§3.1 DISCONTINUOUS SOLUTIONS

In this section, we study discontinuous solutions of hyperbolic equations.

Example 1 For the initial value problem

$$\begin{cases} u_t + cu_t = 0, & x \in R, \quad t > 0 (c > 0) \\ u(x,0) = u_0(x) = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x > 0 \end{cases} \end{cases}$$

The initial data has a discontinuity at $x = 0$. The solution $u = u_0(x - ct)$ also has discontinuities along the characteristic curve $x = ct$

$$u(x,t) = \begin{cases} 0, & \text{if } x > ct \\ 1, & \text{if } x < ct \end{cases}$$

- Discontinuities of linear wave equation propagates along characteristics.
- *Riemann problem*: a hyperbolic system with piecewise constant initial data.

Example 2 Consider the Riemann problem

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x > 0 \end{cases} \end{cases}$$

Along characteristics, determined by $dx/dt = u$, the function $u =$ const. Hence, characteristics are straight lines. Since the initial data $u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$, the characteristics originating from the negative axis have slop 1; the characteristics originating from the positive axis are vertical. These two types of characteristics collide. If we insert a curve to separate these two types of characteristics, we may avoid this difficult. Along this curve, the initial discontinuity at $x = 0$ is carried into the region $t > 0$.

Jump Conditions

Now, let us determine the curve along which the solution has simple discontinuities. We consider a general form of conservation law of the form

$$u_t + \phi_x = 0$$

This equation was derived, under the condition that both u and ϕ are continuously differentiable, by the integral conservation law:

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t)$$

If $x = s(t)$ is a smooth curve in spacetime along which the solution u suffers a simple discontinuity, i.e., 1) u is continuously differentiable for $x > s(t)$ and $x < s(t)$; 2) u and its derivatives have finite one-sided limits as $x \longrightarrow s(t)^-$ and $x \longrightarrow s(t)^+$.

$$\Rightarrow \qquad \frac{d}{dt}\int_a^{s(t)} u(x,t)dx + \frac{d}{dt}\int_{s(t)}^b u(x,t)dx = \phi(a,t) - \phi(b,t)$$

$$\Rightarrow \qquad \int_a^{s(t)} u_t(x,t)dx + \int_{s(t)}^b u_t(x,t)dx + u(s^-,t)s' - u(s^+,t)s'$$

$$= \phi(a,t) - \phi(b,t)$$

where $\lim_{x\longrightarrow s(t)^-} u(x,t) = u(s^-,t), \lim_{x\longrightarrow s(t)^+} u(x,t) = u(s^+,t)$. By taking the limits $a \longrightarrow s(t)^-$ and $b \longrightarrow s(t)^+$, we have

$$u(s^-,t)s' - u(s^+,t)s' = \phi(s^-,t) - \phi(s^+,t)$$

i.e.,

$$\Rightarrow \qquad -s'[u] + [\phi(u)] = 0$$

- The last equation is called the *jump condition.*
- In fluid mechanics, it is known as *Rankine-Hugoniot condition.*
- The discontinuity in u that propagates along the curve $x = s(t)$ is called a *shock wave.*
- The curve $x = s(t)$ is called a *shock path.*
- s' is called the *shock speed.*
- $\big|[u]\big|$ is called the *shock strength.*

Example 3 (continue Example 2) Now we are ready to determine the curve in Example 2 which separates two types of characteristics. In fact, the jump condition for the nonlinear wave equation is

$$-s'[u] + \left[\frac{u^2}{2}\right] = 0$$

$$\Rightarrow \qquad s' = \frac{u_- + u_+}{2} = \frac{1+0}{2} = 1/2$$

So, the curve is a straight line

$$s = t/2$$

and a solution, consistent with the jump condition, to the initial value problem is

$$u(x,t) = \begin{cases} 1, & \text{if } x < t/2 \\ 0, & \text{if } x > t/2 \end{cases}$$

Rarefaction Waves

Example 2 let us consider another type of Riemann problem

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases} \end{cases}$$

The solution u is constant along characteristics. Since $dx/dt = u$, we know that the data $u = 0$ are carried into the region $x < 0$ along vertical characteristics; the data $u = 1$ are carried into the region $x > t$ along the characteristics with speed 1. Hence there is a region $0 < x < t$ completely lacking of characteristics. To define a continuous solution on the whole region $x \in R$ and $t > 0$, we simply select a family of curve to fill the region $0 < x < t$ such that the characteristic curve $x = 0$ continuously varies to $x = t$. These curve must be a family of straight lines, since we need to make sure from the original PDE that $dx/dt = \text{const}$. Thus, we take them to be

$$x = ct, \qquad 0 < c < 1$$

For a fixed value c, along the characteristic $x = ct$, $u = \dfrac{dx}{dt} = c = \dfrac{x}{t}$. Hence, the solution is

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{x}{t}, & \text{if } 0 < x < t \\ 1, & \text{if } x > t \end{cases}$$

- For $t > 0$, the solution is continuous.
- The solution DOES satisfy the PDE in the region $0 < x < t$.
- The solution is not continuously differentiable.
- A solution of the form is called a *rarefaction wave*.

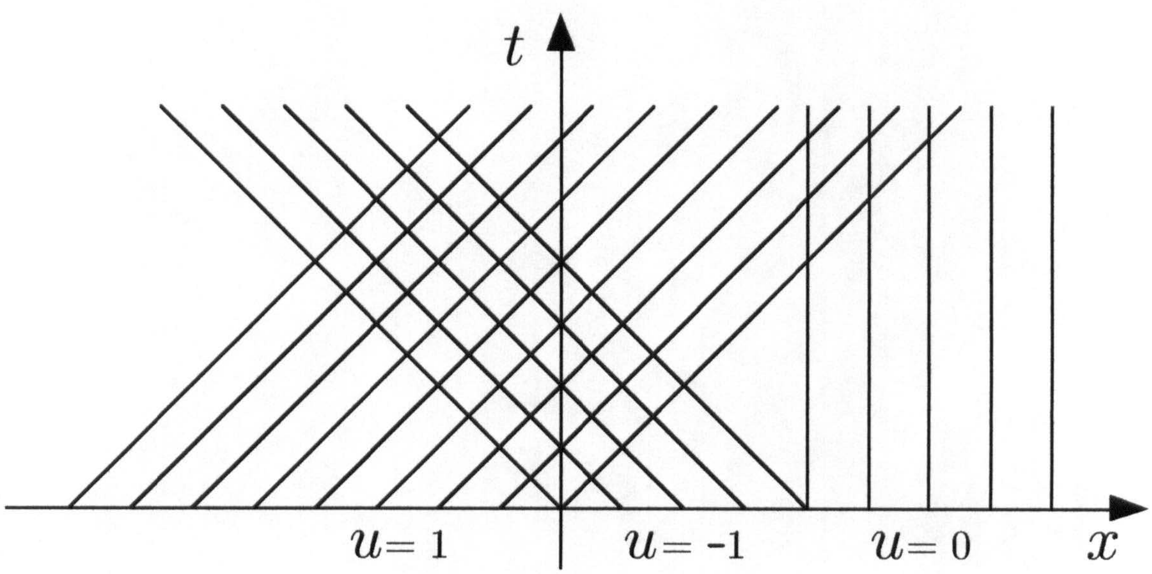
Shock Propagation

- Shock forms if the initial data is discontinuous.
- A solution of a hyperbolic euation may evetually develop discontinuities, even initial it starts a perfectly smooth data.
- The time when the shock forms is called the *breaking time.*

Example 5 Consider a more complicated initial value problem

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x > 0 \end{cases} \end{cases}$$

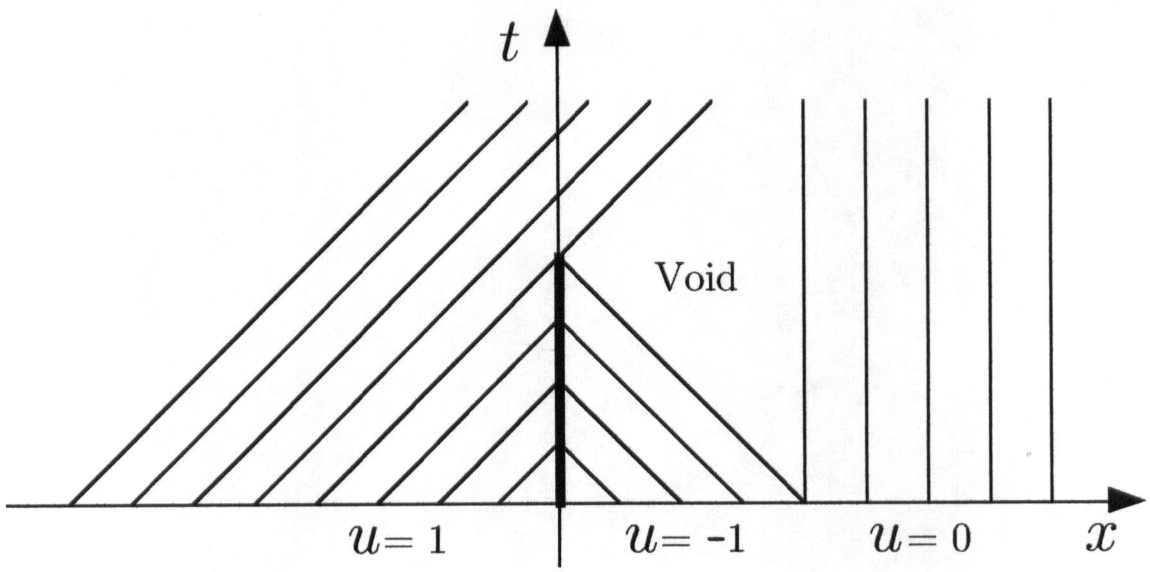
The characteristics are shown here.



Obviously, a shock must form at $t = 0$. The shock path can be determined by the differential equation

$$s' = [u^2/2]/[u] = \frac{u_+ + u_-}{2}$$

Since $u_- = 1$ and $u_+ = -1$, we have $s' = 0 \Rightarrow s = \text{const}$. Since the curve passes through $(0, 0)$, so it is $s(t) = 0$.



The shock propagates until time $t = 1$ with the speed $s' = 0$. Since there is no characteristics in the region bounded by three straight lines: $x = 1$, $x + 1 = t$ and $x - 1 = -t$, we have to insert a family of straight lines in the regions. They have the equation

$$x = -kt + 1, \qquad k = \text{const}$$

As in Example 4, we have

$$u = \frac{x - 1}{t}$$

in the fan region.

Now beyond $t = 1$, the shock will have speed

$$s' = \frac{u_+ + u_-}{2} = \frac{1 + \dfrac{x-1}{t}}{2}$$

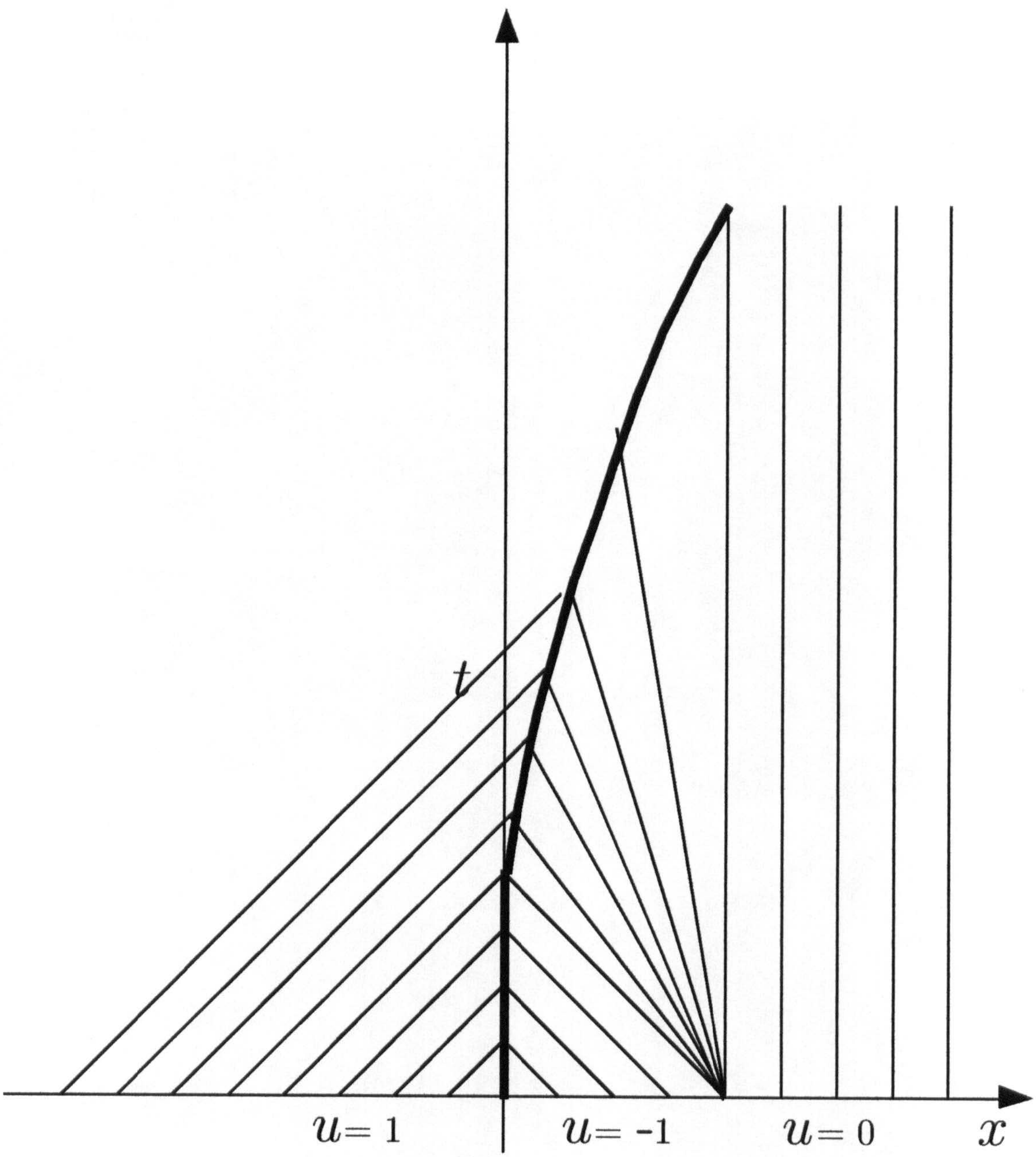
Since $s' = \dfrac{dx}{dt}$, we have a first order ODE

$$\frac{dx}{dt} = \frac{1 + \dfrac{x-1}{t}}{2}$$

with the initial condition $x = 0$ at $t = 1$. This yields

$$x = s(t) = t + 1 - 2\sqrt{t}$$

This curve intersects with $x = 1$ at $t = 4$

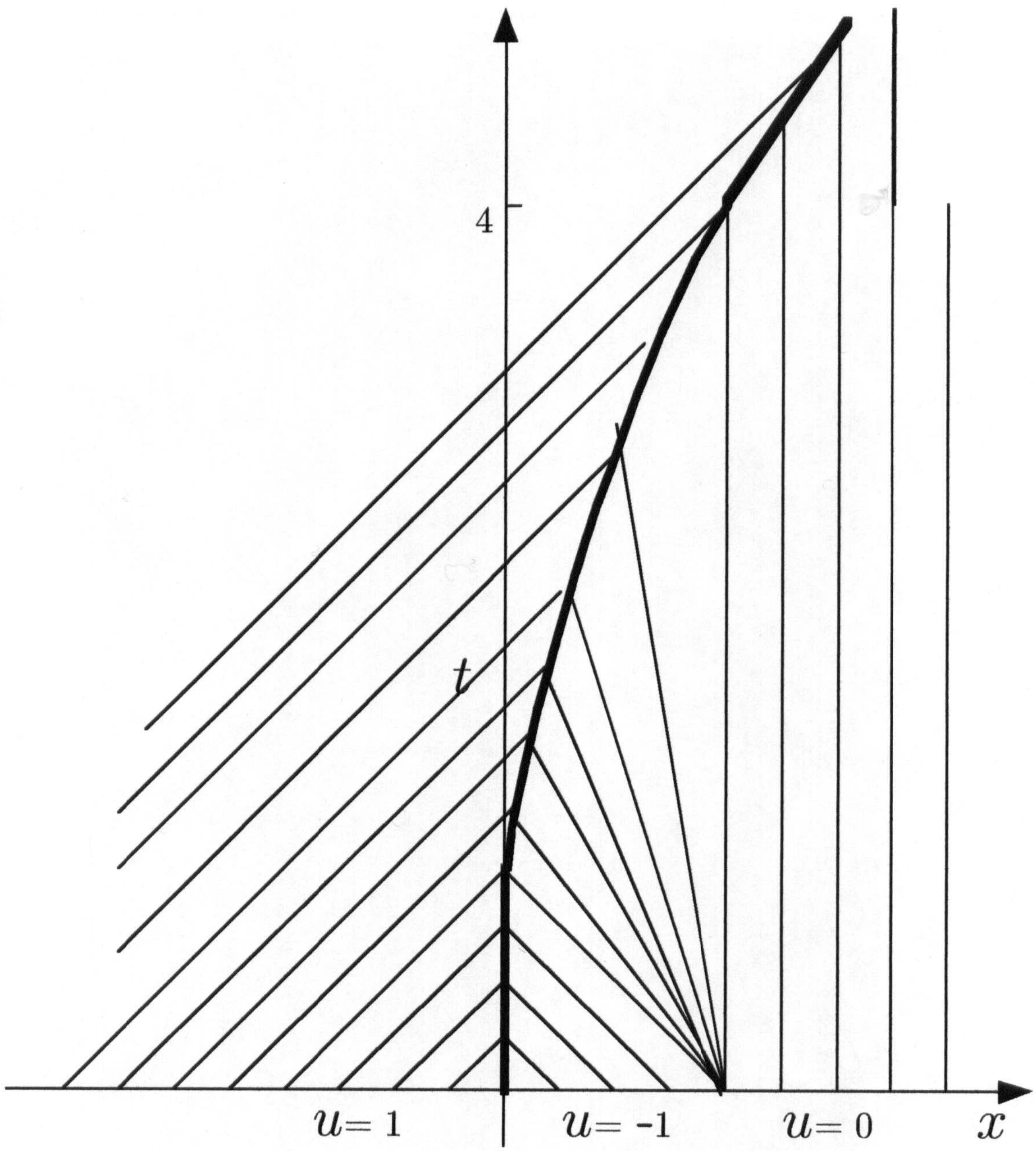


For $t > 4$, the characteristics emanating from $x < 0$ run into the vertical characteristics. The new jump condition for $t > 4$ is

$$s' = \frac{1+0}{2} = 1/2$$

with the initial condition $x = 1$ at $t = 4$. This yields

$$x - 1 = \frac{t-4}{2}$$



Altogether, we have the solution of the initial value problem.

§3.2 SHOCK FORMATION

We have seen that shock waves can form out from discontinuous initial data. In this section, we show that shock waves can also form from smooth data.

Let us consider the initial value problem

$$\begin{cases} u_t + c(u)u_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where $c(u) > 0$, $c'(u) > 0$ and u_0 is C^1. The result in Chapter 2 indicates that, if u_0 is nondecreasing, then a smooth solution of the initial value problem is given implicitly by

$$u(x,t) = u_0(\xi), \qquad \text{where } x - \xi = c(u_0(\xi))t$$

$\Rightarrow$ $\qquad$ u_0 must be strictly decreasing in some open interval

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Without loss of generality, we assume that $u_0'(x) < 0$ on R. The characteristics are a family of straight lines. For any two points on the x-axis, say $(\xi_1, 0)$ and $(\xi_2, 0)$ with $\xi_1 < \xi_2$, we have

$$c(u_0(\xi_1)) > c(u_0(\xi_2))$$

The two characteristics

$$x - \xi_1 = c(u_0(\xi_1))t$$
$$x - \xi_2 = c(u_0(\xi_2))t$$

intersect to each other $\Rightarrow$ a contradiction, since u takes different value on each of the straight lines.

- Even the initial data is smooth, a smooth solution may not exist for all $t > 0$.

Determine the breaking time

Assume that along the characteristic

$$x - \xi = c(u_0(\xi))t$$

the solution breaks down. Write

$$g(t) = u_x(x(t), t)$$

$$\Rightarrow \qquad \frac{dg}{dt} = u_{tx} + c(u)u_{xx}$$

The original PDE $\Rightarrow$

$$u_{tx} + c(u)u_{xx} + c'(u)u_x^2 = 0$$

$$\Rightarrow \qquad \frac{dg}{dt} = -c'(u)g^2$$

Since u is constant along characteristics, we can solve the equation

$$\Rightarrow \qquad g = \frac{g(0)}{1 + g(0)c'(u_0(\xi))t}$$

$$\Rightarrow \qquad u_x = \frac{u_0'(\xi)}{1 + u_0'(\xi)c'(u_0(\xi))t}$$

If c' and u_0' have opposite signs, u_x will blow up at some finite time t along the characteristic. Examine all the characteristics, the breaking time for the wave will be on the characteristic, parametrized by ξ, where

$$1 + F'(\xi)t \equiv 1 + u_0(\xi)c'(u_0(\xi))t = 0$$

- Breaking will occur on the characteristic with $\xi = \xi_b$, for which $F'(\xi) < 0$ and $|F'(\xi)|$ is a maximum.
- The *breaking time* is

$$t_b = -\frac{1}{F'(\xi_b)}$$

Example 1 Consider the initial value problem

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = \exp(-x^2), & x \in R \end{cases}$$

The function

$$F(\xi) = c(u_0(\xi)) = \exp(-\xi^2)$$

Thus

$$F'(\xi) = -2\xi \exp(-\xi^2)$$

The function $|F'|$ has a maximum at $\xi = \xi_b = \sqrt{1/2}$ and the break time is

$\Rightarrow$

$$t_b = -\frac{1}{F'(\xi_b)} \approx 1.16$$

Problem 4 on P.97 Consider the initial value problem

$$\begin{cases} u_t + uu_x = 0, & x > 0, \quad t > 0 \\ u(x,0) = 1, \quad x > 0; & u(0,t) = t+1, \quad t > 0 \end{cases}$$

The characteristics satisfy

$$\frac{dx}{dt} = u$$

Then, the characteristic emanating from the point $(0,\tau)$ on the positive t-axis satisfies

$$\frac{dx}{dt} = \tau + 1$$

$\Rightarrow$

$$x = (\tau + 1)(t - \tau)$$

For any fixed τ, this curve will intersect either with another characteristic

$$x = (\tilde{\tau} + 1)(t - \tilde{\tau})$$

or the characteristic $x = t$ emanating from $(0,0)$. In the first case, we know that these characteristics intersect at time

$$t = \tau + \tilde{\tau} + 1$$

In the second case, the time is

$$t = \tau + 1$$

Therefore, the breaking time $t_b = 1$. The shock forms at $(1,1)$.

To determine the path of the shock, by the jump condition, we have

$$s' = \frac{u_- + u_+}{2} = \frac{\tau + 1 + 1}{2}$$

where τ is determined by

$$x = (\tau + 1)(t - \tau)$$

$$\Rightarrow \qquad \tau = \frac{t - 1 + \sqrt{(t+1)^2 - 4x}}{2}$$

$$\Rightarrow \qquad x' = \frac{t - 1 + \sqrt{(t+1)^2 - 4x}}{4} + 1$$

To solve the equation, denote $(t+1)^2 - 4x = v^2$, then

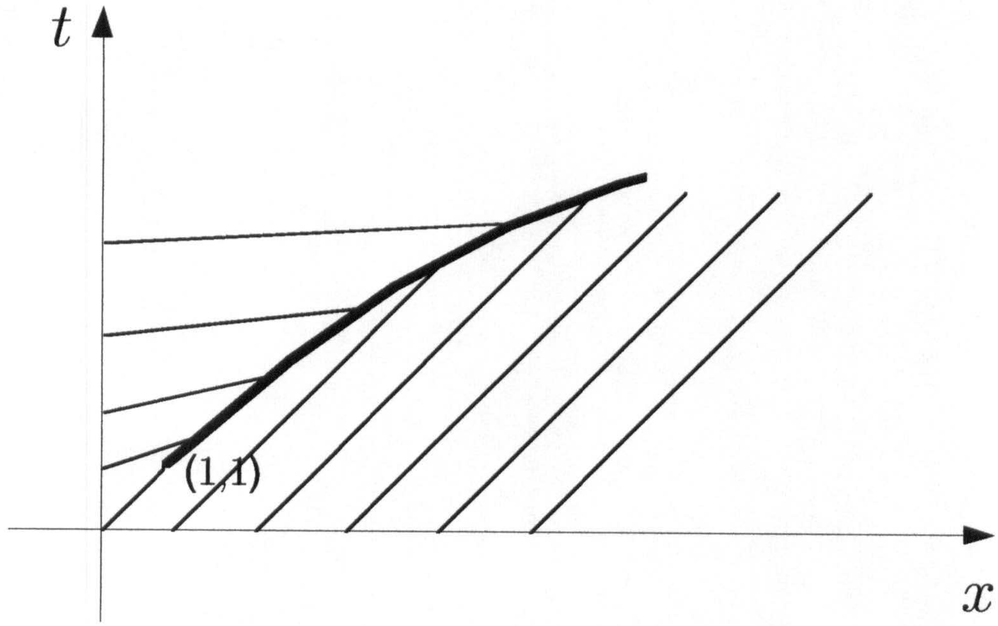
$$t - 1 - v = 2vv'$$

$$\Rightarrow \qquad 2v' = \frac{t-1}{v} - 1$$

We can solve this homogeneous equation by writing $v = (t-1)w$ and have

$$w = 1/2$$

$$\Rightarrow \qquad x = (t+3)(3t+1)/16$$


t
(1,1)
x

§3.4 WEAK SOLUTIONS: A FORMAL APPROACH

Consider the initial value problem

$$\begin{cases} u_t + \phi(u)_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where ϕ is a continuously differentiable function on R.

- u is a *classical* or *genuine* solution if it is smooth.
- A *compact* set in xt spacetime means a closed, bounded set.
- The *support* of f, denoted by supp$f$, is the set

$$\overline{\{(x,t) : f(x,t) \neq 0\}}$$

- The set C_0^1 contains all smooth functions with compact support:

$$C_0^1 = \{f : f_t, f_x \text{ are continuous and supp} f \text{ is compact}\}$$

For a classical solution u of the initial value problem and a function f in C_0^1, take a rectangle

$$D = \{(x,t) : a \leq x \leq b, 0 \leq t \leq T\}$$

such that it contains the support of f. We take D large enough such that f vanishes along$x = a$, $x = b$ and $t = T$. Then, by the PDE, we have

$$\iint_D (fu_t + f\phi_x) dx dt = 0$$

This implies

$$\int_0^T \int_a^b fu_t \; dx \; dt = \int_a^b \int_0^T fu_t \; dt \; dx = \int_a^b \left[fu\Big|_0^T - \int_0^T f_t u \; dt \right] dx$$

$$= -\int_a^b f(x,0)u_0(x) \; dx - \int_a^b \int_0^T f_t u \; dt \; dx$$

and

$$\int_0^T \int_a^b f\phi_x \, dx \, dt = \int_0^T \left[f\phi\Big|_a^b - \int_a^b f_x \phi dx \right] dt = -\int_0^T \int_a^b f_x \phi \, dx \, dt$$

$$\Rightarrow \qquad \iint_{t \geq 0} (uf_t + f_x \phi) \, dx \, dt + \int_{t=0} u_0 f dx = 0$$

- If u is a classical solution and $f \in C_0^1$, then the last identity holds.
- The last equation contains no derivatives of u and ϕ.
- The last equation is called the *week* form of the original IVP.
- The function f is called the *test* function.

Definition 1 A bounded piecewise smooth function $u(x,t)$ is called a *weak solution* of the initial value problem

$$\begin{cases} u_t + \phi(u)_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where u_0 is assumed to be bounded and piecewise smooth, if and only if u satisfies

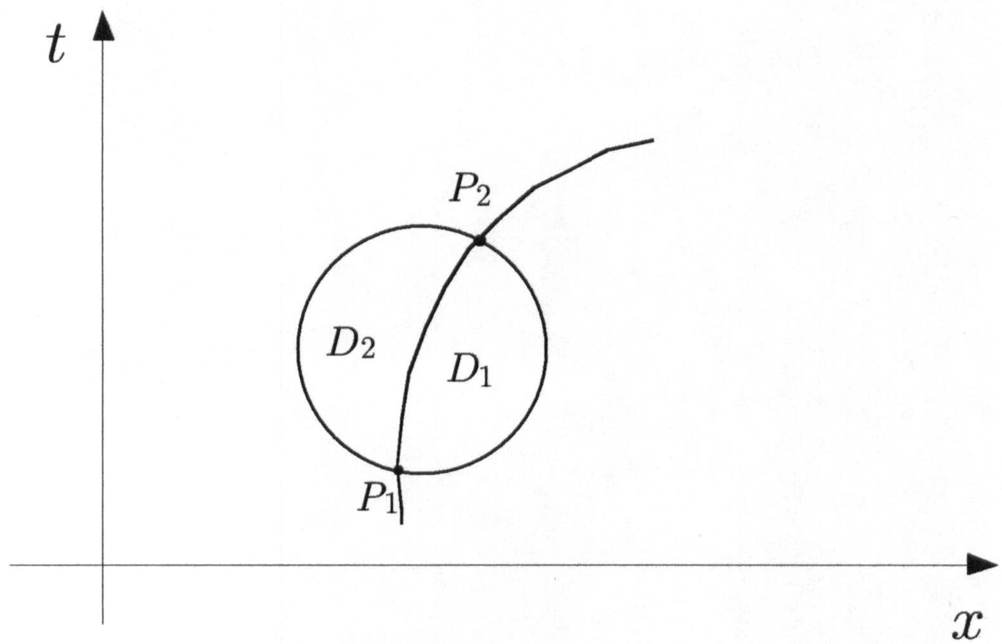
$$\iint_{t \geq 0} (uf_t + f_x \phi) \, dx \, dt + \int_{t=0} u_0 f dx = 0$$

for all smooth functions f with compact support.

- If u is a classical solution, then it is also a weak solution.
- If u is a smooth weak solution and u_0 is also smooth, then u is a classical solution.
- We may change the definition of a weak solution and replace the condition piecewise smooth by *measurable*.

We now show that **the definition of a weak solution leads to the jump condition across a shock.**

Let Γ be a smooth curve in spacetime given by $x = s(t)$ along which u has a simple jump discontinuity. For simplicity, let D be a circular region centered at some point on Γ and lying in the $t > 0$ plane. Let D_1 and D_2 be the disjoint subsets of D on each side of Γ.



Take any $f \in C_0^1(D)$. By the definition of weak solution,

$$\begin{aligned} 0 &= \iint_D [uf_t + \phi(u)f_x]\ dx\ dt \\ &= \iint_{D_1} [uf_t + \phi(u)f_x]\ dx\ dt + \iint_{D_2} [uf_t + \phi(u)f_x]\ dx\ dt \end{aligned}$$

Since u is smooth in D_2, we know that

$$u_t + \phi(u)_x = 0$$

By Green 's theorem,

$$\begin{aligned}\int_{\partial D_2} -uf dx + f\phi dt &= \iint_{D_2} [(uf)_t + (f\phi)_x]\ dx\ dt \\ &= \iint_{D_2} [uf_t + \phi(u) f_x]\ dx\ dt\end{aligned}$$

Since $f \in C_0^1(D)$, the line integral is only nonzero along Γ, hence,

$$\iint_{D_2} [uf_t + \phi(u) f_x]\ dx\ dt = \int_{P_1}^{P_2} -u_2 f dx + f\phi(u_2) dt$$

Similarly,

$$\iint_{D_1} [uf_t + \phi(u) f_x]\ dx\ dt = \int_{P_2}^{P_1} -u_1 f dx + f\phi(u_1) dt$$

$$\Rightarrow \qquad \int_{\Gamma} f\big(-[u]dx + [\phi(u)]dt\big) = 0$$

Since f is arbitrary, we have

$$-[u]dx + [\phi(u)]dt = 0$$

$$\Rightarrow \qquad -s'(t)[u] + [\phi(u)] = 0$$

This is the jump condition.

- PDE alone is not enough to determine the jump condition. There may be more than one jump conditions.
- Weak solutions are not unique.

Example 1 Consider the PDE

$$u_t + uu_x = 0$$

Any positive solution satisfies both

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \text{ and } \left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0$$

Each of the equations yields different jump condition. Thus there are more than one week solutions associate to the PDE.

Since weak solutions are not unique, another condition is required to guarantee uniqueness. One commonly used condition is the so-called *entropy* condition:

$$u(x+a,t) - u(x,t) \le \frac{aE}{t}, \qquad a > 0, \quad t > 0$$

where E is independent of x, t and a.

- As x moves from $-\infty$ to $+\infty$, the solution can only jump down.
- If $\phi'' > 0$, then we have the *entropy inequality*

  $$\phi'(u_2) > s' > \phi'(u_1)$$

  where s' is the shock speed and u_1 and u_2 are the states ahead and behind the shock, respectively. This can be derived from

  $$s = \frac{\phi(u_1) - \phi(u_2)}{u_1 - u_2} = \phi'(\xi)$$

  and the fact that $\phi'(u)$ is increasing with u.

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§3.5 ASYMPTOTTIC BEHAVIOR OF SHOCKS

In this section, we demonstrate how initial waveform evolve over the time.

Equal-Area Principle

Consider the initial value problem

$$\begin{cases} u_t + c(u)u_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

We assume that

- $c(u) > 0$, $c'(u) \geq 0$.
- $u_0(x) \in C^1(R)$ is a bell curve.

The solution is given implicitly by

$$\begin{cases} u = u_0(\xi) \\ x = \xi + F(\xi)t, & F(\xi) = c(u_0(\xi)) \end{cases}$$

As t increasing, the waveform is distorted and eventually the solution becomes multiple-valued.

Lemma 1 Consider a section of the initial waveform between $x = a$ and $x = b$, with $u_0(a) = u_0(b)$, with u_0 a bell curve. Then for any $t > 0$, the area under this section of the wave remains constant as it propagates in time.

Lemma 2 Let a and b be chosen such that $u_0(a) = u_0(b)$, where u_0 is a bell curve, and assume that the shock path is given by $x = s(t)$ with $a + tF(a) < s(t) < b + tF(b)$ for t in some open interval I. Then

$$\int_{a+tF(a)}^{b+tF(b)} u(x,t)dx = \text{const}, \qquad t \in I$$

where $u = u(x,t)$ is the weak solution.

Write $\phi'(u) = c(u)$. Since $u_0(a) = u_0(b)$, we have

$$F(a) = c(u_0(a)) = c(u_0(b)) = F(b)$$

$$u(a + tF(a), t) = u(a, 0) = u_0(a) = u_0(b) = u(b, 0) = u(b + tF(b), t)$$

Thus

$$\begin{aligned}
&\frac{d}{dt}\int_{a+tF(a)}^{b+tF(b)} u(x,t)dx \\
=&\frac{d}{dt}\int_{a+tF(a)}^{s(t)} u(x,t)dx + \frac{d}{dt}\int_{s(t)}^{b+tF(b)} u(x,t)dx \\
=&s'(t)u(s^-(t),t) - F(a)u(a+tF(a),t) + \int_{a+tF(a)}^{s(t)} u_t(x,t)dx \\
&+F(b)u(b+tF(b),t) - s'(t)u(s^+(t),t) + \int_{s(t)}^{b+tF(b)} u_t(x,t)dx \\
=&- s'[u] - \int_{a+tF(a)}^{s(t)} c(u)u_x dx - \int_{s(t)}^{b+tF(b)} c(u)u_x dx \\
=&- s'[u] - \phi(u)\Big|_{a+tF(a)}^{s(t)} - \phi(u)\Big|_{s(t)}^{b+tF(b)} \\
=&- s'[u] + [\phi(u)] = 0
\end{aligned}$$

Equal-Area Principle The location of the shock $x = s(t)$ at time t is the position where a vertical line cuts off equal area lobes of the multivalued wavelet.

Shock Fitting

Consider

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

Once again we assume that the function $u = u_0(x)$ is a bell curve. Now we give an algorithm to determine the shock position.

Assume that the shock is located at $x = s(t)$ at some time at $t > t_b$. To determine $s(t)$, let us assume that the two characteristics intersecting at t are

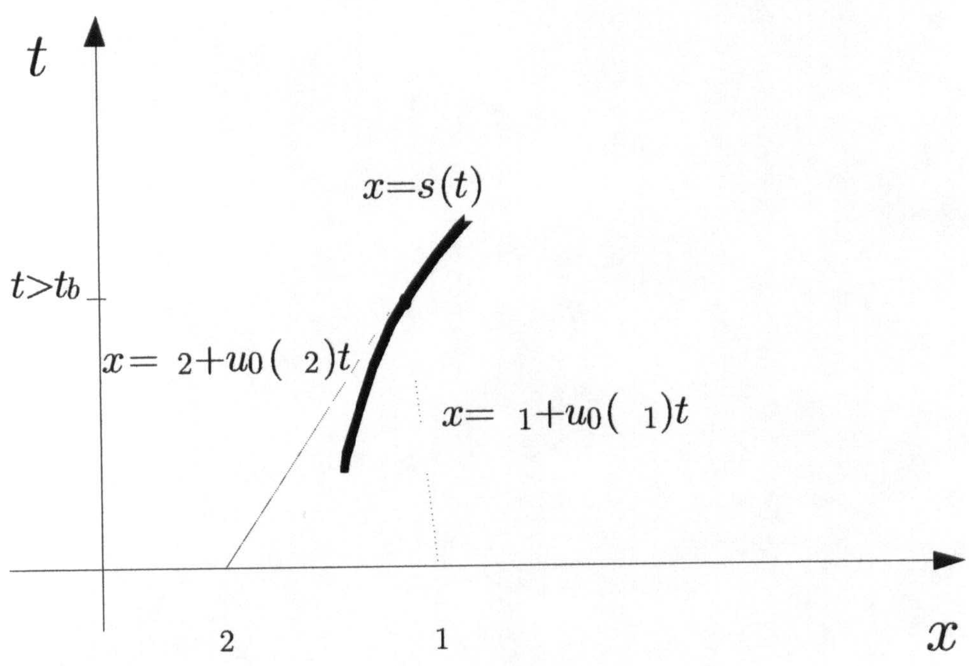
$$x = \xi_1 + u_0(\xi_1)t, \qquad x = \xi_2 + u_0(\xi_2)t$$

Or

$$s(t) = \xi_1 + u_0(\xi_1)t, \qquad s(t) = \xi_2 + u_0(\xi_2)t \tag{8}$$

$\Rightarrow$

$$t = -\frac{\xi_1 - \xi_2}{u_0(\xi_1) - u_0(\xi_2)}$$



The equations in (8) have three unknowns: s, ξ_1, ξ_2. One more equation is need to determine the shock path. It can be obtained by the equal-area principle.

$$\int_{\xi_1}^{\xi_2} u_0(\xi) d\xi = \frac{1}{2} \left[u_0(\xi_1) + u_0(\xi_2) \right] (\xi_1 - \xi_2)$$

$$\Rightarrow \qquad \begin{cases} s(t) = \xi_1 + u_0(\xi_1) t \\ s(t) = \xi_2 + u_0(\xi_2) t \\ \int_{\xi_1}^{\xi_2} u_0(\xi) d\xi = \frac{1}{2} \left[u_0(\xi_1) + u_0(\xi_2) \right] (\xi_1 - \xi_2) \end{cases}$$

Whitham's Rule For a succession of value of time t, draw a chord on the initial profile u_0 of slope $-1/t$ cutting off equal-area lobes. Then shift the endpoints of the chord to the right by the amounts $tu_0(\xi_2)$ and $tu_0(\xi_1)$, respectively, obtaining the location of the shock at the time t.

- If $t \to +\infty$, the slope of the straight line PQ tends to zero.
- The shock strenghth $= u_0(\xi_2) - u_0(\xi_1)$.

Asymptotic Behavior

In this part, we study the long-term behaviors of shock strength, shock path and the solution.

Consider the initial value problem

$$\begin{cases} u_t + uu_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where $u = u_0(x)$ is a bell curve, with $u_0(x) = u_*$ for $x \leq 0$ and for $x \geq a > 0$.

By Whitham's Rule, the slope of the straight line PQ tends to zero as t to $+\infty$. This implies that, for $t \gg 1$,

$$u_0(\xi_1(t)) = u^*$$

$$\Rightarrow \qquad \int_{\xi_2}^{\xi_1} [u_0(\xi) - u^*] d\xi = \frac{1}{2}(\xi_1 - \xi_2)[u_0(\xi_2) - u^*], \qquad \text{for } \xi_1 > a$$

Since, $(\xi_1 - \xi_2) + t[u_0(\xi_1) - u_0(\xi_2)] = 0$, we have

$$\Rightarrow \qquad \int_{\xi_2}^{\xi_1} [u_0(\xi) - u^*] d\xi = \frac{t}{2}[u_0(\xi_2) - u^*]^2, \qquad \text{for } \xi_1 > a$$

Or

$$\Rightarrow \qquad \int_{\xi_2}^{a} [u_0(\xi) - u^*] d\xi = \frac{t}{2}[u_0(\xi_2) - u^*]^2, \qquad \text{for } \xi_1 > a$$

As $t \to +\infty$, since ξ_2 tends to zero, the left hand side tends to the area A and the quantity $u_0(\xi_2) - u^*$ tends to the shock strength. Thus

$$\text{Shock strength} \sim \left(\frac{2A}{t}\right)^{1/2}$$

By the previous discussion, we have

$$u_0(\xi_2(t)) \sim u^* + (2A/t)^{1/2}, \qquad t \gg 1$$

Since as $t \to +\infty$, we have $\xi_2(t) \to 0$, we have

$$\begin{aligned} s(t) &= \xi_2(t) + t u_0(\xi_2(t)) \\ &\sim u^* t + (2At)^{1/2}, \qquad t \gg 1 \end{aligned}$$

The point (0,0) moves at speed u^* and is located at $x = u^* t$.

$$\Rightarrow \qquad u = u^*, \qquad \text{for } x < u^* t$$

Also

$$u = u^*, \qquad \text{for } x > s(t)$$

For $t \gg 1$, notice $s(t) \sim u^* t + (2At)^{1/2}$. In the region $u^* t < x < u^* t + (2At)^{1/2}$, since the solution is given by

$$u = u_0(\xi), \qquad x = \xi + t u_0(\xi)$$

$$\Rightarrow \qquad u = \frac{x - \xi}{t}$$

For $t \gg 1$, $\Rightarrow \xi = \xi_2 \to 0$

$$\Rightarrow \qquad u \sim \frac{x}{t}, \qquad u^* t < x < u^* t + (2At)^{1/2}$$
