The Pancharatnam Phase and CR Geometry

It is a tragedy of Indian Science and Mathematics that there have been many instances in recent History where many outstanding scientists and Mathematicians in India have died young. One easily recalls to mind, the Number theorist S. Ramanujan, the Algebraic Geometer, C. P. Ramanujam, the Differential Geometer, V. K. Patodi and to this list one should add, S. Pancharatnam who died in tragic circumstances in 1969 at the age of 35. Working under the supervision of his uncle, the Nobel Laureate, C. V. Raman, Pancharatnam discovered a phase change at the age of 22, in 1956. Later this fact was re-discovered in a very general quantum mechanical setting by M. V. Berry, the famous Berry phase. The Pancharatnam phase may be viewed as a fore-runner and a discrete analog of the Berry phase. Various links between the two may be found in the tribute article by Berry [1] and also [2].

Our point in this note is to realize the specific example of the Pancharatnam phase as a CR (Cauchy-Riemann) Geometry phenomenon. In particular we wish to establish links with the Webster connection and curvature on S^3, [7]. As a first step we will be considering a $U(1)$ bundle over S^2. The fact that the Berry phase has to do with fiber bundles was already observed by Barry Simon in a paper [3] that appeared before Berry's work by a year. Thus we do not claim any originality in this aspect. But we do point out that the connection used in the Pancharatnam phase on the fibers is indeed Webster's connection as applied to S^3. This link with CR has not been pointed out in the literature. Thus CR Geometry emerges as the natural setup for the study of this Optical phenomenon. We do not know if the general Berry phase can be written in this CR language and how useful this interpretation of this phenomenon is in terms of CR geometry.

To understand the Pancharatnam phase and his formula for the phase shift we need to make a short digression into the Physics of polarized light and the Poincare representation of polarized light by points on the sphere S^2. We do not present Pancharatnam's original derivation of the phase formula except make some cursory remarks. The interested reader can consult Pancharatnam's original article [4] or the exposition by Nityananda [5] in the Pancharatnam memorial issue.

Thus imagine a beam of light traveling along the $z-axis$ in some fixed laboratory frame. One knows that the propagation is given by Maxwell's equation and by the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$. The Poynting vector

$$\mathbf{E} \times \mathbf{B}$$

points along the direction of propagation that is along the z-axis. We only focus on the electric field vector which lies in the instantaneous $x-y$ plane. See Fig 1. The magnetic field vector follows and so we ignore it. As Feynman puts it in his lectures, the "two are in a dance". We resolve the vector $\mathbf{E}$ into its components and for a given frequency ω (we may view the light being monochromatic) we have

$$\mathbf{E} = (A\cos(kz-\omega t),\ B\cos(kz-\omega t+\delta),\ 0).$$

If one eliminates $kz - \omega t$ and sets x to be the first component of $\mathbf{E}$ and y the second component of $\mathbf{E}$ we get the equation of an ellipse, see Fig. 2, Fig. 3:

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 - 2\cos\delta\frac{xy}{AB} = \sin^2\delta.$$

The ellipse makes an angle θ with the x-axis given by

$$\tan 2\theta = \frac{2AB\cos\delta}{A^2 - B^2}. \quad (1)$$

When $\delta = 0$ we say the light is linearly polarized. When $\delta = \pi/2$ and $A = B$ we say the light is circularly polarized. When $A \neq B$ or $A = B$ and $\delta \neq 0, \pm\pi/2$ we say the light is elliptically polarized. Essentially the tip of the $\mathbf{E}$ vector rotates around an ellipse, circle or remains fixed to an observer watching the approaching beam (Lissajous figure). If the tip rotates counterclockwise we say the light is left circularly polarized, while if the tip rotates clockwise we say the light is right circularly polarized. Next we also need an expression related to the eccentricity of the ellipse. Define,

$$\cos\phi = 2\frac{|A||B|}{A^2 + B^2}\sin\delta. \quad (2)$$

Now we represent the polarized light by a point (θ, ϕ) on S^2. Note that for $\delta > 0$ we have left circularly polarized light. This corresponds to points in the Northern hemisphere with the North Pole being circular polarization i.e $A = B$ with $\delta = \pi/2$. The Southern hemisphere corresponds to right circularly polarized light with $\delta < 0$. The Equator corresponds to linearly polarized light with $\delta = 0$. This representation of polarized light is the Poincare representation. Fig. 4. The sphere S^2 used for this representation is called the Poincare sphere.

The Pancharatnam Phase: Now imagine an experiment where we start with a given polarization state P_1 on the Poincare sphere S^2. We next change the polarization state to P_2. This change of state may be represented by moving to P_2 along a great circle. One may view this as an evolution via a Schrodinger equation of some Hamiltonian and because of the least action principle, the state evolves via a geodesic. Next we go to a state P_3 via a geodesic and then return to P_1, see Fig. 5. Pancharatnam observed that when the beam has been brought back to the point P_1, it no longer has the same phase as the original reference beam but is phase shifted by an amount Θ. In fact he observed and proved that the phase shift is given by

$$\Theta = \frac{1}{2}\text{Area of geodesic triangle } P_1P_2P_3. \quad (3)$$

It is a similar situation to the Foucault pendulum and parallel transport, but here there is a phase shift where some other parallel transport is at work. Experimentally one would split the initial beam by a beam splitter with a reference beam. Pass the rest of the beam through polarizers that alter states and then observe an interference pattern when comparing the reference beam and the beam that emerges after passing through various polarizers. Thus it is not only the state that is important but also the phase associated to the state. That is when one evolves via the Schrodinger eqn. and some Hamiltonian we have

$$i\frac{d}{dt}|f>=H|f>$$

Then one studies the evolving state $|f(t)>$ by an ansatz.

$$|f(t)>=e^{i\phi(t)}|f(0)>$$

In general the phase $\phi(t)$ is not detectable. It is a connection. In fact we will identify it as **Webster's** connection on S^3. However the curvature is detectable, it is the field strength. The curvature is Webster's curvature on S^3 which is a constant. Once one makes this identifications, Pancharatnam's formula appears as a holonomy and the Gauss-Bonnet formula. In addition we get some additional insight from the CR viewpoint that we will elaborate next.

CR Geometry and Webster-Tanaka connection: We restrict ourselves to a smooth compact, orientable 3-manifold M^3 with no boundary. We assume that the manifold M is equipped with a contact form θ and an assignment of contact planes given by

$$H=\ker\theta.$$

In addition we are given an endomorphism J on H at each point, the CR structure,

$$J:H\to H,\ J^2=-I.$$

In fact we consider a line sub-bundle $\mathcal{V}$ of the complexified tangent bundle CTM where $Jv=iv$ for $v\in\mathcal{V}$. The elements v are the holomorphic CR vector fields. $\mathcal{V}$ is the CR bundle. We have corresponding co-vectors and a framing of the manifold $(\theta,\theta^1,\theta^{\bar{1}})$. We choose our framing such that,

$$d\theta=i\theta^1\wedge\theta^{\bar{1}}.$$

That is we are assuming a non-degenerate Levi form which has been normalized to be 1. We denote the holomorphic vector field dual to θ^1 by Z_1. We also have the Reeb vector field T dual to θ, with

$$\theta(T)=1.$$

The volume element on the CR manifold is given by

$$dV=\theta\wedge d\theta.$$

We now recall the Webster-Tanaka connection. It is given by

$$d\theta^1=\theta^1\wedge\theta^1_1+\theta\wedge\tau^1.$$

θ_1^1 is the connection form and τ^1 is the Torsion form. The additional properties are,

$$\theta_1^1 + \theta_{\bar{1}}^{\bar{1}} = 0,\ \tau^1 \equiv 0 \bmod \theta^{\bar{1}}.$$

Thus set,

$$\tau^1 = A_{\bar{1}}^1 \theta^{\bar{1}}.$$

Webster-Tanaka curvature R is given by

$$d\theta_1^1 = R\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A_{\bar{1},\bar{1}}^{\bar{1}} \theta^1 \wedge \theta).$$

We have taken the covariant derivative of $A_{\bar{1}}^1$ by the anti-holomorphic vector field $\bar{Z}_1$ to arrive at $A_{\bar{1},\bar{1}}^{\bar{1}}$ in the formula above.

For our purposes we focus on S^3 which is a rigid CR structure and thus torsion τ^1 vanishes by a result of Webster [7], Prop. 2.2. Recalling Proposition 2.5 in [6] we list some facts.

Write S^3 as $|z_1|^2 + |z_2|^2 = 1$, with $z_i \in \mathbf{C}$. Then using the defining function $u = |z_1^2| + |z_2|^2 - 1$, and setting $u = 0$ we have

$$\theta = \frac{i}{2}(\bar{\partial} u - \partial u)|_{S^3}$$

$$\theta^1 = z_2 dz_1 - z_1 dz_2.$$

$$Z_1 = \bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2}.$$

$$\theta_1^1 = -2i\theta \tag{4}$$

Lastly we have

$$d\theta_1^1 = R\theta^1 \wedge \theta^{\bar{1}} = 2d(-i\theta) = 2\theta^1 \wedge \theta^{\bar{1}} \tag{5}$$

Thus $R \equiv 2$.

We are now in a position to describe the Pancharatnam phase. We may view a closed loop on S^2 which is a cycle representing the change of states. Associated with each state is a phase. Thus we may think of a circle bundle over S^2 where the phase represents a point in the fiber over each point in the base. This is the famous Hopf Fibration of S^3. The total space of the circle bundle over S^2 is then identified with S^3. For completeness we recall the Hopf fibration. For (z_1, z_2) with $z_i \in \mathbf{C}$ we consider points on S^3 given by $|z_1|^2 + |z_2|^2 = 1$ and the projection map,

$$(z_1, z_2) \to (2z_1\bar{z}_2,\ |z_1|^2 - |z_2|^2).$$

As we transition over states on S^2 in the base, Fig. 6. we describe a curve in the total space which is a horizontal curve. Let us recall what this means. On the total space S^3 we pick the Webster connection θ_1^1. The Horizontal vectors are given by

$$W = \{v | \theta_1^1(v) = 0\}.$$

The lifted curve in Fig 4. is such that it projects to the base curve and at each point its tangent vector lies in W. But we observed (4)

$$\theta_1^1 = -2i\theta$$

Therefore the horizontal tangent plane consists of vectors in $\ker\theta$. That is the horizontal vectors are spanned exactly by the real and imaginary parts of the CR Vector field Z_1 on S^3 which we have written down above. More precisely such curves on S^3 whose tangent directions lie along the $\ker\theta$ are called Legendrian curves. We remind the reader that $\ker\theta$ consists exactly of vectors that lie in the plane spanned by the real and imaginary parts of the CR vector field Z_1 on S^3. The final point B on the horizontal lift may not coincide with the starting point A (see Fig. 6), though of course they will lie on the same fiber. See Fig. 6. To move from the initial point to the final point on the fiber we need to apply an S^1 action, $e^{i\Theta}$ and this Θ is exactly the holonomy or Pancharatnam shift. Note that the assignment of contact planes is exactly the tangent planes of S^2 and the CR structure on S^3 is related to the complex structure of S^2.

To compute this holonomy one may apply Gauss-Bonnet and relate the curvature of the sphere S^2 to Webster's curvature $d\theta_1^1$ which is given by:

$$d\theta_1^1 = -2id\theta = 2\theta^1 \wedge \theta^{\bar{1}}$$

Making these identifications one gets Pancharatnam's phase shift formula.

Another way to understand the holonomy is to recall that the lifted curve Γ is Legendrian (see Fig. 6). Thus the tangent vectors lie in the contact plane that is in $\ker\theta$. As we move along Γ the contact plane rotates since contact structures are not integrable since by definition of a contact form

$$\theta \wedge d\theta \neq 0.$$

Thus we have the a difference between the angles between the contact planes at A and B. This difference in angles is called the Maslov factor. One can prove that the Maslov factor is the same as the Pancharatnam phase shift Θ. More precisely we may denote the points on S^3 by (z_1, z_2) with $z_i \in \mathbf{C}$ and $|z_1|^2 + |z_2|^2 = 1$. Now assume that the point A in Fig. 6 is given by (z_1, z_2) and B by (w_1, w_2) with $w_i = e^{i\Theta} z_i,\ i = 1,\ 2$. An easy computation shows

$$\bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2} = e^{2i\Theta}(\bar{w}_2 \frac{\partial}{\partial w_1} - \bar{w}_1 \frac{\partial}{\partial w_2}).$$

This last identity shows the relation between the CR vector field Z_1 at the points A and B. In particular, one sees that there has been a rotation by 2Θ between the CR planes at A and B which is the Maslov factor.

It is an astonishing fact that Pancharatnam's pioneering work is not known to many Physicists, mathematical or otherwise.

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Sagun Chanillo,

Deptt. Of Math. Rutgers Univ.,

Piscataway, NJ 08854, USA

chanillo@math.rutgers.edu

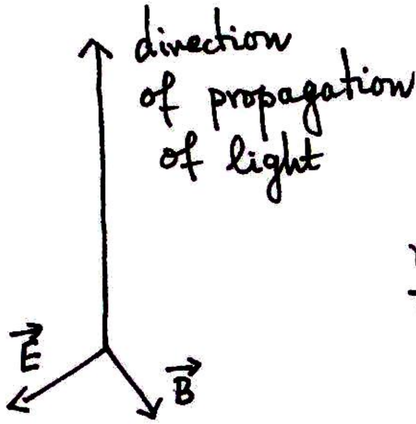


Fig. 1.

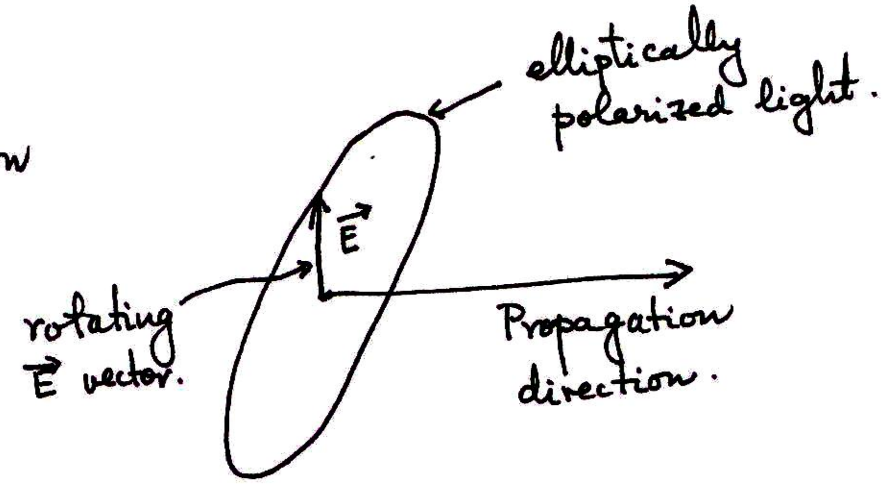


Fig. 2 .

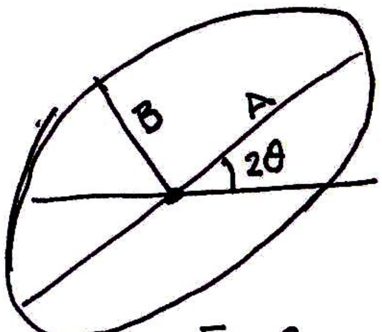
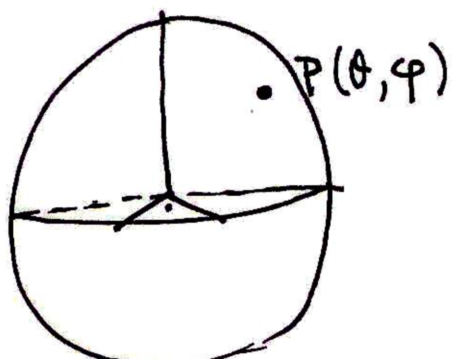


Fig. 3



Poincare sphere

Fig. 4.

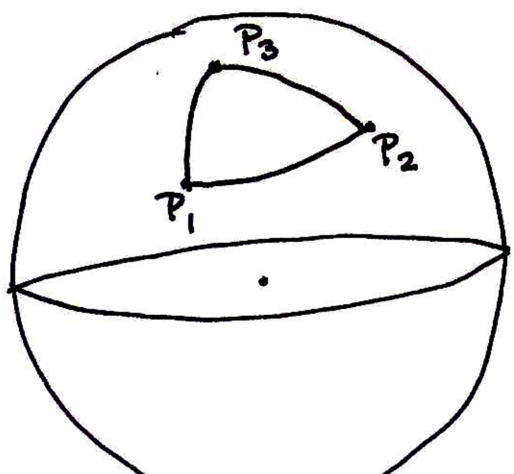


Fig. 5.

$P_1 P_2 P_3$ the geodesic triangle.
Returning to P_1 does NOT bring us back to the original phase. Light gets phase <u>shifted</u> .

S^3 (With Webster connection).

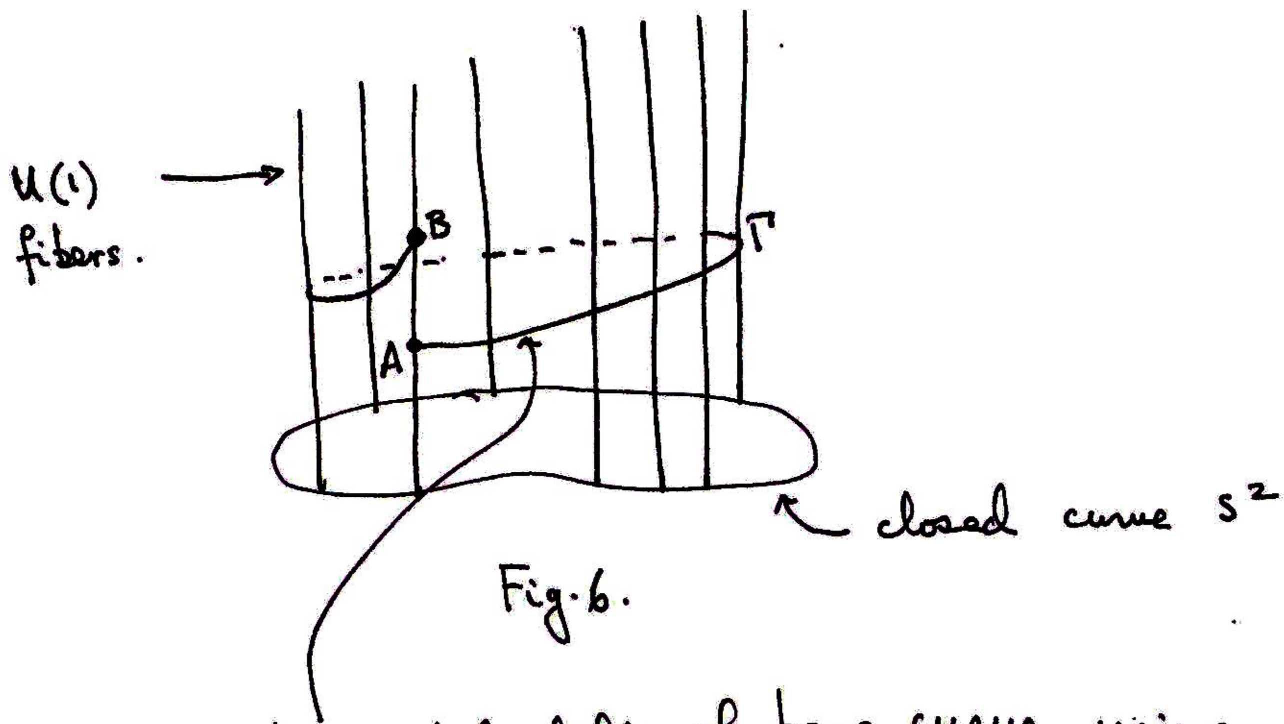


Fig. 6.

horizontal lift of base curve, using Webster's connection $\theta_1' = -i\theta$ on S^3 (eqn. (4)).

The lifted curve Γ is perforce Legendrian, tangent vectors lie along CR direction.

The lifted curve is NOT closed. $\exists$ Ⓗ, $\ni$

$$e^{i \text{Ⓗ}} A = B.$$

Ⓗ is the shift of Pancharatnam, the holonomy.