Fibonacci Numbers in the Music and Metre of Indian Music and Poetry

Both Music and Metre play an important role since Vedic times(5000 BCE) in India. The *Sama Veda* is an early account of this tradition. Reciting Sanskrit *shlokas* according to a metre with various beats not only is pleasing to the ear, but also aids in memorizing verses in a tradition where the propagation was largely done by oral means. It is thus natural that various mathematical questions will arise in such a tradition that leads to the natural investigation of the partition of numbers. This investigation leads to the modern culmination in the celebrated theorem of Hardy-Ramanujan on partitions and the extraordinary rich mathematics that flows from this theorem. It is not our purpose here to give even a rudimentary account of this theory and the powerful method known as the Hardy-Ramanujan circle method that was born from these investigations, but rather to focus on another partition question directly related to Music and Metre that was understood and answered by Hindu mathematicians long ago.

Those of us who have experienced an Indian classical music concert would have wondered about the complex notes that a percussionist produces whether on a *tabla* an instrument favored in the North Indian school or the *mridangam* and *ghatam* used primarily in the Carnatic tradition of South India. The complexity stems not only from the beats as we will explain, but also from the construction of the drum and percussion instrument which is composed of different types of leather, giving rise to a composite instrument, rich in producing harmonics. The treatment of the harmonics is a non-trivial problem in Mathematical Physics involving Bessel functions which are natural for a circular membrane, but now with the added complexity of a composite material which may or may not be centered. The Indian Nobel laureate (1930) C. V. Raman wrote many papers investigating both stringed and percussion instruments employed in Indian music. It is said that the Nobel laureate Rutherford, the British Physicist once quipped *Does Raman think he will fiddle his way into the Royal Society ?* This Raman later not only did, but also acquired a Nobel prize too for his discovery of the famous quantum-optical effect called the Raman Effect that is a primary tool nowadays in the study of Molecular structure.

Coming back to our problem, many a time the percussionist in a concert will call out the short and long notes/beats and pauses known as *bols*. Calling out the beats he will use is sometimes also used to challenge another percussionist on the stage to replicate it. Typical calls are *Dhin* and *Dha*. The *Dhin* is a short note corresponding to 1 beat, while the *Dha* a long note corresponding to 2 beats. All the beats are combined into a rhythmical cycle of say 16 beats the *teen tal* or the *rupak taal* a rhythmical cycle of 7 beats, *pancham taal* and other more exotic rhythmic cycles. It may not be misplaced to remark that *teen taal* though a rhythmic cycle of 16 beats gets its name from the fact that the audience sometimes is encouraged to keep time at the first, 5th, 9th and 13th beat. Hence the word *teen* or three in Hindi.

The question we have thus is how many possible ways can you form a cycle of say 5 beats using short and long notes. The mathematical reader is to be cautioned that unlike Hardy-Ramanujan's partition problem, permutations are allowed. Each permutation has its own cadence and can be heard distinctly in Music and Poetry.

The complete list of distinct cycles with 5 beats is as follows:

$$2+2+1,\ 1+1+1+1+1,\ 1+2+2,\ 2+1+2,\ 1+2+1+1,$$

$$1+1+1+2,\ 2+1+1+1,\ 1+1+2+1.$$

Thus we see there are exactly 8 ways/cycles to partition 5 beats in a sequence of long and short beats. A percussionist may well say in a concert:

Dhin, Dha, Dha, Dhin

corresponding to a cycle $1+2+2+1$ that he has or is going to play on his percussion instrument. Thus our question is

Question: Given a cycle of n beats. How many ways $p(n)$ is it possible to play this cycle by using short and long beats?

The answer to this question was known to Hindu mathematicians long ago and appears certainly in the works of Hemchandra (1089-1173) and even earlier Pingala(200 BCE). The solution to this problem is encoded in the following theorem whose proof we supply.

Theorem 1: The number $p(n)$ of different beat patterns of short and long beats that arise in a rhythmic cycle of n beats is exactly F_n the n-th Fibonacci number.

The Fibonacci numbers are the number sequence

$$1, 2, 3, 5, 8, 13, 21, \cdots.$$

In particular the numbers are generated by the recurrence relation for the n-th Fibonacci number given by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 3. \tag{1}$$

(1) simply means that the n-th number is generated by adding the previous 2 numbers. Thus in our list $F_5 = 8$ which is exactly the same as the number of ways of using short and long beats in a rhythmical cycle of 5 beats, something we discovered above in this special case $n = 5$. The theorem states that this is true for any n in complete generality.

The Fibonacci numbers were well understood and known in India certainly 200 years before Fibonacci and the reason for this familiarity is linked to the question of beats as explained above. Fibonacci is also credited with the introduction of Hindu numerals of place value and decimals to the Western world. It is also quite fascinating to observe that the so-called Fibonacci numbers appear in Nature.

We now end with a rigorous proof of our theorem.

Proof: Our proof is based on induction on n as is to be anticipated. We wish to show,

$$p(n) = F_n.$$

We check this for $n = 1$ and it is obvious $F_1 = 1 = p(1)$. In fact since for two beats $n = 2$ we also have $1+1, 2$ we also see $p(2) = F_2 = 2$. Next we will establish that $p(n)$ satisfies the same recurrence relation as F_n given by (1), that is we show

$$p(n) = p(n-1) + p(n-2), n \geq 3. \tag{2}$$

This then finishes our proof once we check (2). Now given **any** rhythmic cycle of $n-1$ beats we can add 1 short beat at the start and then play the $n-1$ beats to give rise to a cycle of n beats. Since there are $p(n-1)$ cycles with $n-1$ beats, we would have created $p(n-1)$ cycles of n beats. Let us call the cycles we have created as Type I.

There is another way to create cycles with n beats. We could take a cycle of $n-2$ beats and add at the start a long beat (corresponding to 2 short beats). Since there are $p(n-2)$ cycles with $n-2$ beats, we have by this process created $p(n-2)$ cycles with n beats. Let us label cycles created in this manner Type II.

We next observe that the cycles in Type I and Type II are distinct. That is no cycle is found in both lists. To check this, note Type I cycles begin with a short beat and cycles in Type II begin with a long beat so at the first place the cycles in List I and List II differ. So the cycles in both lists are distinct. Next we need to show that in each category Type I and Type II there is no repetition. To see this note each list either I or II was generated from a previous list that had no repetitions to which we added a short or long beat at the start. Thus by adding an additional short beat or long beat we still arrive at a set of cycles no two of which agree at all places. Lastly we make an observation that we have listed all possible cycles of n beats in our list which is the list consisting of both Type I and Type II. To check this pick any cycle with n beats. It either starts with a short or long beat. Assume it starts with a short beat. The rest of the cycle must then be composed of $n-1$ beats and must therefore be in the list of cycles that make up $n-1$ beats. Thus the cycle we selected of n beats must be one of them in the list Type I. A similar argument obviously works if we had selected a cycle of n beats which started with a long beat. We would have found it was already counted in the Type II list. Thus in more precise mathematical language: the list Type I and Type II are disjoint and their union is exactly **all** the possible cycles of n beats. Adding the cardinalities together due to disjointness, we see easily,

$$p(n) = p(n-1) + p(n-2).$$

But this is exactly what we wanted to prove (2) to finish the proof of our theorem.
QED

Remark: The method of proof displayed above also yields an amusing observation. In the number of cycles with n beats of which there are $p(n) = F_n$ in number, $p(n-1) = F_{n-1}$ of them will start with a short beat and $p(n-2) = F_{n-2}$ will start with a long beat. Thus for example if we take the number of cycles with 6 beats, we know there are 13 of them. Out of which 8 begin with a short beat and 5 with a long beat.

It is well-known that the Fibonacci numbers are given by an explicit, exact formula,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],\ n \geq 1. \tag{3}$$

Combining the identity (3) with Theorem 1, we get an exact formula for $p(n)$. We record this in the next theorem.

Theorem 2: The number of partitions $p(n)$ is given by

$$p(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],\ n \geq 1.$$

Sagun Chanillo
Deptt. of Math., Rutgers Univ.
110 Frelinghuysen Rd.,
Piscataway, NJ 08854, USA
chanillo@math.rutgers.edu