Kloosterman Sums

A standard Kloosterman sum is defined as

$$S(m,n,c) = \sum_{(c,h)=1} e^{2\pi i\left(\frac{mh+n\bar{h}}{c}\right)},$$

where $(c,h)=1$ means one is summing over those elements h, such that $1 \leq h < c$ that are co-prime to c and $\bar{h}$ is the inverse to h, i.e

$$h\bar{h} \equiv 1 (\text{mod } c).$$

It is enough to prove bounds on $S(m,n,c)$ for c prime and the general case follows by symmetry properties of the sum. In obtaining his improvements on the Hardy-Ramanujan-Littlewood circle method Kloosterman introduced these sums in 1926. When $n=0$, the degenerate Kloosterman sums are exactly Ramanujan sums and Ramanujan gave an explicit formula for them in 1917. For p a prime, we have

Theorem:(Kloosterman)

$$\left|S(m,n,p)\right| \leq Cp^{3/4}.$$

Theorem:(Davenport)

$$\left|S(m,n,p)\right| \leq Cp^{2/3}.$$

Theorem:(Andre Weil(1948))

$$\left|S(m,n,p)\right| \leq 2\sqrt{p}.$$

Taking the result of Weil and using symmetry properties the sums have, we conclude

Theorem:

$$\left|S(m,n,c)\right| \leq (m,n,c)^{1/2}\tau(c)c^{1/2}.$$

$\tau(c)$ is the number of divisors of c and it is standard that for any $\epsilon > 0$,

$$\tau(c) \leq C(\epsilon)c^{\epsilon}.$$

(m,n,c) is the greatest common divisor of m,n,c. Using elementary properties of $S(m,n,c)$ it is easy to reduce the problem to c a single prime power $c=p^a$ and then further reduce to $a=1$ and apply Weil's theorem.

Analogs of Kloosterman sums for other discrete subgroups of $SL(2,R)$ are studied in Iwaniec's book, but the estimates there are very far from optimal. Kloosterman sums correspond to $SL(2,Z)$ and to the Farey points that were used by Hardy-Ramanujan in their celebrated paper on number of partitions. The Hardy-Littlewood-Ramanujan method is very powerful but is still unable to prove the Lagrange four squares theorem, it needs lots of variables to make the estimates work for oscillatory sums and integrals. Loo-Keng Hua(1938) obtained some improvements on the minor arcs but in general to solve the Waring problem, where one is asked to represent a number N as sums of k th powers:

$$x_1^k + x_2^k + \cdots + x_n^k = N,$$

one needs $n > 2^k + 1$. In fact Hardy-Littlewood used Hermann Weyl's method of differencing to obtain bounds on the oscillatory integrals/sums over the minor arcs in the circle method. Weyl's method is built on induction and Hölder's inequality. It leads to $n > k2^k + 1$. By partly differencing and partly using Parseval formula, Loo-Keng Hua obtained his lemma that is better and allows him to show that $n > 2^k + 1$.

Kloosterman by introducing his sums obtained Lagrange's theorem, $n = 4, k = 2$. Since then Kloosterman sums have appeared in the expression for the scattering matrix in the Spectral theory of automorphic forms and other areas where Analysis, Geometry and Arithmetic interact.

Kloosterman Sums

Theorem(Kloosterman 1926): Consider for p prime, and $m\overline{m} \equiv 1(\text{mod } p)$, the Kloosterman sum,

$$S(a,b,p) = \sum_{m=1}^{p-1} e(\frac{am + b\overline{m}}{p}).$$

Then for any $\epsilon > 0$,

$$|S(a,b,p)| \leq C(\epsilon)p^{\frac{3}{4}+\epsilon}.$$

Proof: First note that

$$\sum_{l,k=1}^{p} |\sum_{m=1}^{p-1} e(\frac{lam + b\overline{m}\overline{k}}{p})|^4 \geq \sum_{l=k=1}^{p-1} |\sum_{m=1}^{p-1} e(\frac{lam + b\overline{m}\overline{l}}{p})|^4 \geq (p-1)|S(a,b,p)|^4. \quad (1)$$

We now open the left side out. We get it is exactly,

$$\Big(\sum_{l=1}^{p} \sum_{m_1,m_2,m_3,m_4=1}^{p-1} e(\frac{la}{p}(m_1 + m_2 - m_3 - m_4))\Big)\times$$
$$\Big(\sum_{k=1}^{p} \sum_{\overline{m_1},\overline{m_2},\overline{m_3},\overline{m_4}=1}^{p-1} e(\frac{\overline{k}b}{p}(\overline{m_1} + \overline{m_2} - \overline{m_3} - \overline{m_4}))\Big). \quad (2)$$

Summing over l,k we see the quantity in brackets will vanish unless we have the system of equations,

$$m_1 + m_2 - m_3 - m_4 = 0,\ \overline{m_1} + \overline{m_2} - \overline{m_3} - \overline{m_4} = 0.$$

We also note then the quantity (2) is exactly for $1 \leq m_i, \overline{m_i} \leq p-1$ the expression,

$$p^2\#\{(m_1,m_2,m_3,m_4)|\ m_1 + m_2 - m_3 - m_4 = 0,\ \overline{m_1} + \overline{m_2} - \overline{m_3} - \overline{m_4} = 0\}. \quad (3)$$

To find the number of integral solutions of our system we re-write it as

$$m_1 + m_2 = m_3 + m_4,\ m_3m_4(m_1 + m_2) = m_1m_2(m_3 + m_4).$$

And so we arrive at by solving the two together to

$$m_1m_2 = m_3m_4.$$

Since m_3, m_4 is arbitrary and m_2 is a divisor of m_3m_4, the number of m_2 is $\tau(m_3m_4)$ the number of divisors. Now $\tau(m_3m_4) << (m_3m_4)^\epsilon << p^\epsilon$. Thus cardinality of the set in (3) is $p^{2+\epsilon}$. Hence putting everything together,

$$p|S(a,b,p)|^4 << p^{4+\epsilon}.$$

Thus we obtain

$$|S(a,b,p)| << p^{\frac{3}{4}+\epsilon}.$$

This ends the proof.

Notes on Kloosterman Sums-Part 2

Definition: The Kloosterman sums are defined by

$$S(m,n,q)=\sum_{(h,q)=1} e(\frac{mh+n\overline{h}}{q}), \overline{h}h\equiv 1(\text{mod } q).$$

Kloosterman sums should be viewed as a discrete analog of Bessel functions. The sum defining Kloosterman sums are the finite place analog at primes of the integral representation of Bessel functions found in standard treatises like the book by G. N. Watson, *A Treatise on the Bessel Functions*. Degenerate Kloosterman sums are Ramanujan sums, for which Ramanujan gave an explicit formula.

Definition(Ramanujan Sums): The Ramanujan sum is defined by a sum over co-prime residue classes h:

$$S(m,0,q)=\sum_{(h,q)=1} e(\frac{mh}{q}).$$

Theorem(Ramanujan): Let μ be the Möbius function. Then

$$S(m,0,q)=\sum_{d/(m,q)} d\mu(\frac{q}{d}).$$

Proof: First we note that

$$\sum_{h=1}^{q} e(\frac{mh}{q})=\begin{cases} 0, & q \text{ does not divide } m \\ q, & \text{if } q/m \end{cases}.$$

Next we may write $\frac{h}{q}=\frac{h_1}{q_1}$, where $(h_1,q_1)=1$, where h_1,q_1 is unique and obviously q_1/q. By the uniqueness we get

$$\sum_{h=1}^{q} e(\frac{mh}{q})=\sum_{q_1/q} S(m,0,q_1)=\begin{cases} 0, & q \text{ does not divide } m \\ q, & \text{if } q/m \end{cases}.$$

We also note that $q_1/(q,m)$ for the sum to be non-zero above. Next we note that if $(q_1,q_2)=1$ then

$$S(m,0,q_1q_2)=S(m,0,q_1)S(m,0,q_2),$$

that is $S(m,0,q)$ is a multiplicative function. This follows because the co-prime residue classes for q_1q_2 are given by $h_2q_1+h_1q_2$, with $(h_1,q_1)=(h_2,q_2)=1$. Thus by Möbius inversion we get our result:

$$S(m,0,q)=\sum_{d/(m,q)} d\mu(\frac{q}{d}).$$

Ramanujan sums have a generating formula that is useful in writing down the coefficients of Eisenstein series in the decomposition of the continuous wave packets in the spectral decomposition of the Laplacian. See H. Iwaniec (Theorem 3.4 and eqn. 3.25) *Spectral Methods of Automorphic Forms*, Graduate Studies in Math., AMS publications. We have:

Proposition:

$$\sum_{q=1}^{\infty} \frac{S(m,0,q)}{q^s} = \zeta(s)^{-1} \sum_{d/m} d^{1-s}.$$

Proof: We simply insert the formula derived by Ramanujan in the left side above to get,

$$\sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d/(q,m),\ dk=q} d\mu(k) = \sum_{k} \frac{\mu(k)}{k^s} \sum_{d/m} d^{1-s}.$$

This ends the proof.

The identity above can be used to show that the coefficients of Eisenstein series indeed satisfy the Ramanujan-Petersen conjecture quite trivially. The Ramanujan-Petersen conjecture is open for Maass/cusp forms. P. Deligne established the conjecture for automorphic forms. Maass forms are real-analytic and not holomorphc and this is the difficulty.

Next we wish to derive an identity for Kloosterman sums first noted by Selberg.

Proposition 1: (a) $S(m,n,q) = S(n,m,q)$.

(b) If $(m,q) = 1$ then

$$S(m,n,q) = S(1,mn,q).$$

Proof: (a) is obvious. As far as (b) goes, set $x = mh$. Then $m\overline{x} = \overline{h}$. Then,

$$S(m,n,q) = \sum_{(x,q)=1} e(\frac{x + mn\overline{x}}{q}) = S(1,mn,q).$$

This proves everything.

Assume that $q = q_1 q_2$ such that $(q_1, q_2) = 1$. Thus for any n since $(q_1^2, q_2^2) = 1$ we may decompose

$$n = n_1 q_2^2 + n_2 q_1^2. \tag{1}$$

We have

Proposition 2:

$$S(m,n,q) = S(m,n_1,q_1) S(m,n_2,q_2).$$

Proof: We check the identity by multiplying the terms on the right which is

$$\sum_{(h_1,q_1)=1,\ (h_2,q_2)=1} e(\frac{mh_1+n_1\overline{h_1}}{q_1})e(\frac{mh_2+n_2\overline{h_2}}{q_2}).$$

Each summand is

$$e(\frac{m(h_1q_2+h_2q_1)+(n_1q_2\overline{h_1}+n_2q_1\overline{h_2})}{q_1q_2}). \tag{2}$$

But $h=h_1q_2+h_2q_1$ is a list of all the residue classes mod $q=q_1q_2$. Thus we will be done if we show mod $q=q_1q_2$

$$n\overline{h}=n_1q_2\overline{h_1}+n_2q_1\overline{h_2}.$$

We check by computing and verifying mod $q=q_1q_2$,

$$h(n_1q_2\overline{h_1}+n_2q_1\overline{h_2})=n.$$

We see that the expression above is

$$(h_1q_2+h_2q_1)(n_1q_2\overline{h_1}+n_2q_1\overline{h_2}).$$

This is

$$n_1q_2^2h_1\overline{h_1}+n_2q_1^2h_2\overline{h_2}. \tag{3}$$

But

$$h_1\overline{h_1}\equiv 1(\text{mod } q_1),\ h_2\overline{h_2}\equiv 1(\text{mod } q_2).$$

Inserting the last fact in (3) we get from (3) that mod $q=q_1q_2$ it is from (1)

$$n_1q_2^2+n_2q_1^2=n,$$

which completes the proof.

This proposition helps to reduce Kloosterman sums to single prime powers $q=p^a$ to which we may comfortably apply Weil bounds and obtain the result

Theorem(A. Weil):

$$|S(m,n,q)|\leq (m,n,q)^{1/2}\tau(q)\sqrt{q}.$$

We now obtain an identity of A. Selberg which bears a close resemblance to the composition formulae for Hecke operators. In fact Kuznestov used the composition formula to give a proof of what we obtain by elementary means. The classical Hecke operators are defined on functions invariant under the modular group by setting

$$T_nf(z)=\frac{1}{\sqrt{n}}\sum_{ad=n,\ d>0}\sum_{0\leq b<d} f(\frac{az+b}{d}).$$

The operators T_n are self-adjoint in the inner product of $L^2(\Gamma\setminus H)$ and also commute with the Laplacian and so a basis for T_n can be given using cusp forms. The composition formula one has is

$$T_nT_m=\sum_{d/(m,n)} T_{\frac{mn}{d^2}}.$$

Note if $(m,n)=1$ then

$$T_m T_n = T_{nm}.$$

We will prove

Proposition 3(A. Selberg):

$$S(m,n,q) = \sum_{d/(m,n,q)} dS(1, \frac{mn}{d^2}, \frac{q}{d}).$$

We will use Proposition 2 to reduce the proof to single prime powers $q = p^a$. We also need a few simple facts that we collect.

Lemma 1: Let $a \geq 2$. Let p/r, then

$$S(1,r,p^a) = 0.$$

Proof: For $(h,p^a)=1$, we observe that $(h + r\overline{h}, p^a) = 1$. For if $p/(h + r\overline{h})$, since p/r it follows p/h a contradiction. Next the residue classes $h + r\overline{h}$, do not repeat. In fact if

$$h_1 + r\overline{h_1} = h_2 + r\overline{h_2},$$

then

$$h_1 - h_2 = r\frac{h_1 - h_2}{h_1 h_2} (\text{mod } p^a),\ (h_1 - h_2)(h_1 h_2 - r) \equiv 0 (\text{mod } p^a).$$

Since $h_1 \not\equiv h_2$, we must have $p/h_1 h_2$ since p/r. But this is nonsense. Thus

$$S(1,r,p^a) = \sum_{(x,p^a)=1} e(\frac{x}{p^a}), \tag{4}$$

a Ramanujan sum. We write (4) as a difference

$$\sum_{l=1}^{p^a} e(\frac{l}{p^a}) - \sum_{k=1}^{p^{a-1}} e(\frac{kp}{p^a}) = 1 - 1 = 0.$$

We used $a \geq 2$ in the second sum above.

Using the Lemma we first establish Proposition 3 for prime powers $q = p^a$. We first consider Case 1.

Case 1: $q = p$. One of m, n is co-prime to p. If n is co-prime we employ Proposition 1 and obtain

$$S(m,n,p) = S(n,m,p) = S(1,mn,p)$$

and since if $d/(m,n,p)$ then $d=1$ we have established Proposition 3 in this case.

Case 2: $q = p$ but p/m **and** p/n. In this case by definition,

$$S(m, n, p) = p - 1.$$

The right side of our Proposition is

$$S(1, mn, p) + pS(1, \frac{mn}{p^2}, 1). \tag{5}$$

Note now

$$S(1, mn, p) = \sum_{h=1}^{p-1} e(\frac{h + mn\overline{h}}{p}) = \sum_{x=1}^{p-1} e(\frac{x}{p}) = \sum_{x=1}^{p} e(\frac{x}{p}) - 1 = -1.$$

Thus (5) is $p - 1$ and we are done again.

Next we establish our result for $q = p^a$, $a \geq 2$. Obviously with no loss of generality we can suppose that p/m and p/n or else we are done again by the previous argument. We have to thus consider

$$S(1, mn, p^a) + \sum_{d/(m,n,p^a),\ d=p^k>1} dS(1, \frac{mn}{p^{2k}}, \frac{p^a}{p^k}).$$

By Lemma 1, the first term above vanishes. We may factor p from the second sum and using induction write the sum as

$$pS(\frac{m}{p}, \frac{n}{p}, \frac{p^a}{p}) = p \sum_{(y,p^{a-1})=1} e(\frac{my + n\overline{y}}{p^a}).$$

But

$$S(m, n, p^a) = \sum_{(x,p^a)=1} e(\frac{mx + n\overline{x}}{p^a}).$$

But between p^{a-1} and p^a there are p blocks of length p^{a-1} of the type $(kp^{a-1}, (k+1)p^{a-1})$. Thus each y above will also appear once as $y + kp^{a-1}$ in the list of x above with $(x, p^a) = 1$ since $(y, p^a) = 1$ too and so $(y + kp^{a-1}, p^a) = 1$. Thus indeed

$$S(m, n, p^a) = pS(\frac{m}{p}, \frac{n}{p}, \frac{p^a}{p}).$$

Thus Proposition 3 is established for single prime powers. Now let $q = q_1 q_2$, $(q_1, q_2) = 1$. Then by Proposition 2 and induction

$$S(m, n, q) = S(m, n_1, q_1) S(m, n_2, q_2)$$

and so we have

$$S(m, n, q) = \sum_{d_1/(m,n_1,q_1),\ d_2/(m,n_2,q_2)} d_1 d_2 S(1, \frac{mn_1}{d_1^2}, \frac{q_1}{d_1}) S(1, \frac{mn_2}{d_2^2}, \frac{q_2}{d_2}).$$

Set $d = d_1 d_2$. Then $d/q_1 q_2$ that is d/q. Next we observe from (1) that if d_1/n_1 and d_1/q_1 then d_1/n and likewise d_2/n and thus $d/(m,n,q)$. We claim that

$$S(1, \frac{mn_1}{d_1^2}, \frac{q_1}{d_1}) S(1, \frac{mn_2}{d_2^2}, \frac{q_2}{d_2}) = S(1, \frac{mn}{d^2}, \frac{q}{d}). \tag{6}$$

This will finish the proof of our proposition. To check (6) we apply Prop. 2 again and compute

$$\frac{mn_1}{d_1^2} \frac{q_2^2}{d_2^2} + \frac{mn_2}{d_2^2} \frac{q_1^2}{d_1^2} = \frac{mn}{d^2}.$$

Thus Prop.2 immediately yields (6) and thus Prop. 3 is proved.

Now we extend the estimate of the first part to all prime powers p^k, with $k \geq 2$. When $k = 1$ this is the deep result of Andre Weil. We follow the exposition of T. Estermann. We have

Theorem: We have: Assume p a prime that does not divide mn. Then,

$$|S(m, n, p^k)| \leq 2p^{\frac{k}{2}}, \; k \geq 2, \; p > 2$$

and

$$|S(m, n, 2^k)| \leq 4 \cdot 2^{\frac{k}{2}}, \; k \geq 2.$$

We begin with an elementary lemma whose proof is left to the reader:

Lemma 2: Let $k \geq 2$. Then if $(a, p) = 1$ we have

$$x^2 \equiv a (\text{mod } p^k), \; p > 2,$$

has at most 2 solutions for any k.

For a odd,

$$x^2 \equiv a (\text{mod } 2^k),$$

has at most 4 solutions for any k.

We now split the proof of our theorem into the case where k is odd or even. We consider the case k even first. Set $j = \frac{k}{2}$. Write

$$h = s + tp^j,$$

where p is any prime including 2 and $0 \leq t < p^{k-j}$. s is given mod p^j. Since h is invertible we have

$$\overline{h} = (s(1 + t\overline{s}p^j))^{-1} = \overline{s}(1 - t\overline{s}p^j + t^2\overline{s}^2 p^{2j}).$$

Thus

$$\overline{h} = \overline{s} - t\overline{s}^2 p^j + t^2 \overline{s}^3 p^{2j}, \tag{7}$$

since the higher terms vanish mod p^k. Since $k = 2j$ we have

$$\overline{h} = \overline{s} - t\overline{s}^2 p^j.$$

We may now write our Kloosterman sum using our representation for h as

$$\sum_{s,t} e(\frac{m(s+tp^j) + n(\overline{s} - t\overline{s}^2 p^j)}{p^k}) = \sum_s e(\frac{(ms + n\overline{s})}{p^k}) \sum_{0 \le t < p^{k-j}} e(\frac{t(m - n\overline{s}^2)}{p^{k-j}}).$$

The inner sum is bounded by p^{k-j} provided $m \equiv n\overline{s}^2 (\mathrm{mod}\ p^{k-j})$ or else it vanishes. But $0 < s < p^j$. Thus from above

$$|S(m,n,p^k)| \le p^{k-j} \sum_{0 < s < p^j:\ m \equiv n\overline{s}^2 (\mathrm{mod}\ p^{k-j})} 1.$$

The right side is

$$p^{\frac{k}{2}} \sum_{0 < s < p^{\frac{k}{2}}:\ m = n\overline{s}^2 (\mathrm{mod}\ p^{\frac{k}{2}})} 1.$$

Applying Lemma 2 to the sum we see it is bounded by $2p^{\frac{k}{2}}$ if $p > 2$ and by $4 \cdot 2^{\frac{k}{2}}$ if $p = 2$. This ends the proof in this case.

We now consider the case of odd k and set $j = \frac{k-1}{2}$, that is $k = 2j+1$. We also recall a standard fact about Gauss sums which we state as a lemma.

Lemma 3: Let $(a,p) = 1$. Then

$$\left|\sum_{t=1}^{p-1} e(\frac{at^2 + bt}{p})\right| \le p^{\frac{1}{2}}.$$

Since $2j < k$ we will now have to use all three terms on the right in (7). We will further split up t and write $t = u + lp$. Thus $0 \le u < p$ and $0 \le l < p^{k-j-1} = p^j$. Our Kloosterman sum becomes after simplification

$$\sum_s e(\frac{ms + n\overline{s}}{p^k}) \sum_u e(\frac{u(m - n\overline{s}^2)}{p^{k-j}} + \frac{u^2\overline{s}^3}{p}) \sum_l e(\frac{l(m - n\overline{s}^2)}{p^{k-j-1}}).$$

The sum in l vanishes or is equal to p^{k-j-1} if $m \equiv n\overline{s}^2 (\mathrm{mod}\ p^{k-j-1})$. In the non-zero case case the Kloosterman sum becomes (substituting what we just observed into the sum in u)

$$p^{k-j-1} \sum_s e(\frac{ms + n\overline{s}}{p^k}) \sum_u e(\frac{bu}{p} + \frac{u^2\overline{s}^3}{p}).$$

The sum in u is now seen to be a Gauss sum to which Lemma 3 applies. Noting that $p^{k-j-1} = p^j$ and since $m \equiv n\bar{s}^2 (\bmod\ p^{k-j-1})$ is the same congruence as $m \equiv n\bar{s}^2 (\bmod\ p^j)$, the Kloosterman sum is bounded by

$$p^{k-j-1+\frac{1}{2}} \sum_{s:\ m\equiv n\bar{s}^2 (\bmod\ p^j)} 1 = p^{\frac{k}{2}} \sum_{s:\ m\equiv n\bar{s}^2 (\bmod\ p^j)} 1.$$

Lastly applying Lemma 2 to the sum in s above, we obtain our Theorem in the case k odd.

Remarks:

1. In the first part of these notes we saw Kloosterman's proof of bounds for the sums, by introducing an averaging device that led to more points and a sort of decoupling. This idea again appears in another form in the famous theorem by D. Burgess on short character sums. Remarkably at the core of the proof by Burgess he resorts to the A. Weil result we have quoted above. Specifically Burgess proved that

Theorem(D. Burgess(1962): Let $\chi(n)$ denote **any** non-principal Dirichlet characters to a prime modulus p. The modulus condition can be relaxed. Then we have

$$\Big|\sum_{M\leq n\leq M+N} \chi(n)\Big| \leq cN^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}, 1 < r < \infty,$$

and where c is an absolute constant.

Note the theorem of Burgess is non-trivial provided

$$c_1 p^{\frac{1}{4}+\frac{1}{4r}} \log p \leq N \leq c_2 p^{\frac{1}{2}+\frac{1}{4r}} \log p.$$

The left inequality follows from the trivial bound N on the character sum, and the right inequality from the Polya-Vinogradov bound $p^{\frac{1}{2}} \log p$ on character sums.

2. The Kloosterman sums $S(n, n, c)$ appear in the work of H. Iwaniec(*J. fur der Reine and Angewandte Matematik* **349**(1984)) on Prime Geodesics in estimating the error term in the Weyl law for Geodesics in the spirit of the Prime Number theorem. In this result a decisive use is made of Burgess's theorem. The Kloosterman sums appear via the Bruggeman-Kuznestov formula which is a form of the Poisson summation formula, with Bessel transforms replacing the Fourier transform and Kloosterman sums appearing via the formula for the automorphic Green function. See Ch. 9. in the book Spectral Methods of Automorphic Forms, by H. Iwaniec, Graduate Studies in Math. American Mathematical Society.

3. Kloosterman sums appear in the formula for the scattering matrix on $\Gamma \setminus H$ where H is the Poincaré upper half plane, and Γ is a discrete Fuchsian group of Type I which only has cusps and no funnels at infinity. Exact formulae can be obtained in the case of Γ being the modular group or other Arithmetic groups of congruent subgroup type. In the general case

not much is known for the bounds for Kloosterman sums. For example one knows that the scattering matrix is meromorphic of order 2 and for the modular group meromorphic of order 1 due to the appearance of the Riemann zeta function. However it is unknown as to how to construct groups Γ where the scattering matrix is of order $3/2$ say. The order matters in determining Weyl laws for the zeros of the scattering matrix which determine the non-scattering frequencies or Transmission eigenvalues as in the paper by F. Cakoni and S. Chanillo, *Transmission Eigenvalues and the Riemann Zeta Function in Scattering Theory for Automorphic Forms on Fuchsian Groups of Type I*, Acta Mathematica Sinica, English Series **35**(6)(2019), 987-1010.

Sagun Chanillo

Deptt. of Math., Rutgers Univ.

110 Frelinghuysen Rd.,

Piscataway NJ 08854 USA

`chanillo@math.rutgers.edu`