Primes in Short Intervals

The principal focus here is to understand the problem of determining intervals $[a, b]$ so that one always finds a prime in this interval. This problem is connected to the Riemann Hypothesis and certain zero density estimates for the Riemann zeta function. We will begin with a result due to Chebychev which can be proved by elementary means by consideration of the central value of the binomial coefficients. This result is also called Bertrand's postulate. Next we will prove Hoheisel's theorem which allows for better results and which makes a link with zero density estimates for the Riemann zeta function.

Theorem 1(Chebychev): For any $n \geq 1$, there is always a prime in the interval $[n, 2n]$.

The proof relies on several lemmas.

Lemma 1: The binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

satisfies the estimates,

$$\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n.$$

Proof: The binomial coefficient $\binom{2n}{n}$ is a coefficient that appears in the expansion for $(1+1)^{2n}$. Moreover it is the central binomial coefficient and hence the largest. Thus we have,

$$4^n = (1+1)^{2n} \leq (2n+1)\binom{2n}{n}.$$

This gives the left hand inequality. The right hand inequality is obtained by observing,

$$\binom{2n}{n} \leq (1+1)^{2n} = 4^n.$$

Definition: Let $[x]$ denotes the integral part of x. We define the **primorial** $P(x)$ as the product of primes p given by

$$P(x) = \Pi_{p \leq [x]} p.$$

Lemma 2: We have the estimate

$$P(x) \leq 2^{2x-3}.$$

Proof: With no loss of generality we prove this estimate with x a natural number. Let $x = 2m - 1$ and so odd. We will employ induction and show

$$P(2m-1) \leq 2^{2(2m-1)-3} = 2^{4m-5}.$$

Now in the binomial expansion $(1+1)^{2m-1}$ the central largest value occurs twice at,

$$\binom{2m-1}{m}, \binom{2m-1}{m-1}.$$

Thus

$$\binom{2m-1}{m-1} = \frac{1}{2}(\binom{2m-1}{m-1} + \binom{2m-1}{m}) \leq \frac{1}{2}(1+1)^{2m-1} = 2^{2m-2}. \tag{1}$$

Now we have bounds for the primorial,

$$\frac{P(2m-1)}{P(m)} = \Pi_{m<p\leq 2m-1} p \leq \binom{2m-1}{m-1} = \frac{(m+1)(m+2)\cdots(2m-1)}{(m-1)!}. \tag{2}$$

Note all the primes p such that $m < p \leq 2m-1$ are present in the numerator on the extreme right above, and will not cancel with the denominator, since being prime their only divisor is themselves and 1. We may now use induction on m and by the inductive hypothesis assume

$$P(m) \leq 2^{2m-3}.$$

So using (1) we have via (2) and the induction hypothesis,

$$P(2m-1) \leq 2^{2m-2} 2^{2m-3} = 2^{4m-5}.$$

This proves our claim. If $x = 2m$ even, note as we are dealing with primes

$$P(2m-1) = P(2m) = \Pi_{p\leq 2m} p = \Pi_{p\leq 2m-1} p.$$

Thus from the case already proved

$$P(2m) = P(2m-1) \leq 2^{4m-5} \leq 2^{2(2m)-3}.$$

This proves the lemma.

For p a prime, we want to consider the largest exponent $D(p,n)$ such that $p^{D(p,n)}$ divides $n!$. We have the following formula

Lemma 3:(Legendre)

$$D(p,n) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

Here $[\cdot]$ denotes the integral part of a number. The sum above is a finite sum and will terminate for i when $p^i > n$.

Proof: Given a prime $p < n$. Each $kp \leq n$, $k = 1, 2, \cdots \left[\frac{n}{p}\right]$ will contribute once to $D(p,n)$. Likewise we consider p^2 and sp^2, $s = 1, 2, \cdots, \left[\frac{n}{p^2}\right]$ will contribute once to $D(p,n)$. Summing over powers of p we obtain our formula.

The summands of our formula are large and unwieldy and we would like them to be under control and be no larger than 1. So instead of considering $n!$ we consider instead $\binom{2n}{n}$. We will use Lemma 3 to prove

Lemma 4: (a)The largest exponent $D(p,n)$ such that $p^{D(p,n)}$ divides $\binom{2n}{n}$ satisfies

$$p^{D(p,n)} \leq 2n. \tag{3}$$

(b)Moreover, there are no prime divisors p of $\binom{2n}{n}$ such that $\frac{2n}{3} < p < n$.

(c) If $D(p,n) \geq 2$, then $p \leq \sqrt{2n}$.

Proof: We use Legendre's formula. Since $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$, use of Legendre's formula gives us that the largest exponent $D(p,n)$ such that $p^{D(p,n)}$ divides $\binom{2n}{n}$ satisfies

$$D(p,n) = \sum_{i=1}^{\infty} \left\{ [\frac{2n}{p^i}] - 2[\frac{n}{p^i}] \right\}. \tag{4}$$

Now

$$\frac{n}{p^i} = k + \gamma,$$

where γ is a fraction and k a natural number. Thus $\left[\frac{n}{p^i}\right] = k$. If $\gamma < 1/2$ then, $\left[\frac{2n}{p^i}\right] = 2k$ and the summand in (4) vanishes. If $\gamma \geq 1/2$ then, $\left[\frac{2n}{p^i}\right] = 2k+1$ and the summand in (4) is 1. So in no case does the summand in (4) exceed 1. Now we also notice that the index i stops at $p^i \leq 2n$, $i \leq \log_p 2n$. Since the summands never exceed 1, we get

$$D(p,n) \leq \log_p 2n, \text{ which yields } p^{D(p,n)} \leq 2n.$$

This proves the first part of the lemma.

To prove the second part of the lemma, we use (4) again. In the range of p under consideration $1 < \frac{n}{p} < \frac{3}{2}$. So, $\left[\frac{n}{p}\right] = 1$. We also obtain $2 < \frac{2n}{p} < 3$. Thus $\left[\frac{2n}{p}\right] = 2$. Thus the summands are all zero for primes p which satisfy $\frac{2n}{3} < p < n$. This proves the second part of the Lemma.

For the last part assume $D(p,n) \geq 2$. Then by part (a), $p^2 \leq 2n$. Taking square roots we obtain $p \leq \sqrt{2n}$. This proves (c).

Now we have assembled all the parts to prove our theorem.

Proof of Chebychev's theorem: We proceed by contradiction and assume there are no primes in $[n, 2n]$. In view of Lemma 4, part (b), we will then have there are **no** primes in the interval $[\frac{2n}{3}, 2n]$.

Next we write the prime factorization of $\binom{2n}{n}$,

$$\binom{2n}{n} = \Pi p_i^{a_i}.$$

We separate those prime factors whose exponents satisfy $a_i \geq 2$ which we know from part(c) lemma 4 satisfy $p \leq \sqrt{2n}$. Moreover by Lemma 4(a) $p_i^{a_i} \leq 2n$. Thus

$$\binom{2n}{n} = \Pi p_i^{a_i} = \Pi_{p_i \leq \sqrt{2n}} p_i^{a_i} \Pi_{p_i > \sqrt{2n}} p_i \leq (2n)^{\sqrt{2n}} \Pi_{2n \geq p_i > \sqrt{2n}} p_i. \tag{5}$$

But by Lemma 4(c) we control the finite product on the extreme right in terms of the primorial, that is we have,

$$\Pi_{2n \geq p_i > \sqrt{2n}} p_i = \Pi_{\frac{2n}{3} \geq p_i > \sqrt{2n}} p_i \leq P(\frac{2n}{3}).$$

Thus from (5) we have using Lemma 1,

$$\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq (2n)^{\sqrt{2n}} P(\frac{2n}{3}).$$

Thus by Lemma 2,

$$\frac{2^{2n}}{2n+1} \leq (2n)^{\sqrt{2n}} 2^{\frac{4n}{3}-3}.$$

We get

$$8 \times 2^{\frac{2n}{3}} \leq (2n+1)(2n)^{\sqrt{2n}}.$$

Taking logs,

$$\log 8 + \frac{2n}{3} \log 2 \leq \log(2n+1) + \sqrt{2n} \log(2n).$$

One can check that this inequality is valid provided $n \leq 168$. Thus the remaining cases are the finitely many intervals $[1,2], [2,4], [4,8], [8,16], [16,32], [32,64], [64,128], [128,256]$ which are easily seen to contain primes. This proves Chebychev's theorem.

Our considerations so far allow us to prove an upper bound with a crude constant in the Prime Number theorem.

Proposition: Let $\pi(n)$ denote the number of primes less than n. Then,

$$\pi(n) \leq C \frac{n}{\log n},$$

where C is an absolute constant.

Proof: We consider the Primorial again. We have,

$$n^{\pi(2n)-\pi(n)} \leq \Pi_{n \leq p \leq 2n} p \leq P(2n) \leq 2^{4n-3}.$$

Taking logs we get,

$$\pi(2n) - \pi(n) \leq C \frac{n}{\log n}.$$

Thus adding this estimate,

$$\pi(2^k) = \sum_{j=1}^{k} (\pi(2^j) - \pi(2^{j-1})) \leq C \sum_{j=1}^{k} \frac{2^j}{j}.$$

We sum by parts the right side and we get

$$\sum_{j=1}^{k} \frac{2^j}{j} \leq \frac{2^k}{k} + \sum_{j=1}^{k} \frac{2^j}{j^2} \leq \frac{2^k}{k} + \frac{1}{2} \sum_{j=1}^{k} \frac{2^j}{j}.$$

Thus we obtain

$$\pi(2^k) \leq \sum_{j=1}^{k} \frac{2^j}{j} \leq C \frac{2^k}{k}.$$

Now for any n, choose k such that $2^{k-1} < n \leq 2^k$ and using

$$\pi(n) \leq \pi(2^k) \leq C \frac{2^k}{k} \leq C \frac{n}{\log n}.$$

This proves the proposition

To go further than these elementary arguments we need to examine the Riemann zeta function. A solution of the Riemann hypothesis will yield the following conjecture.

Conjecture: Under the Riemann Hypothesis, there exists an absolute constant $C > 0$ so that every interval $[n, n + C\sqrt{n}], n \geq 1$ contains a prime.

Zero Density Estimates

Consider the Riemann zeta function for $\Re s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ s = \sigma + \imath t, \ \imath = \sqrt{-1}.$$

This function is known to have an analytic continuation to the entire complex plane. The zeros of this function play an important role in what follows. There are the trivial zeros at $s = -2k$ and the non-trivial ones that are all supposed to lie on $\Re s = 1/2$. We first have the theorem of Riemann. The non-trivial zeros(those with non-zero imaginary part) are known to lie in the strip $0 < \Re s < 1$.

Theorem 2(Riemann): Let

$$N(T) = \#\{s_j | \ \zeta(s_j) = 0, \ 0 < \Re s_j < 1, \ |\Im s_j| < T\}$$

denote the counting function of the zeros of the Riemann zeta function in a box of height T above the real axis and height T below the real axis. Then we have the asymptotics

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T),\ T \geq 2.$$

The region which is described in the Theorem above is sketched in Fig 1. A proof of this result is to be found in [3], [4].

We now state a result that links the Riemann zeta function with primes that is key to the whole matter. It is known by the name of the **explicit formula**. It is proved by using the well known Perron formula that is a contour integration to isolate partial sums of Dirichlet series. For a proof see [3], [4].

Theorem 3(Explicit Formula): Let us recall the Chebychev function $\psi(x)$. Set

$$\psi(x) = \sum_{2\leq n\leq x}\Lambda(n).$$

Here $\Lambda(n)$ is the von Mangoldt function which is defined for p any prime by,

$$\Lambda(n) = \begin{cases} \log p,\ n = p^m \\ 0, n \neq p^m \end{cases}.$$

Then for $T \geq 3$,

$$\psi(x) = x - \sum_j \frac{x^{s_j}}{s_j} + O\big(\frac{x\log^2 x}{T}\big),$$

where s_j are the (non-trivial) zeros of the Riemann zeta function in the box $0 < \Re s_j < 1,\ |\Im s_j| < T$.

Since we have no proof of the Riemann hypothesis that all the non-trivial zeros lie on $\Re s = \frac{1}{2}$, we would like to get by with some weaker information. This is what a zero density density is all about. The zero here being the zeros of the Riemann zeta function. We now consider a zero counting function for the box given by Fig. 2 that is the box,

$$R_\sigma = \{s|\ \sigma < \Re s < 1,\ |\Im s| < T\}.$$

We set

$$N(\sigma, T) = \#\{s_j|\ s_j \in R_\sigma, \zeta(s_j) = 0\}.$$

We shall reason under the assumption

Zero Density Estimate Assumption: There are absolute constants c, C, B such that

$$N(\sigma, T) \leq CT^{2(1+2c)(1-\sigma)}\log^B T.$$

In the sequel we shall set $2(1 + 2c) = b$.

We now discuss the zero density estimates. The growth of the Riemann zeta function on the critical line is key to establishing a zero density estimate. In fact one has

Theorem 4 (Ingham)[2]: Assume that the Riemann Zeta function satisfies

$$|\zeta(\frac{1}{2}+\imath t)| \leq C(1+|t|)^c,$$

then,

$$N(\sigma,T) \leq CT^{2(1+2c)(1-\sigma)} \log^B T.$$

Using the best known growth rate of the Riemann zeta function due to Hardy-Littlewood at that time, that is $c = 1/6$, Ingham arrived at the estimate

$$N(\sigma,T) \leq CT^{\frac{8}{3}(1-\sigma)} \log^B T. \tag{6}$$

This was a remarkable result. Zero density estimates go back to Carlson [3], [4] and relied on two ideas. A suitable mollifier to make the Riemann zeta function small and Complex analysis involving a contour integral. Typically the mollifier is the reciprocal of the Riemann zeta function given by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Möbius function. One usually truncates the sum above and considers the partial sum and then product

$$\Big(\sum_{n=1}^{N} \frac{\mu(n)}{n^s}\Big)\zeta(s) = 1 + E(s).$$

$E(s)$ on the right side consists of Dirichlet polynomials and is an error. Now at a zero s_j of the Riemann zeta function, the left side vanishes. If the right side has to vanish, then obviously $E(s)$ has to be large. One considers the set s where $|E(s)| > 1/2$ and uses $E(s)$ as a zero detecting device. Now instead of using Complex analysis, Real variables are preferred in the modern treatment based in part on ideas of Gallagher in his treatment of the Large sieve. One applies Sobolev type inequalities analogous to those used by Gallagher to Large Sieve inequalities, but here adapted to the Dirichlet series for $E(s)$. One obtains L^2 estimates that are used to control $E(s)$ and thus obtain information on the set s where it is big. A treatment of the zero density estimates by these methods by Huxley and others is to be found in the appropriate chapter in [4].

The **Lindelöf conjecture** which is tied to the Riemann Hypothesis is a conjecture on the growth rate of the Riemann zeta function on the critical line. The conjecture states for any $\epsilon > 0$ one has,

$$|\zeta(\frac{1}{2}+\imath t)| \leq C(\epsilon)(1+|t|)^{\epsilon}.$$

If we thus choose $\epsilon = 0$ yielding $c = 0$ we arrive at the zero density estimate

$$N(\sigma, T) \le CT^{2(1-\sigma)} \log^B T.$$

For $\sigma = 1/2$, this agrees with Riemann's theorem, Theorem 2 upto logarithmic factors.

We lastly need a result by Chudakov (see [3], [4] for proofs)on the zero free regions of the Riemann zeta function. See Fig. 3

Theorem 5(Chudakov): For every $T \ge 1$ and $s = \sigma + \imath T$, $\zeta(s) \neq 0$ if

$$\sigma > 1 - A\frac{\log\log T}{\log T}, \tag{7}$$

where A is an absolute constant.

We are now in a position to state and prove Hoheisel's theorem and using Ingham's result stated above we will be able to improve the Chebychev theorem. Advances in zero density estimates when used with Hoheisel's theorem give results better than what Ingham's result yields. These results are due to Huxley, Baker and Pintz and others.

Theorem 6(Hoheisel): Assume the zero density estimate,

$$N(\sigma, T) \le CT^{b(1-\sigma)} \log^B T,\ b > 1$$

Then for θ given by

$$\theta > 1 - \frac{1}{b + \frac{B}{A}},$$

where A is the constant in Chudakov's theorem (7), there exists an absolute constant M, such that for any $x \ge M$, there is a prime in the interval $[x, x + x^\theta]$.

We immediately obtain a corollary combining Ingham's result (6) and Hoheisel's result. Taking $b = 8/3$, we obtain for $\theta > 5/8$

Corollary 1: For any $\theta > 5/8$, there exists M such that for any $x \ge M$, the interval $[x, x + x^\theta]$ contains a prime. Thus Chebychev's result Theorem 1 has been improved significantly. In Chebychev's result $\theta = 1$.

Proof: Our goal is to show that, for $h = x^\theta$, θ as given in the theorem, we have for an absolute constant C, $x \ge M$,

$$|\frac{\psi(x+h) - \psi(x)}{h}| \ge C > 0.$$

This inequality proves the theorem. By Theorem 3, the explicit formula,

$$\frac{\psi(x+h) - \psi(x)}{h} = 1 + \sum_j \frac{(x+h)^{s_j} - x^{s_j}}{hs_j} + O(\frac{x\log^2 x}{hT}).$$

Our goal is to prove, as h goes to infinity we have

$$A(x) = \sum_j \frac{(x+h)^{s_j} - x^{s_j}}{hs_j} + O(\frac{x \log^2 x}{hT}) = o(1).$$

By the mean value inequality, (since $0 < \Re s_j = \sigma_j < 1$ and $h \leq x$)

$$|\frac{(x+h)^{s_j} - x^{s_j}}{hs_j}| \leq x^{\sigma_j - 1}.$$

Thus

$$A(x) \leq \sum_j x^{\sigma_j - 1} + O(\frac{x \log^2 x}{hT}).$$

We re-write the right side using Theorem 2 to get

$$A(x) \leq \frac{T \log T}{x} + \sum_j \left(x^{\sigma_j - 1} - x^{-1}\right) + O(\frac{x \log^2 x}{hT}). \tag{8}$$

Since

$$\frac{d}{d\sigma}(x^\sigma) = \log x \; x^\sigma,$$

we can write

$$x^{\sigma_j - 1} - x^{-1} = \int_0^{\sigma_j} \log x \; x^{\sigma - 1} d\sigma.$$

Using the last identity in (8) we get,

$$A(x) \leq \frac{T \log T}{x} + \sum_j \int_0^{\sigma_j} \log x \; x^{\sigma - 1} d\sigma + O(\frac{x \log^2 x}{hT}).$$

Now we rewrite the integral using Theorem 5 of Chudakov,

$$\sum_j \int_0^{\sigma_j} \log x \; x^{\sigma-1} d\sigma = \sum_j \int_0^{1 - A\frac{\log\log T}{\log T}} \chi_{\{\sigma \leq \sigma_j\}} \log x \; x^{\sigma - 1} d\sigma.$$

Exchanging the summation and integration, we get

$$\int_0^{1 - A\frac{\log\log T}{\log T}} \sum_j \chi_{\{\sigma \leq \sigma_j\}} \log x \; x^{\sigma - 1} d\sigma.$$

But the inner sum is exactly $N(\sigma, T)$ and so we have

$$A(x) \leq \frac{T \log T}{x} + \int_0^{1 - A\frac{\log\log T}{\log T}} N(\sigma, T) \log x \; x^{\sigma - 1} d\sigma + O(\frac{x \log^2 x}{hT}).$$

Using the zero density hypothesis for $N(\sigma, T)$ the right side above is bounded by

$$\frac{T\log T}{x} + C\log^B T \int_0^{1-A\frac{\log\log T}{\log T}} \left(\frac{T^b}{x}\right)^{1-\sigma} \log x \; d\sigma + O(\frac{x\log^2 x}{hT}).$$

Performing the integration we get,

$$\frac{T\log T}{x} + \log^B T\left(\left(\frac{T^b}{x}\right)^{A\frac{\log\log T}{\log T}} + \frac{T^b}{x}\right) + O(\frac{x\log^2 x}{hT}). \tag{9}$$

Choose $T = x^\alpha, h = x^\theta$, with

$$\alpha + \theta > 1,\; \alpha < 1,\; b\alpha < 1. \tag{10}$$

Thus

$$O(\frac{x\log^2 x}{hT}) = O(x^{1-\alpha-\theta}\log^2 x) = o(1), x \to \infty,$$

and

$$\frac{T\log T}{x} = O(x^{\alpha-1}\log x) = o(1), x \to \infty.$$

Next, by (10) again

$$\log^B T\frac{T^b}{x} = O(x^{b\alpha-1}\log^B x) = o(1), x \to \infty.$$

Thus

$$A(x) \le C\log^B T\left(\frac{T^b}{x}\right)^{A\frac{\log\log T}{\log T}} + o(1), x \to \infty.$$

Thus

$$A(x) \le C\log^B x \exp\big(\frac{A}{\alpha}(\alpha b - 1)\log\log x\big) + o(1).$$

Simplifying further,

$$A(x) \le C(\log x)^{B+\frac{A}{\alpha}(\alpha b-1)} + o(1).$$

Thus if

$$B + \frac{A}{\alpha}(\alpha b - 1) < 0,$$

we will achieve $A(x) = o(1)$. Simplifying we obtain,

$$\alpha < \frac{1}{b + \frac{B}{A}}.$$

Since $b > 1$ by hypothesis, it follows that $\alpha < 1$ and part of the conditions in (10) are met. Clearly

$$b\alpha < \frac{b}{b + \frac{B}{A}} < 1,$$

and so another condition in (10) is fulfilled. Lastly we take

$$\theta > 1 - \alpha > 1 - \frac{1}{b + \frac{B}{A}},$$

This fulfills the remaining condition in (10) and proves Hoheisel's theorem.

Remark 1: If the Riemann Hypothesis were true and the attendant Lindelöf conjecture were true, then we may take $b = 2$ in the zero density estimate. Using this value of b in Hoheisel's theorem, we readily see we can then take $\theta > \frac{1}{2}$. This should be compared with the conjecture stated earlier.

Remark 2: A result similar to the explicit formula, Theorem 3 was proved by H. Iwaniec(Crelle J. 1984) for prime geodesics on the modular surface. His result was improved by Bykovskii[1], using L-functions that appear in the work of D. Zagier.

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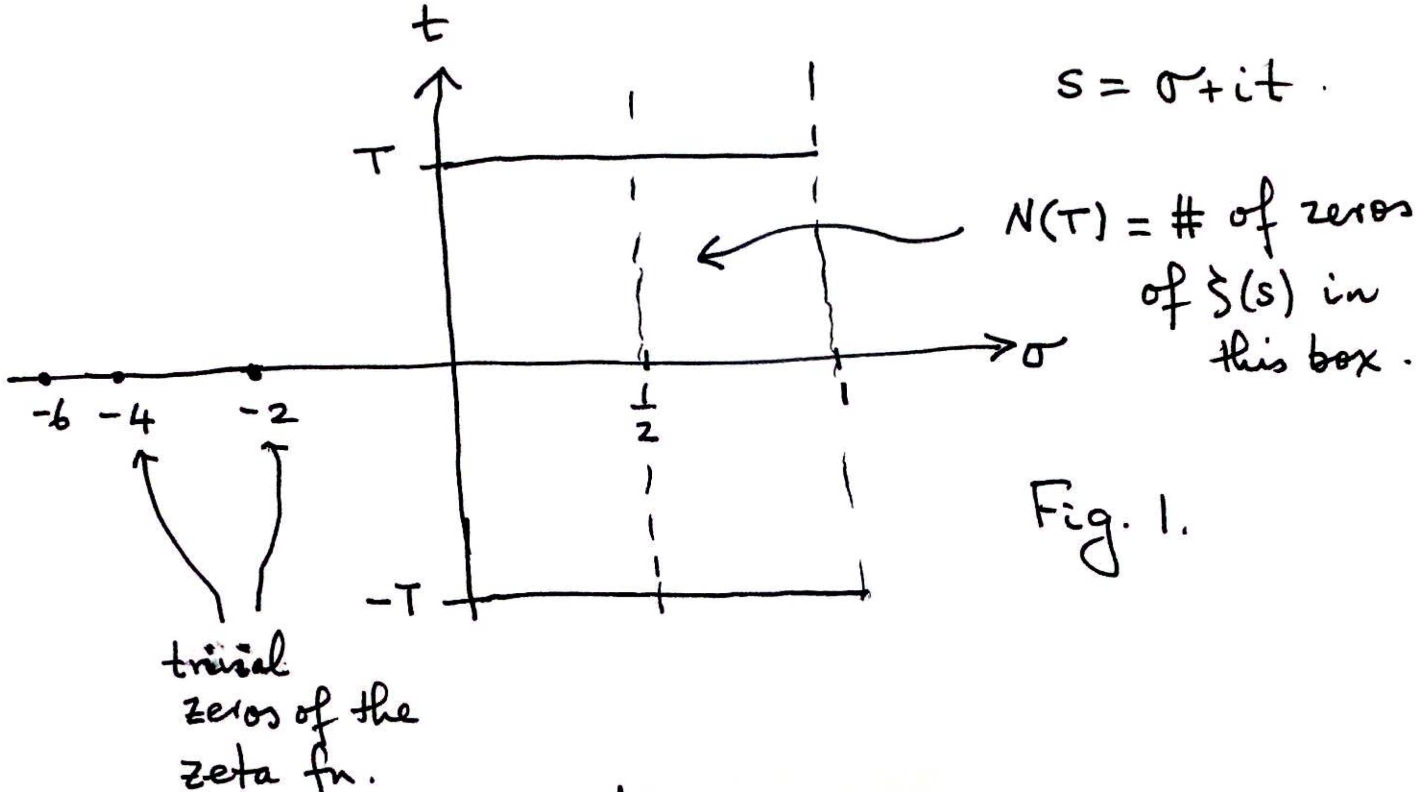

Fig. 1.

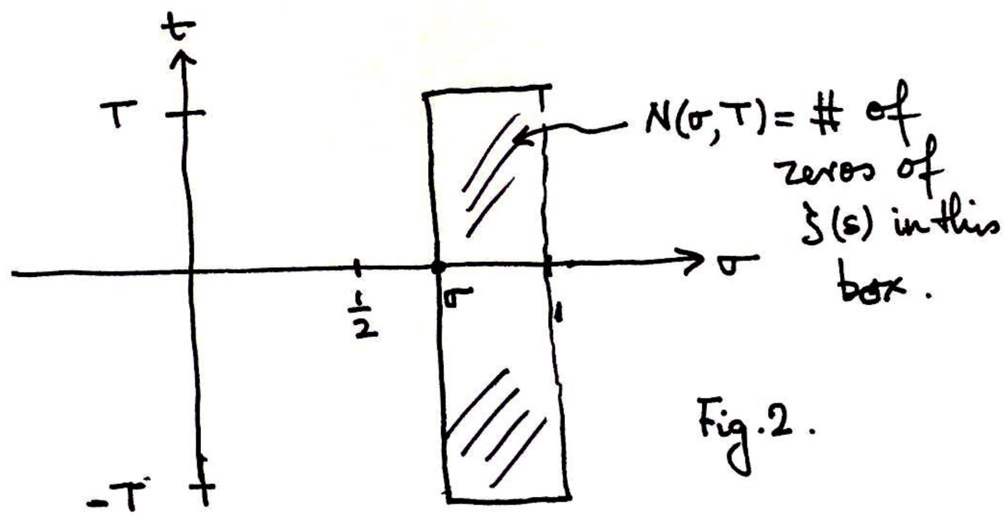

Fig. 2.

If Riemann hypothesis is true $N(\sigma, T) = 0$, for $\sigma > 1/2$.

<u>Chudakov's theorem.</u>
There are NO zeros of the Riemann zeta fn. in the shaded region.

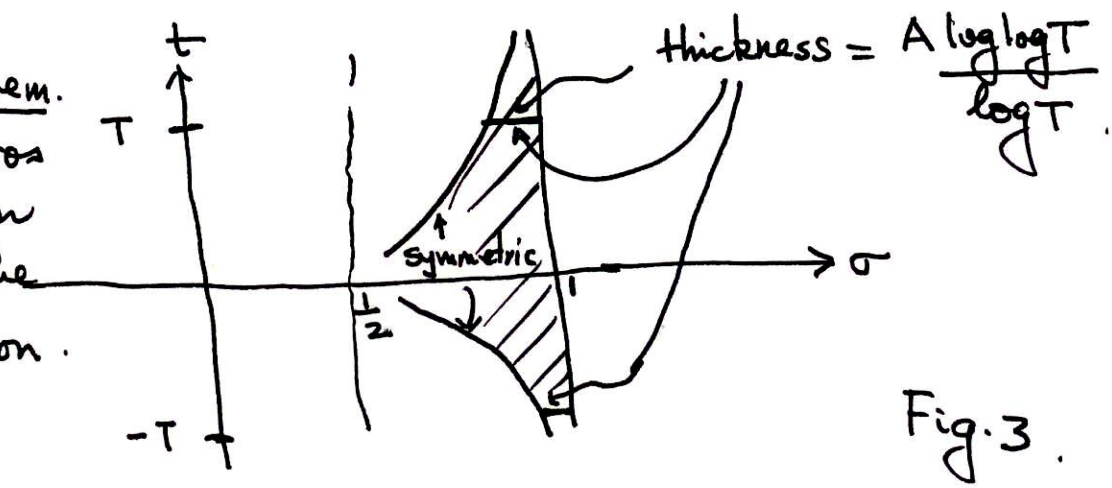

Fig. 3.