Thoughts on Probability, Analysis, Number Theory Mathematical Physics and Algebra, Mathematical Recreation

The aim of these notes is to point out that increasingly in Modern mathematics, a good mathematician must have deep knowledge across several areas and the old divisions have become blurred. The arguments I offer in this note are heuristic, they can be made rigorous provided you have a sound knowledge of Chap 8. of your textbook by Rudin and know facts about **infinite products**. You do know what infinite sums are from Chap 3 of your book. The idea in a nutshell is to convert infinite products to sums by taking **logarithms**. This will be developed in Chap. 8. as we will soon see. Chap 8. in Rudin has many hidden treasures even in the problem set. For example one of the problems is a form of the Birkhoff-Von Neumann **Ergodic Theorem**. This is **not** pointed out by Rudin. This theorem is important in the study of Statistical Physics of Ideal gases(see the book on Quantum Mechanics by the Nobel Laureate Max Born–grandfather of the singer Olivia Newton-John), in the Number theory arising from the action of Discrete groups(see the book by David Borthwick). The Ergodic theorem has the following important consequence. Think of an ideal gas, with all sorts of molecules bumping into each other and so on. What we observe on the macroscopic level, Pressure, temperature and so on is the statistical average behavior of the gas molecules. Now suppose we take one particular molecule and take its momentum and average it over time, follow it and take the momentum and average it for a very long time, in effect to infinite time, then the Ergodic theorem says, that value will be the average over space of the momentum of all the molecules, that is take the average value of the momentum over the box in which the gas is stored. The average over the box is at any one given time. Thus what I am saying is that the **time** average over infinite time of the momentum of a **single** particle is the same as the **space** average over **all** particles at a fixed time.

Now coming back to the current note, we will soon see it cycles back to the Riemann Zeta function.

Definition: We are given a finite set of natural numbers $\{n_1, n_2, \cdots, n_k\}$. We say the numbers are co-prime or relatively prime if their greatest common divisor, hereafter abbreviated gcd, is 1.

Examples are $\{2, 3, 5\}$.

Our aim is to show at least heuristically:

Theorem: Given a set M containing N numbers. Now consider all subsets of M containing exactly s numbers. There are exactly $\binom{N}{s}$ such subsets. Now in this list of subsets containing $\binom{N}{s}$ elements, let L be the number of subsets having exactly s numbers which are co-prime, i.e whose gcd=1. We are interested in the probability,

$$\lim_{N\to\infty} \frac{L}{\binom{N}{s}}.$$

We want to show that this probability is:

$$\frac{1}{\zeta(s)},$$

and where the Riemann-Zeta function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In standard books of Fourier Series or if you take Math 421 or Math 423, you will see that one can easily compute

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Thus the probability any two given numbers are co-prime is

$$\frac{6}{\pi^2}.$$

Remark 1: The values of the Zeta function at s which is a natural number is rather poorly understood. A few years ago by an ingenious argument Apéry proved that $\zeta(3)$ is irrational. This caused a buzz and people thought that the methods used by Apéry could be used for other natural numbers, but this initial excitement has faded since the methods of Apéry do not extend beyond $s = 3$. It is still unclear if $\zeta(3)$ is algebraic or transcendental. However it is known that infinitely many of the $\zeta(2k+1)$ are irrational, that is for infinitely many odd natural numbers the zeta function evaluated at the odd numbers is irrational. The point is: Is this true for **all** odd natural numbers?

Okay let us get back to our result. First fix a prime p. Let us consider the residue classes we get mod p, or in other words consider $\mathbf{Z}_p$. This is just the set,

$$S = \{0, 1, 2, \cdots, p-1\}$$

In other words when we divide a number by p, the remainder is some number in the set S which is called the residue class or remainder class. Thus the probability that a number is divisible by the prime p is just $1/p$. This is because the set S has exactly p numbers and we assume that we have a dart board with the p numbers of the set S arranged like the numbers around a clock on the dart board and we are shooting darts and finding the probability that a dart lands on one such number. Thus the probability of s numbers are all divisible by p is the s-fold product:

$$\frac{1}{p} \times \frac{1}{p} \times \cdots \times \frac{1}{p} = \frac{1}{p^s}.$$

Thus the probability that **at least one** of the s numbers is not divisible by p is

$$1 - \frac{1}{p^s}.$$

This finishes the analysis for a single prime. Now lets take two prime p_1, p_2. If a set of s numbers is **not** divisible by p_1 then the fact that this set is not divisible by p_2 is independent of the fact that the numbers in the set is not divisible or divisible by p_1 for that matter. In other words divisibility by a certain prime is **independent** of divisibility by another prime. Thus the probability that a given a set of s numbers cannot be all divided by p_1 and p_2 is

$$\left(1 - \frac{1}{p_1^s}\right)\left(1 - \frac{1}{p_2^s}\right). \tag{1}$$

Now the fact is that a set consists of numbers whose gcd is 1 if the set of numbers cannot be divided by any prime. Thus the probability that a set of s numbers consists of co-prime numbers is to extend the product over two primes (1) to an infinite product over all primes:

$$\Pi_{\text{all primes}}\left(1 - \frac{1}{p_j^s}\right). \tag{2}$$

Thus we will be done if we can show that

$$\frac{1}{\zeta(s)} = \Pi_{\text{all primes}}\left(1 - \frac{1}{p_j^s}\right) \tag{3}$$

First let me convince you that the infinite product on the right actually makes sense, by the logarithm trick alluded to above. To achieve our aims we will need the following fact. For $0 \le x < \frac{1}{2}$,

$$2x \ge -\log(1-x) \ge x \tag{4}$$

This can be proved by the **method used in class** while proving the Weierstrass theorem. Set,(we only prove one of the inequalities in (4), the remaining one is left to the reader)

$$F(x) = -\log(1-x), \; G(x) = x$$

Note $F(0) = G(0)$. Next,

$$F'(x) = \frac{1}{1-x}, G'(x) = 1.$$

For $0 \le x < 1/2$, we see easily,

$$F'(x) \ge G'(x)$$

Thus

$$F(x) \ge G(x).$$

We re-write (4) as

$$-2x \le \log(1-x) \le -x \tag{5}$$

Thus using (5)

$$-2\sum_{\text{primes}} \frac{1}{p_j^s} \le \log\left(\Pi_{\text{all primes}}\left(1 - \frac{1}{p_j^s}\right)\right) = \sum_{\text{primes}} \log\left(1 - \frac{1}{p_j^s}\right) \le -\sum_{\text{primes}} \frac{1}{p_j^s}.$$

But now we can use the comparison test and note

$$0 < \sum_{\text{primes}} \frac{1}{p_j^s} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty.$$

Note $s \geq 2$ and so the series on the right and extreme left converges to finite numbers. Thus the series in the middle is finite by the so-called **Pinching theorem** of **Freshman Calculus**.

Thus we see how infinite products can be turned to a problem of standard series of Chap. 3 of Rudin. More about this when we discuss Chap. 8. Now the last order of business is to hook this up with the Zeta function and (3). First we recall an identity of Euler. This identity can be made rigorous exactly by the technique above of converting products into sums.

The identity reads,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \cdots \frac{1}{p_1^{ks}} + \cdots\right) \cdots \left(1 + \frac{1}{p_m^s} + \frac{1}{p_m^{2s}} + \cdots \frac{1}{p_m^{ks}} + \cdots\right)\left(\cdots\right) \quad (6)$$

That is we take every single prime, and form the Geometric series and multiply out the the product of the series. Note because every natural number n is a product of primes

$$n = p_{m_1}^{a_1} \cdots p_{m_k}^{a_k},$$

on multiplication one gets all the natural numbers raised to s. But each Geometric series is convergent as $p_k > 1$, p_k being prime. In fact

$$1 + \frac{1}{p_m^s} + \frac{1}{p_m^{2s}} + \cdots \frac{1}{p_m^{ks}} + \cdots = \left(1 - \frac{1}{p_m^s}\right)^{-1}.$$

Thus from (6)

$$\zeta(s) = \Pi_{\text{all primes}}\left(1 - \frac{1}{p_m^s}\right)^{-1}.$$

At which point we see our identity (3). There is also another nice way of stating (3) that is left to the reader. It involves the **Möbius function** of Algebra.

Definition: The Möbius function is defined on the **Natural numbers N**.

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k, & n = p_1 p_2 \cdots p_k, \text{ where } p_j \text{ is prime} . \\ 0, & \text{if } p_k^2/n, \end{cases}$$

Thus the Möbius function is zero if n is divisible by a square of a prime.

Proposition: Prove,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

There is an important consequence of the Proposition above. Observe from the definition, $|\mu(n)| \leq 1$. Thus from the proposition above, for $\Re s > 1$

$$\left|\frac{1}{\zeta(s)}\right| \leq \sum_{n=1}^{\infty} \left|\frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| < \infty,$$

We thus obtain

Corollary: $1/\zeta(s)$ is finite for $\Re s > 1$.

Thus the Zeta function has **no zero** in the right half plane $\Re s > 1$. Thus the only possible zeros are in the critical strip $0 < \Re s < 1$. A slightly harder argument shows there are no zeros on $\Re s = 1$. This fact that there are no zeros in $\Re s \geq 1$ immediately allows one to deduce the Prime number theorem by a Tauberian theorem argument. There is a simple Tauberian theorem argument that is elementary by Donald Newman[1] and appears in an undergraduate Math. journal the Monthly, read it. It is far easier than the traditional Ikehara Tauberian theorem and quickly gives a proof of the Prime Number theorem based on the fact in the Corollary above. In fact the argument in the article [1] can be itself simplified to a page of hand-written notes, but that is my trade secret!!!

Remark 2: Note that for s a **real number**,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} > 1$$

and so $\zeta(s) > 1$ for $s > 1$ and s real. Thus indeed

$$\frac{1}{\zeta(s)} < 1,$$

and so we do have a true probability. Next the terms

$$\frac{1}{2^s}, \cdots \frac{1}{n^s}$$

get smaller and smaller as $s \to \infty$ and so the probability goes to 1 as $s \to \infty$. Thus given a list of s randomly collected numbers, as $s \to \infty$, it is highly likely the numbers are co-prime. In fact, one has again from Fourier series,

$$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} = 1.0832\cdots$$

So already for $s = 4$, the probability is already very close to 1 that the numbers are co-prime. Since we began with Statistical Physics of gases, let us end with it. $\zeta(4)$ that we have written out above, appears in Blackbody radiation and the Stefan-Boltzmann law, see Chap 1. of [2] for a highly entertaining and invigorating account.

[1] Donald J. Newman, Simple Analytic Proof of the Prime Number Theorem, *American Mathematical Monthly*, **87** (1980), 693-696.

[2] Richard P. Feynman, *Statistical Mechanics: A Set of Lectures*, Addison-Wesley Publishing Co. (1990).