice was always careful and well thought out and calmly given. Generous with his ideas and time, he was the ideal colleague. Nothing prepared me for the phone call I received from Sharon the morning of April 10th, informing me that Dick was no more.

2. R. L. Wheeden's Contributions to Integral Inequalities on Metric Spaces and Degenerate Elliptic PDE

Bruno Franchi

In the 80s, Wheeden started to apply weighted norm inequalities for classical operators (like the Riesz potential described in the article by E. Sawyer) to the study of elliptic PDE

$$\nabla \cdot \big(A(x)\nabla u\big) = f, \qquad (1)$$

where $A(x)$ is a symmetric $n \times n$ non-negative matrix with measurable functions as entries. One imposes an ellipticity condition, that is, the quadratic form associated with $A(x)$ satisfies, for $c > 0$,

$$c\, w(x)|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le c^{-1}\, v(x)|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$. The functions $w(x)$ and $v(x)$ are allowed to vanish in a certain controlled manner. Typically, one assumes that $w \in A_2$ (see Sawyer's contribution for the definition). Thus the quadratic form does degenerate. Regularity properties to solutions of these elliptic degenerate PDE were investigated in the 1970s by Murthy and Stampacchia, Trudinger, and later, when $v = w$, by Fabes, Kenig, and Serapioni [7]. The heart of the methods used to understand the regularity of solutions lies in a method known as the Moser iteration scheme, where successive control is established on larger and larger powers of the solution u. To implement this method one needs a pair of inequalities collectively known as Sobolev-Poincaré inequalities. For a ball $B \subset \mathbb{R}^n$, of radius $r > 0$, one needs, for all $f \in C_0^1(B)$ (or for all $f \in C^1(B)$ with vanishing average), the inequality

$$\Big(\frac{1}{v(B)}\int_B |f|^q\, v dx\Big)^{1/q} \le C\, r\Big(\frac{1}{w(B)}\int_B |\nabla f|^p\, w dx\Big)^{1/p} \qquad (2)$$

for suitable $q > p \ge 1$, where

$$v(B) = \int_B v\, dx, \qquad w(B) = \int_B w\, dx,$$

and $c > 0$ is a universal constant. Such inequalities were obtained by Chanillo and Wheeden in 1985 and applied to the study of regularity of solutions of degenerate elliptic PDE ([6]). Wheeden went further with Gutiérrez to study the heat equation versions of these equations.

The next problem that Wheeden attempted to answer was to consider (2) when the gradient on the right side was replaced by vector fields or some first-order differential operators, and to replace the Euclidean space $\mathbb{R}^n$ itself by some metric space. The metric space setup leads to the study of what today are called Carnot-Carathéodory (or control) metrics. These metrics are not Riemannian and appear in diverse settings of optimal control, Lie group theory, non-holonomic mechanics, robotics, theoretical computer science, geometry of Banach spaces, and mathematical models in neuroscience. Roughly speaking, the new metric is given by the minimum time to go from one point to another point along integral curves of some vector fields, as in non-holonomic mechanics we move between two configurations respecting the constraints of the mechanical system. The notion of the Carnot-Carathéodory metric is already implicit in the seminal papers by Hörmander [10] and J.-M. Bony, where a key role is played by the integral curves of a family of vector fields associated with a differential operator. Later, this approach enabled Franchi and Lanconelli to adapt Moser's technique to this new metric setting, by proving a suitable form of the Sobolev-Poincaré inequalities.

In the Euclidean space, the case $p = 1$ in (2) is known to be equivalent to the isoperimetric inequality. In works done with Franchi and Gallot [8] and with Franchi and Lu, Wheeden established via inequalities of type (2) suitable isoperimetric inequalities for Carnot-Carathéodory metrics.

A result that highlights some of Dick's finesse is a result of equivalence he obtained with Franchi and Lu ([9]). It concerns what is called a representation formula. A version of this result reads as follows. Assume we have a metric space (X, ρ) endowed with a Radon measure μ that is

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doubling, i.e.,

$$\int_{B(x,2r)} d\mu \le c \int_{B(x,r)} d\mu,$$

where $B(x,r)$ denotes any metric ball centred at x of radius r and c is an absolute constant. Assume also that one has a Poincaré-type inequality for Lipschitz continuous functions u with vanishing average in a metric ball $B = B(x,r)$, i.e.,

$$\frac{1}{\mu(B)} \int_B |u|\, d\mu \le C\, r \frac{1}{\mu(B)} \int_B |Xu|\, d\mu, \qquad (3)$$

where c is an absolute constant and $|Xu|$ is some version of the gradient in the metric space. Then (3) is equivalent to proving

$$|u(x) - u_B| \le c \int_{B(x,\tau r)} |Xu(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\, d\mu(y),$$

where $\tau > 1$ and

$$u_B = \frac{1}{\mu(B)} \int_B u(y)\, d\mu(y).$$

Wheeden and Lu proved subsequently that τ can be taken to be 1, and such equivalences are valid for higher-order derivatives.

Dick had a vast culture and marvelous technical ability which were not ends in themselves, but served to give breadth to the general picture he had in his mind. Dick also had a sort of modesty towards his love of mathematics, and hid it at times behind a veiled lightness—like a musician who is almost ashamed of telling of the depths he goes through with his performances and lingers on particular minutes.

For me, his understated personality can be summarized by this small anecdote. One day, we were talking in his office in front of a blank blackboard. Sharon his wife called, and Dick said “We are looking at the blackboard. Yes, we’re having fun.” Mathematics was for him an intellectual pleasure, not a competition or a gymnastic exercise.

When I met Dick in Berkeley over thirty years ago, I felt as if I had known him forever. I received so much from him: not just math, but also many small and great lessons in life. Dick has been and remains for me a model scientist and human being. Finally, let me quote his first e-mail at the very beginning of our collaboration: “Come si può vedere, non sono un uomo di questo secolo! (in Italian: As you can see, I am not a man of this Century!) I had not checked my mail in some time, but finally received your note. . . ”

Figure 2. From left to right: Ben Muckenhoupt, Richard Wheeden, Richard Gundy.

3. The Work of R. Wheeden on Potential Theory in Lipschitz Domains

Carlos E. Kenig

I got to know Dick Wheeden very well in the period 1978–1980, when I was an instructor at Princeton University (my first job). At that time, there was a flurry of activity in harmonic analysis at Rutgers, centered around Muckenhoupt and Wheeden, and their very active research on weighted norm inequalities. My own thesis (defended in 1978) had been on weighted Hardy spaces on Lipschitz domains in the plane, and so I was very interested in the subject. Thus, every week, I drove to Rutgers to attend the seminar organized by them. I learned a lot from attending this seminar, and also from the many mathematical discussions that I had with Dick during this period. Dick was extremely friendly and generous with ideas, with a passion for mathematics that stayed with him his whole life. During this period we became good friends, and this remained so, even though after I left Princeton we did not see each other very often. His warmth and generosity will always stay with me.

To put the work that Wheeden (with R. Hunt) did on potential theory in Lipschitz domains in perspective, I recall that the 1950s saw the beginning of a spectacular development of harmonic analysis and its applications in higher dimensions. This was through the introduction of new real-variable methods that replaced the classical tools from complex analysis. This development was in large part due to the works of Calderon, Zygmund, and Stein, and their students and collaborators. One of the early results

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in this development, due to Calderon (1950), was the extension to higher dimensions of the classical theorem of Fatou (1906) (in the upper half-plane in $\mathbb{R}^2$) on the almost everywhere existence of finite non-tangential limits (i.e., limits through cones that are non-tangential to the boundary) for bounded harmonic functions. Calderon's proof highlighted the use of "saw-tooth" domains, obtained as unions of cones, with vertices in a boundary set E, a basic example of a Lipschitz domain, and the study of harmonic functions in these domains. Another influential result in this direction was due to Carleson [11], who extended Fatou's theorem and Calderon's extension of it to non-negative harmonic functions (one-sided boundedness). Carleson obtained this result through a careful study of "Green's functions" and "harmonic measure" for saw-tooth domains. Carleson's analysis was influenced by works of Calderon (1950) and Stein (1961) on "square functions." This is the background for the very influential works [13], [14] of Hunt and Wheeden, which initiated the study of harmonic functions in Lipschitz domains in higher dimensions. A bounded domain D is a Lipschitz domain if it is locally given as the domain above the graph of a Lipschitz function. It is not difficult to see that such domains have, for every point in the boundary, exterior and interior (truncated) cones, with vertex at the boundary point. The existence of exterior cones implies that Lipschitz domains are regular for the Dirichlet problem. This means that for any continuous function f, on the boundary of D, there exists a unique harmonic function u in the domain which is continuous up to the closure of D and which equals f on the boundary of D. The "harmonic measure" of the boundary of D is the family of measures $\{d\omega^X\}$, parametrized by points X in D, which give the values of u by integration of f on the boundary of D, against $d\omega^X$, i.e., $u(X) = \int_{\partial D} f d\omega^X$. The existence of $\{d\omega^X\}$ is guaranteed by the maximum principle, the Riesz representation theorem, and the regularity of the Dirichlet problem in D. If we fix a point $X_0 \in D$, by an abuse of notation, we sometimes call the harmonic measure at X_0 the harmonic measure $d\omega$, with $d\omega = d\omega^{X_0}$. It is easy to see, as a consequence of Harnack's principle, that harmonic measures at different points are mutually absolutely continuous. The main result in [13] is that if u is a non-negative harmonic function in a bounded Lipschitz domain D in n-dimensional space, then, for almost every boundary point (with respect to harmonic measure) u has finite non-tangential limits. Here, non-tangential limits are well defined, due to the existence of interior truncated cones with a vertex at each boundary point. This is the exact analog of Carleson's result [11], but the exceptional set, instead of having zero surface measure, has zero harmonic measure. This is an optimal result because, given a boundary set of zero harmonic measure, one can construct a non-negative harmonic function which is infinite on the set. The proof of this result proceeded by reducing it to the case when D is a star-like Lipschitz domain and u is bounded in D. This reduction is ingenious, but not difficult. To prove this last case, if D is star-like with respect to 0, and we choose $X_0 = 0$, then Hunt and Wheeden showed that we can obtain a representation for u of the form

$$u(X) = \int_{\partial D} K(X,Q) f(Q) d\omega^0(Q), \tag{4}$$

where f is a bounded function and $K(X,Q)$ is the Radon-Nykodym derivative of $d\omega^X$ with respect to $d\omega^0$. This exploits the star-like character of D. The formula (4) reduces the proof to studying properties of the measure $d\omega^0$ and the analog of the Poisson kernel, the kernel $K(X,Q)$. The key property of $d\omega^0$ is that it is a doubling measure: for $0 < r < r_0$,

$$\omega^0(B(Q,2r) \cap \partial D) \le C\omega^0(B(Q,r) \cap \partial D) \tag{5}$$

for all $Q \in D$. This implies that the Vitali covering lemma can be used to study the maximal function

$$M_\omega(f)(Q) = \sup_{0<r<r_0} \frac{1}{\omega^0(B(Q,r)\cap\partial D)} \int_{B(Q,\Omega)\cap\partial D} |f| d\omega^0 \tag{6}$$

and to prove its $L^p(\partial D, d\omega^0)$ estimates. The estimates proved for the kernel $K(X,Q)$ are such that if $\Gamma(Q)$ is a truncated cone with vertex Q, contained in D, then

$$\sup_{x\in\Gamma(Q)} |u(X)| \le CM_\omega(f)(Q), \tag{7}$$

when u is given by (4).

Once (4), (5), (6), (7) are shown, the proof can be concluded as in the classical proof of the Lebesgue differentiation theorem, using the Hardy-Littlewood maximal theorem. The proofs of (5) and (7) are the crucial steps. They are inspired by some of the arguments in [11] and involve clever applications of the Harnack principle, the maximum principle, and barriers.

In the second paper [14] of Hunt and Wheeden, they gave an important connection between extensions of the technical results in [13] and the abstract theory of Martin, dealing with an "ideal" boundary of D and the corresponding topology in $\overline{D}$ for the case of bounded Lipschitz domains D. If $Q \in \partial D$ is fixed, a positive harmonic function in D, continuous on $\overline{D}\backslash\{Q\}$, 0 on $\partial D\backslash\{Q\}$, and 1 at $X_0 \in D$, is called a kernel function (at Q). In [14], Hunt and Wheeden showed that for D a bounded Lipschitz domain, a kernel function at Q is unique, and they identified it with $K(X,Q)$, the Radon-Nykodym derivative of $d\omega^X$ with respect to $d\omega^{X_0}$. The proofs of these results use the methods developed in [13]. As a consequence of these results, the Martin "ideal" boundary can be identified with the topological boundary of D for bounded Lipschitz domains, and $K(X,Q)$ is a continuous function of Q. Using also the abstract theory of Martin and the result in [13], Hunt and Wheeden showed that, for any non-negative harmonic function u in a bounded Lipschitz domain D, there exists a unique Borel measure $d\mu$ on ∂D, such that

$$u(X) = \int_{\partial D} K(X,Q)d\mu(Q),$$

which is an important extension of (4) above, which held for bounded harmonic functions in star-like Lipschitz domains.

The papers [13] and [14] have been extremely influential and have led to many other important results in a number of different directions. Here I mention a few sample results. A natural question that arose from [13], [14], and which remained open for some time, was whether harmonic measure and surface measure are mutually absolutely continuous on Lipschitz domains, so that the exceptional set in the Fatou-type theorem in [13] has zero surface measure. This was resolved in the affirmative by B. Dahlberg in [12], an important breakthrough. Moreover, Dahlberg showed that the Radon-Nykodym derivative of harmonic measure with respect to surface measure, on a Lipschitz domain, is a weight of the type studied extensively by Muckenhoupt and Wheeden! This was used by Dahlberg (1979) to obtain optimal solvability results for the Dirichlet problem on bounded Lipschitz domains with data in $L^p(\partial D, d\sigma)$.

Another important development that followed from the work in [13], [14] was the work of Caffarelli, Fabes, Mortola, and Salsa (1981), where they replaced the Laplacian in a bounded Lipschitz domain by a divergence form second-order elliptic operator with bounded measurable coefficients in the unit ball. These authors established, in this setting, results analogous to the ones in [13], [14], in which the exceptional set has zero "elliptic measure," where the "elliptic measure" is the analog of harmonic measure in this context. A simple change of variables shows that these results generalize those in [13], [14] for the Laplacian in a star-like Lipschitz domain. However, examples due to Caffarelli-Fabes-Kenig (1981) and Modica-Mortola (1981) show that the "elliptic measure" can be singular with respect to surface measure. This has led to a vast, ever-growing literature on determining when the two measures are mutually absolutely continuous.

The last development motivated by the work of Hunt and Wheeden that I mention here is the introduction by Jerison-Kenig [15] of the class of non-tangentially accessible domains (NTA domains) as a very general class of domains which generalize Lipschitz domains, in which the results of Hunt-Wheeden still hold, with exceptional sets having zero harmonic measure. NTA domains need not have interior or exterior cones, but there is a natural way to define non-tangential convergence in them. NTA domains need not have rectifiable boundaries and hence surface measure plays no role here. The class of NTA domains has proven to be very useful in the study of free boundary problems, and also in geometric measure theory. As indicated by this small sample of further results (and even whole new areas of research) that were spawned by the remarkable works of Hunt and Wheeden, the influence of these works has been enormous.

Dick Wheeden was an outstanding mathematician whose works continue to be extremely influential. Dick had a deep love of mathematics, and a passion for sharing it with friends, colleagues, and students. Dick was also a warm, fun-loving person. I miss him greatly.

4. Richard L. Wheeden and His Influence on Weighted Norm Inequalities and Degenerate Elliptic Equations

Eric T. Sawyer

4.1. The Hunt-Muckenhoupt-Wheeden theorem: The flourishing of weighted norm inequalities. I met Dick Wheeden in 1987, during my first sabbatical leave from McMaster University. I spent about six weeks at Rutgers University discussing weighted norm inequalities for maximal functions, fractional integrals, and Calderón-Zygmund operators with Dick and his colleague Ben Muckenhoupt, who had opened up the theory of A_p weights with his fundamental 1972 paper on the maximal function [18]. This was an exciting time for me as a newly minted mathematician, and Dick was quick to "take me under his wing."

Before commenting more on Dick's personality, I'd like to discuss in some detail this new area of research propelled by the highly cited 1973 signature article of Hunt, Muckenhoupt, and Wheeden [17], in which they extended the classical theorem of M. Riesz on L^p-boundedness of the conjugate function on the unit circle $\mathbb{T}$ (we identify $e^{i\theta} \in \mathbb{T}$

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with $\theta \in [-\pi, \pi)$),

$$\widetilde{f}(\theta) \equiv \mathrm{pv}\,\frac{1}{\pi}\int_{\mathbb{T}} \frac{f(\theta-\phi)}{2\tan\frac{\phi}{2}}d\phi$$

to weighted spaces $L^p(w)$. They showed that $f \to \widetilde{f}$ is bounded on $L^p(w)$ if and only if w is an A_p weight on the unit circle $\mathbb{T}$, i.e., $A_p(w) < \infty$; more precisely,

$$\int_{\mathbb{T}} \left|\widetilde{f}(\theta)\right|^2 w(\theta)\,d\theta \le C\int_{\mathbb{T}} |f(\theta)|^2 w(\theta)\,d\theta$$

for some $C < \infty$ and all $f \in L^p(\mathbb{T})$ if and only if

$$A_p(w) \equiv \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(x)^{p'-1}}dx\right)^{p-1} < \infty$$

for all intervals I. The conjugate function $\widetilde{f}$ arises as the imaginary part of the boundary values of a holomorphic function $F(z)$ in the unit disk (satisfying a mild growth condition at the boundary) having real part f, namely $F(e^{i\theta}) = f(e^{i\theta}) + i\widetilde{f}(e^{i\theta})$, and is thus an important real-variable link to the theory of holomorphic functions in the unit disk. Moreover, boundedness of $f \to \widetilde{f}$ on $L^p(\mathbb{T})$ is the key to establishing the classical inequality

$$\|S_n f\|_{L^p(\mathbb{T})} \le C_p \|f\|_{L^p(\mathbb{T})},$$

which gives mean $L^p(\mathbb{T})$ convergence of the partial sums $S_n f$ of the Fourier series for f.

This remarkable characterization deserves some background context before proceeding. Through the work of Helson and Szegö [16] in the case $p = 2$, it was known at the time that the boundedness of the conjugate operator on $L^2(w)$ was equivalent to a decomposition $\log w = u + \tilde{v}$, where u and v are bounded measurable functions on the circle and v satisfies the *strict* inequality $\|v\|_\infty < \frac{\pi}{2}$. While this result reveals a beautiful connection between two seemingly disparate questions in function theory, it sheds little light on how to detect if a weight $w(\theta)$, presented as a function of $\theta \in [-\pi, \pi)$, actually satisfies the weighted norm inequality.

What is most spectacular regarding the above characterization of Hunt, Muckenhoupt, and Wheeden is the *unreasonable simplicity* of the A_p condition, the case $p = 2$ being a bound on the product of averages of w and $\frac{1}{w}$ uniformly over intervals, amounting to a reversal of the Cauchy-Schwarz inequality applied to $1 = \sqrt{w}\frac{1}{\sqrt{w}}$ uniformly on intervals. Also notable is that while the classical proof of the Helson-Szegö theorem relied on complex function theory, the proof of the Hunt-Muckenhoupt-Wheeden theorem exploited the relatively recent transition to real-variable methods championed by Calderón, Zygmund, and Stein. Moreover, the A_p condition satisfies a large number of surprising and useful properties, foremost among them being the open-ended nature of dependence on the index p—namely $w \in A_p \Longrightarrow w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$ depending only on the supremum defining $A_p(w)$. It is interesting to note that after half a century, there is still no "direct" proof that A_2 is equivalent to the Helson-Szegö decomposition.

Figure 3. Richard Wheeden and Vladimir Maz'ya.

All of this made weighted norm inequalities ripe for investigation. For example, right after this breakthrough result characterizing boundedness of the conjugate function and Hilbert transform on weighted L^p spaces, Dick turned from singular integrals to fractional integrals,

$$I_\alpha f(x) \equiv c_{\alpha,n}\int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy,$$

which (isotropically) antidifferentiate a function f to fractional order α, with $0 < \alpha < n$. The case $\alpha = 2$ is the Newtonian potential of f which inverts the Laplacian, $\triangle I_2 f = f$, at least on compactly supported smooth functions f. This potential operator plays a key role in constructing the Green's function associated to sufficiently nice domains Ω in $\mathbb{R}^n$, which then leads to a solution to the classical Dirichlet boundary value problem for Ω.

In joint work with Ben [19] in 1974, Dick showed that the weighted norm inequality

$$\left(\int |w(x) I_\alpha f(x)|^q\,dx\right)^{\frac{1}{q}} \le C\int |w(x) f(x)|^p\,dx,$$

where $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, holds *if and* *only if* the $A_{p,q}$ condition holds:

$$\left(\frac{1}{|Q|}\int_Q w(y)^q\,dy\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w(y)^{-p'}\,dy\right)^{\frac{1}{p'}} \leq C.$$

Their proof used a good λ inequality relating I_α to the fractional maximal function M_α, a clever remodeling of an analogous inequality used earlier that same year by Coifman and Fefferman to extend the H-M-W result to a large class of Calderón-Zygmund operators in Euclidean space.

Turning then to two-weight inequalities for potential operators I_α,

$$\int_{\mathbb{R}^n} |I_\alpha f\sigma|^p\,d\omega \leq C\int_{\mathbb{R}^n} |f|^p\,d\sigma,$$

Dick extended the one-weight theory by establishing, in joint work with the author [20], a variety of conditions of $A_{p,q}$ type on a weight pair sufficient for boundedness of I_α.

4.2. An engaging personality. Returning to my first meetings with Dick in 1987, I found him to be a welcoming researcher who was excited to share his insights with a newbie, and moreover, to patiently explain in clear language his and others' arguments in detail. I learned early on that while he was freewheeling in his interpretation of the big picture in mathematics, his dedication to clear and complete proofs was indispensable to someone like me starting out in the field of weighted norm inequalities. I have vivid memories of attending several of Dick's talks over the ensuing decades in which, unlike other presenters who used either prepared transparencies or Beamer files, Dick would give a beautiful blackboard presentation with impeccably written formulas, and just the right amount of explanation leading from one line to the next. He was a true master of the vanishing art of blackboard presentations in mathematical research.

Dick was also a charmer. Everyone he met felt comfortable around him, and indeed, he always remembered the tiniest of details about the lives of those he encountered, and his ensuing enquiries as to how things were going made everyone feel appreciated and important in his presence. I spent many days in his office that first year and for many years after, learning about and solving new problems with him related to weighted norm inequalities. Every lunch hour, without fail and despite other responsibilities, he made the time to go swimming, or jogging with Ben around the campus, and left me the keys to his office. Then at the end of the day, my family and I were often invited to his home for dinner, where we experienced the warm hospitality of Dick and his wife Sharon. My then preschool daughter still remembers the kind and engaging man who fussed over her and her brother at his house in New Brunswick, and never failed to exchange a few words with them when picking me up from breakfast at the hotel where we stayed weeks at a time. These are fond memories that will live with me and my family always.

4.3. Subsequent work in weighted norm inequalities and degenerate elliptic equations. Here is a sampling of Dick's subsequent work in this and related areas, joint with S. Chanillo, S. Chua, B. Franchi, D. Kurtz, G. Lu, C. Pérez, C. Rios, J.-O. Strömberg, the author, and far too many others to mention all of them here. Topics include:

1. weighted norm inequalities for fractional integrals, Fourier multiplier operators of Hörmander-Mihlin type, square functions,
2. Fefferman-Stein inequalities, weighted Peano derivatives, and Harnack's inequality for solutions to degenerate elliptic equations,
3. Sobolev-Poincaré inequalities permitting weights to vanish to high order,
4. analogues of the classical subrepresentation formula

$$|f(x)-f_B| \leq C\int_B |\nabla f(y)|\,\frac{|x-y|}{|x-y|^n}dy, \qquad x\in B,$$

   for general vector fields $\mathcal{X}f = (X_1 f,\dots,X_m f)$; these were shown to be simple corollaries of, and often equivalent to, appropriate weighted L^1 Poincaré inequalities of the form

$$\frac{1}{|B|_\nu}\int_B |f(x)-f_{B,\nu}|\,d\nu \leq Cr(B)\,\frac{1}{|B|_\mu}\int_B |\mathcal{X}f(y)|\,d\mu(y),$$

   where μ and ν are measures and B is a ball of radius $r(B)$ with respect to the control metric for the vector fields (this was one of Dick's favourite results),
5. self-improving properties for Poincaré inequalities,
6. failure of the Besicovitch covering lemma for the Heisenberg group,
7. a construction of a dyadic grid for spaces of homogeneous type,
8. results on regularity of solutions to rough subelliptic equations,
9. smoothness of solutions u to the subelliptic Monge–Ampère equation

$$\det D^2 u(x) = k(x,u,Du), \qquad x\in\Omega,$$

   where $u\in C^2$ is convex, $k\approx|x|^{2m}$, and the elementary $(n-1)$st symmetric curvature k_{n-1} of u is positive.

The legacy of Dick's work in all of these areas combined has continued to grow with subsequent investigations by many other authors in the world of *two-weight* norm inequalities including,

- investigation of "bumped up" A_p conditions for a pair of weights to be sufficient for two-weight norm inequalities

$$\int_{\mathbb{R}^n} |Tf\sigma|^p \, d\omega \le C \int_{\mathbb{R}^n} |f|^p \, d\sigma,$$

- the introduction of weight-adapted Haar functions and random grids by Nazarov, Treil, and Volberg, and orthogonality in the case $p = 2$,
- the optimal power of the A_p constant in weighted norm inequalities,
- matrix-valued analogues of some of the above results.

All of this activity has cemented Dick's position as one of the founding fathers of the theory of weights in analysis.

4.4. A reluctant goodbye. The last email I received from Dick, on Thursday, April 9, 2020, 10:45 a.m., was typical of his good-natured determination to press on in the face of obstacles, and his clear descriptions of them:

> Hi Eric. I realized a snag in assuming that |\partial_1 a_2| satisfies the A_1 condition (on the line), so I haven't typed anything while thinking about it.
>
> ⋮
>
> The trouble is that A_1 doesn't fit together much at all with a Lipschitz condition. For example, if a_2(x_1, x_2) = |x_1|^{alpha}, then |\partial_{x_1} a_2| is both A_1 and Lip only when alpha =1.
>
> Darn it! Dick

I can picture Dick saying these latter words to me in person, and it is with a bittersweet heart that I know there will be no more. I miss you Dick for your math, your outlook on life, and your kindness.

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Credits

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