Castelnuovo's Rationality Theorem

The aim is to give a simple proof of this famous and important theorem which is fundamental in the Enriques-Kodaira classification of complex surfaces. Modern textbooks like Griffiths and Harris(Principles of Algebraic Geometry) or Arnaud Beauville(Complex Algebraic Surfaces) usually present the proof by Kodaira, in the fourth of his series of papers on classification of Complex Algebraic Surfaces[1]. Here we present a proof more in the spirit of Mori theory. I do not claim any originality, to experts this proof must be well known, though I have not seen it in books. As a consequence we get the version of Lüroth theorem for two dimensions. The higher dimensional version of the Lüroth theorem is false by the counterexamples of Griffiths-Clemens(using the intermediate Jacobian), Michael Artin-Mumford(using ideas of the Indian Algebraic Geometer C. P. Ramanujam who died tragically at a young age) and Ishkovskih and Manin who independently found examples around 1971. We also comment briefly on Enriques surfaces which provides an example to show that one cannot relax the conditions in Castelnuovo's theorem.

Theorem(Guido Castelnuovo, 1896): Let S be a smooth, compact, complex surface. Assume that the surface S has the numerical invariants, $q = 0$, $P_2 = 0$ and necessarily then the geometric genus $P_1 = p_g = 0$. Then the surface is rational, that is, it is birational to complex projective space CP^2.

Corollary: We get the following analog of Lüroth's theorem. Let the surface S be unirational. That is there is a rational map Φ such that

$$\Phi : CP^2 \to S.$$

Then Φ is birational.

The classical Lüroth theorem is a statement for curves/Riemann surfaces. We state and prove it for completeness of this discussion.

Theorem (Lüroth)(1876): Let Φ be a unirational map to a Riemann surface C, that is

$$\Phi : CP^1 \to C.$$

Then C is CP^1 and Φ is bi-rational.

Proof: There is an algebraic proof using the function field [7]. However we can also prove the theorem by using the Riemann-Hurwitz formula which we recall. Let

$$\Phi : X \to Y$$

where X, Y are compact Riemann surfaces. Then,

$$2g_X - 2 = d(2g_Y - 2) + \sum_i (b_i - 1)$$

where b_i is the branching order of the map Φ and g_X, g_Y denotes the genus of X, Y respectively and d the degree of the map Φ. We apply this result with the choice $Y = C$ and $X = CP^1$. Thus $g_X = 0$ and so we have,

$$d(2g_C - 2) + \sum_i (b_i - 1) = -2.$$

So we must have $b_i = 1$ for all i and $g_C = 0$ and $d = 1$ This ends the proof. **QED**

Proof of the Corollary: Pulling back any holomorphic 1-form ω on S via Φ we get a 1-form $\Phi^\star(\omega)$ on CP^2. But CP^2 has none and so we conclude that $q = 0$ for S. Likewise we conclude that $p_g = P_2 = 0$ for S and so S is rational by Castelnuovo's theorem. We conclude that S is rational and Φ is a bi-rational map.

We now consider the proof of Castelnuovo's theorem. We note the Hodge diamond for S by hypothesis in our theorem is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & b_2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Thus the holomorphic Euler characteristic $\chi(\mathcal{O}) = 1$. First a few small remarks before we begin the proof.

Remark 1: The surface S having the Hodge diamond above is automatically projective algebraic. This follows from the very basic results in Kodaira 1 [2] on numerical invariants. First since $p_g = 0$ in the Hodge diamond and hypothesis and the first Betti number is even(the remark works even under this weak assumption) we conclude from Kodaira's lemma that

$$b_2^+ = 2p_g + 1 = 1.$$

Thus there is an element of the second Betti class that has positive self-intersection. Now consider the sequence

$$0 \to Z \to \mathcal{O} \to \mathcal{O}^\star \to 0,$$

and the long exact sequence of cohomology associated to it,

$$H^1(S, \mathcal{O}^\star) \to H^2(S, Z) \to H^2(S, \mathcal{O}) = \{0\}.$$

Thus the map on the left is onto and since $b_2^+ = 1$, there exists a line bundle L such that,

$$L \cdot L = c_1^2(L) > 0. \tag{1}$$

This immediately implies that the surface S is algebraic. See for example [2], theorem 9.

We will always assume in the sequel that our surface S is minimal. That is it has no (-1) curves that are rational. Any such curve can of course be blown down by general results of Grauert and Castelnuovo in the algebraic case. We have.

Lemma 1: Assume the Kodaira dimension κ of the surface satisfies $\kappa \geq 0$. Then the canonical line bundle K_S is numerically effective (nef). That is for any effective divisor D we have

$$K_S \cdot D \geq 0.$$

Proof: This is like Mori theory. We prove by contradiction. Assume K_S is not nef. Then there exists an effective divisor D such that

$$K_S \cdot D < 0. \tag{2}$$

Since $\kappa \geq 0$, some plurigenera has sections and so we can also assume that K_S is effective and write,

$$K_S = \sum_i a_i C_i,$$

with $a_i > 0$ and C_i irreducible curves. We can also assume that the effective divisor D in (2) above is one of the curves C_i, or else there is no intersection. Thus if we remove this C_i we get,

$$(K_S - a_i C_i) \cdot C_i = (\sum_{j \neq i} a_j C_j) \cdot C_i \geq 0.$$

It follows from above and the contradiction hypothesis that,

$$a_i C_i^2 \leq K_S \cdot C_i < 0.$$

Thus by the adjunction formula where g is the genus of C_i

$$2g - 2 = K_S \cdot C_i + C_i^2$$

we see right away that $g = 0$ and $C_i^2 = -1$. Thus C_i is an exceptional curve. But S is minimal and has no exceptional curves by hypothesis. Thus K_S has to be nef.

Lemma 2: Under the hypothesis that $\kappa \geq 0$ we have

$$K_S \cdot K_S = K_S^2 \geq 0.$$

This is automatic. We know K_S is an effective divisor as $\kappa \geq 0$ and we also know K_S is nef. Thus the Lemma follows.

In what follows below, we shall consistently use the notation

$$NK_S = K_S \otimes K_S \otimes \cdots \otimes K_S,$$

where the tensor product is taken N times.

Now we are ready to finish the proof of our theorem by using the Riemann-Roch theorem. We assume by contradiction that the Kodaira dimension $\kappa \geq 0$ as opposed to being rational which is $\kappa = -\infty$. That is we are assuming some plurigenera has a section. We will now arrive at a contradiction. First we recall the Riemann-Roch theorem for line bundles on complex surfaces. It reads

$$h^0(L) - h^1(L) + h^2(L) = \frac{1}{2}(L^2 - K_S \cdot L) + \chi(\mathcal{O}), \tag{3}$$

where

$$h^i(L) = \dim H^i(S, L).$$

We apply the theorem with $L = 2K_S$. Note now by Serre duality,

$$\dim H^2(S, 2K_S) = \dim H^0(S, -K_S) = h^0(-K_S).$$

Thus using the last remark in (3) above we get,

$$h^0(2K_S) + h^0(-K_S) = K_S^2 + \chi(\mathcal{O}) + h^1(2K_S)$$

Applying Lemma 2 we see now from the Hodge diamond and the hypothesis that

$$P_2 = h^0(2K_S) = 0$$

we have,

$$h^0(-K_S) \geq 1.$$

Thus $-K_S$ has a section and we can move in the linear system and obtain that $-K_S$ is effective, that is

$$-K_S \geq 0.$$

But K_S is also effective since we are assuming that $\kappa \geq 0$. Thus we have both,

$$-K_S \geq 0, K_S \geq 0.$$

Thus $K_S \equiv 0$. But this means the function 1 is a global section of K_S. Thus $p_g = h^0(K_S) = 1$ and violates our assumption that it is zero. Thus we have a contradiction and so the Kodaira dimension κ cannot be ≥ 0 and has to be $-\infty$. That is **NO** plurigenera P_N has a section.

QED

The idea of Castelnuevo was the famous idea of *termination of adjunction.* This is how Kodaira exposes it in [1]. Nowadays people instead follow Mori theory as applied to surfaces. The proof then goes via the Mori Rationality lemma. The aim in all approaches is to first prove:

Lemma 3: Assume that an algebraic surface with $q = P_1 = 0$ has a rational curve C such that $C^2 \geq 0$. Then S is geometrically ruled. That is there is a map,

$$\phi : S \to B$$

where B is a smooth Riemann surface and the fibers of Φ are connected rational curves.

Since in the Castelnuevo theorem we are assuming $q = 0$, if B had positive genus, we could pull back a holomorphic one-form on B via Φ to S, and this would contradict $q = 0$. Thus $B = CP^1$. Lastly we can then apply the theorem of Noether-Enriques to finish matters.

Theorem(Noether-Enriques): If a surface S is geometrically ruled, then it is ruled, that is it is bi-rational to

$$CP^1 \times B$$

In our case since $B = CP^1$, we conclude that S is bi-rational to

$$CP^1 \times CP^1$$

Proof: A simple but penetrating proof of the Noether-Enriques theorem is found in the book by Griffiths and Harris [5], pg. 514, there is also a proof in Beauville [4] but more algebraic, see pg. 25.

Conversely note that a rational surface has automatically $q = p_g = P_2 = 0$. Thus the assumptions in Castelnuovo's theorem are necessary and sufficient.

We now prove Lemma 3.

Proof of Lemma 3: We first claim that,

$$h^0(C) \geq 2.$$

We apply Riemann-Roch to get,

$$h^0(C) + h^0(K_S - C) \geq \frac{1}{2}(C^2 - K_S \cdot C) + 1.$$

Next note by the adjunction formula and because C is rational,

$$-2 = C^2 + K_S \cdot C$$

Thus

$$-K_S \cdot C = C^2 + 2$$

Hence the rhs of Riemann-Roch inequality is bounded below by,

$$2 + C^2 \geq 2$$

because $C^2 \geq 0$. Next by contradiction assume that

$$h^0(K_S - C) \neq 0.$$

Then there is a section and and so since C is a curve and a positive divisor, it follows that if

$$(s) + K_S - C \geq 0$$

then

$$(s) + K_S \geq C \geq 0$$

and so $P_1 \neq 0$. This is a contradiction to the hypothesis. Thus

$$h^0(K_S - C) = 0.$$

Thus from Riemann-Roch we conclude that

$$h^0(C) \geq 2.$$

So we have found a meromorphic function F/G such that after blow up to remove indeterminacy(and by abusing notation we still call the blown up surface S) we have a may

$$\Phi : S \rightarrow CP^1$$

given by,

$$\Phi(z) = [F(z) : G(z)].$$

We can use the Stein factorization theorem and use assume then that the map Φ is to a curve B and the fibers of Φ are **connected** and the fibers belong to the linear system $|C|$, the linear system of the rational curve C and so the fibers of Φ are rational curves $\{C_\lambda\}$. This proves the lemma.

Thus the goal is to prove the hypothesis of Lemma 2. This can be done as remarked via termination of adjunction as Castelnuevo originally did, or the new approach via Mori theory which we will follow. To prove the fundamental Mori Rationality lemma we need a consequence of a theorem in Number Theory. This consequence is a corollary to Theorem 167 in the book by Hardy and Wright [6].

Lemma 4: For any irrational number $x > 0$, we can find positive natural numbers p, q as large as we please such that

$$\frac{p}{q} - \frac{1}{3q} < x < \frac{p}{q}.$$

Proof: The proof relies on some elementary properties of Continued fractions. First note we can always express any positive irrational number as a continued fraction with **natural number entries** in the following way. Write

$$x = [x] + y = a_1 + y_1$$

where y_1 is a decimal and a_1 a natural number. Then we write

$$y_1 = 1/(1/y_1).$$

Now $1/y_1 > 1$ and so we can write,

$$\frac{1}{y} = a_2 + y_2$$

where a_2 is a natural number and y_2 is a decimal. We now inductively apply the previous step to y_2. We then get a continued fraction

$$x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}.$$

Next we note by induction that the partial summands of our continued fraction satisfy,

$$p_n = a_n p_{n-1} + p_{n-2}, \; q_n = a_n q_{n-1} + q_{n-2},$$

where the partial summand is

$$\frac{p_n}{q_n}.$$

This is very general and we do not have to assume that the a_i are natural numbers. Let us show the argument for p_n. We have by induction:

$$p_{n+1} = (a_n + \frac{1}{a_{n+1}})p_{n-1} + p_{n-2} = \frac{1}{a_{n+1}}(a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}).$$

By the induction hypothesis again we get,

$$p_{n+1} = \frac{1}{a_{n+1}}(a_{n+1} p_n + p_{n-1}).$$

Similarly,

$$q_{n+1} = \frac{1}{a_{n+1}}(a_{n+1} q_n + q_{n-1}).$$

Thus we have proved the recursive formula for the summands.

Next an elementary consequence of the recursive formula is

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}. \tag{1}$$

This is easily checked by plugging in for p_n, q_n from the recursion to get,

$$p_n q_{n-1} - p_{n-1} q_n = -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}).$$

Applying this repeatedly we get,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} (p_2 q_1 - p_1 q_2).$$

In the special case of integers, $p_1 = a_1, q_1 = 1$ and

$$p_2 = a_1 a_2 + 1, q_2 = a_2$$

We obtain our formula. We re-write this as

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

Note from this formula if n is odd it follows that the odd summand is more than the even summand,

$$\frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n}.$$

Next we show that the odd summands decrease and the even summands increase and thereby for n odd,

$$\frac{p_{n-1}}{q_{n-1}} < x < \frac{p_n}{q_n}.$$

To see this last fact we simply observe that

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

This is obtained by inserting the recursive formula for p_n, q_n and then applying the result (1) above. It follows that for n even,

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} > 0,$$

that is the even summands increase. By taking n odd we can see the odd summands decrease. Thus we have also convergence of the summands. The last remark is that for natural numbers we have from the recursive formulae,

$$q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + q_{n-2}$$

We know $q_1 \geq 1$ and so we claim $q_n \geq n$. From the inequality above we always have

$$q_n \geq q_{n-1} + 1$$

Thus our claim $q_n \geq n$ follows by induction. Thus we have

$$|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}| = \frac{1}{q_n q_{n-1}} \leq \frac{1}{n(n-1)} \to 0,\ n \to \infty.$$

Thus both the odd and even summands converge to the same limit value, while the odd summands converge from above, the even summands converge from below. Thus for n odd and ≥ 5 we have,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{1}{q_n q_{n-1}} \leq \frac{1}{3q_n}.$$

So

$$\frac{p_n}{q_n} - \frac{1}{3q_n} \leq \frac{p_{n-1}}{q_{n-1}} < x < \frac{p_n}{q_n}.$$

Lastly note that we have shown that q_n increases to infinity. If p_n were to remain bounded, then $x = 0$ which is a contradiction and so p_n also increases to infinity. This finishes the proof.

Mori's Rationality Lemma: Assume S is an algebraic surface such that K_S is **not** nef. Let H be an ample line bundle on S. Then the number

$$b = \sup\{t|\ t > 0,\ H + tK_S \text{ is ample}\},$$

is a rational number.

Simple Remark: Note that $H + bK_S$ is nef. Simply because for any effective divisor D we have for any $t < b$ by definition of being ample,

$$(H + tK_S) \cdot D > 0.$$

So, letting $t \to b$ we get,

$$(H + bK_S) \cdot D \geq 0.$$

Proof: First note that $t = 0$ belongs to the set $H + tK_S$ ample. So the set is non-empty. Next under the hypothesis that K_S is **not** nef, we can find a curve C such that

$$K_S \cdot C < 0.$$

Hence for large enough t we have,

$$(H + tK_S) \cdot C < 0.$$

So for such a value of t, t does not belong to the set. Thus under the hypothesis that K_S is not nef we conclude that the value of b in the Mori rationality lemma is finite. We now show it is rational. We consider the Hilbert polynomial,

$$P(u, v) = h^0(uH + vK_S) - h^1(uH + vK_S) + h^2(uH + vK_S).$$

This is a non-trivial polynomial that does not vanish identically. Simply take $u = n$ and $v = 1$ with n very large, and we see then that $nH + K_S$ is very ample and hence the Hilbert polynomial cannot be identically zero. Next if $(v-1)/u < b$ we see that

$$H + \frac{v-1}{u} K_S$$

must be ample by definition. Thus by the Kodaira vanishing theorem, if $(v-1)/u < b$ we have,

$$h^1(uH+vK_S) = h^1(uH+(v-1)K_S+K_S) = h^2(uH+vK_S) = h^2(uH+(v-1)K_S+K_S) = 0.$$

Thus for $(v-1)/u < b$,

$$P(u,v) = h^0(uH + vK_S) \geq 0.$$

We are now in a position to apply the lemma from Number theory. Assume by contradiction that b is irrational. Choose natural numbers p, q very, very large such that,

$$\frac{p}{q} - \frac{1}{3q} < b < \frac{p}{q}.$$

Consider the line M given by $u = tq,\ v = tp$. Now the Hilbert polynomial is quadratic in u, v as the right side of the Riemann-Roch formula shows that

$$P(u,v) = \chi(\mathcal{O}) + \frac{L^2 - L \cdot K_S}{2}.$$

Thus the line M cannot meet the curve $P(u,v) = 0$ in more than two points. The curve $P(u,v) = 0$ is a conic. If the line M met the curve $P(u,v) = 0$ in three points then $(up - vq)$ is a factor of $P(u,v)$. But p, q can be taken to be as large as we please. So if we select them suitably large, it would mean the coefficients of $P(u,v)$ are also very large. But that is not possible from the Riemann-Roch formula. The coefficents only depend on the holomorphic Euler characteristic and the intersection numbers of H and K_S. Thus M only meets $P(u,v) = 0$ in two points. Now note that for points on the line M,

$$\frac{v-1}{u} = \frac{p}{q} - \frac{1}{tq}.$$

By the Number theory lemma for $t = 1, 2, 3$ we have

$$P(u,v) \geq 0$$

For at most two $P(u,v)$ can possibly vanish. Thus for for one of the values $t = 1, 2, 3$ we must have,

$$P(u,v) > 0.$$

But for this value,

$$h^0(uH + vK_S) = P(u,v) > 0.$$

Thus for this choice $p/q = v/u$ of a rational we find that $H + \frac{p}{q}K_S$ is effective and by the Number theory lemma again since $p/q > b$ we also find that $H + \frac{p}{q}K_S$ cannot be nef. If it were nef, the nef cone is the closure of the ample cone and we would then find a number bigger than b, call it c where $H + cK_S$ is ample. Next since $H + b_oK_S$ is effective, where $b_0 = p/q$ we can write,

$$H + b_oK_S = \sum_j a_j\Gamma_j,\ a_j > 0,\ a_j \in Q \tag{1}$$

But $H + b_oK_S$ is not nef, so

$$H + b_0K_S \cdot \Gamma_j < 0.$$

Next note that for every j

$$H + bK_S \cdot \Gamma_j > 0$$

The reason is that by choice

$$H + bK_S \cdot \Gamma_j \geq 0.$$

But because b is irrational and intersection numbers are integers, one cannot have

$$H + bK_S \cdot \Gamma_j = 0.$$

Since $b_0 > b$, we can thus find a **rational number** s_0 with $b_0 > s_0 > b$ such that for every j.

$$H + s_0K_S \cdot \Gamma_j > 0.$$

We now claim that $H + s_0K_S$ is nef and this contradicts the choice of b. So let C be any other curve distinct from Γ_j. We have,

$$(H + s_0K_S) \cdot C = s_0(\frac{1}{s_0}(H + K_S) \cdot C$$

$$= s_0(\frac{1}{b_0}H + K_S + (\frac{1}{s_0} - \frac{1}{b_0}H) \cdot C = (H + b_0K_S) \cdot C + s_0(\frac{1}{s_0} - \frac{1}{b_0})H \cdot C.$$

The curve C does not appear in the list Γ_j and so the first term on the extreme right vanishes identically. H is ample and so $H \cdot C > 0$. Also by choice

$$\frac{1}{s_0} - \frac{1}{b_0} > 0,$$

and so,

$$(H + s_0K_S) \cdot C > 0.$$

Since $s_0 > b$ we have violated the definition of b. This is a contradiction and so b has to be rational. **QED**

We have now completed the most difficult part of the Castelnuevo Rationality theorem.

Now we will put all the pieces together. We always reason under the assumption $q = 0 = P_2$. Note by Lemma 1, we already know by hypothesis that K_S is not nef. Thus the Mori rationality applies for us. Our goal is to show:

Lemma 5: Under the hypothesis $q = P_2 = 0$, there is a rational curve C on S, such that $C^2 \geq 0$. In view of Lemma 3, this will finish matters.

Proof: We first note that we always have,

$$0 \to Z \to \mathcal{O} \to \mathcal{O}^\star \to 0.$$

Under the hypothesis $q = P_2 = 0$ we have that $q = P_1 = 0$ and hence by the long exact sequence of cohomology,

$$0 \to H^1(S, \mathcal{O}^\star) \to H^2(S, Z) \to 0. \tag{1}$$

The proof is divided into some cases.

Case 1. We first assume that all the ample line bundles lie on the positive half-ray spanned by $-K_S$ over the rationals. Now any line bundle L is the difference of two ample bundles. This is elementary and proved in these notes elsewhere. Simply take any ample bundle H and then for any line bundle L we have $L + nH$ is ample. Call $L + nH = H_1$. Then $L = H_1 - nH$. Since all ample bundles lie along the half ray in the direction $-K_S$, it now follows from (1) that there is a **single** generator for $H^2(S, Z)$, since all line bundles can be generated from the ample ones by taking integral linear combinations and hence $b_2 = 1$. If this generator is called g. We must have also,

$$-K_S = 3g$$

with $g^2 = 1$. The reason is the Noether formula. If $-K_S = mg$ we would have by the Noether formula as $b_2 = 1$ and hypothesis,

$$m^2 + 3 = 12$$

Thus $m = 3$. Thus we have the Hodge diamond of projective space or fake projective space,

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Now in view of (1) there is a line bundle L with $c_1(L) = g$. The system corresponding to $|L|$ cannot be reducible or else the generator for $H^2(S, Z)$ would be ag with $a < 1$ and this is a contradiction. So there are no fixed components. Next we can write

$$L = L - K_S + K_S$$

and recalling that L and $-K_S$ is ample it follows by Kodaira vanishing theorem that,

$$h^1(L) = h^2(L) = 0$$

Since $L^2 = 1$ and $L \cdot K_S = 3$, it follows by Riemann-Roch that

$$h^0(L) = 3.$$

The sections of L have no base points and so we can define a map with no indeterminancy from S into CP^2 using the sections of L. We leave to the reader to verify that the map is an isomorphism.

Case 2. Thus we can now assume that there is an ample line bundle H that does not lie on the positive half-ray through K_S. By the Mori rationality theorem we can find a rational number $s = p/q$ such that,

$$L = qH + pK_S$$

is nef. We now claim that $L^2 = 0$. Since L is nef we already have $L^2 \geq 0$. So assume that $L^2 > 0$. Let C be any curve. Then by the nef property

$$LC \geq 0$$

If $LC = 0$, then since $L^2 > 0$, by the index theorem, $C^2 \leq 0$. Since $HC > 0$ and $LC = 0$, it follows that $K_S C < 0$. Thus by the adjunction formula, C is a exceptional (-1) curve and the minimality of S is contradicted. Thus $LC = 0$ cannot happen and for all curves $LC > 0$. But if $L^2 > 0$ and $LC > 0$, then for example by the Nakai criterion it follows that L is ample. But now we have violated the choice of the rational number p/q in the Mori rationality theorem(see the proof, it is the largest rational). Thus we must have $L^2 = 0$

Case 3. $L^2 = 0$. Our goal is that mL for m large will give us a ruling by providing us a rational curve C in the linear system for $|mL|$. This rational curve will have the property that $C^2 \geq 0$. The goal as always is to show $h^0(mL)$ is effective. One does this by applying Riemann-Roch as a tool. So consider

$$mL - K_S = m(H + \frac{mp-1}{mq}K_S)$$

But $(mp-1)/mq < p/q$ and so by the choice of p/q in the Mori lemma, we conclude that $mL - K_S$ is ample. Thus by Kodaira vanishing

$$h^1(mL) = h^2(mL) = 0.$$

Thus since $L^2 = 0$ we have by Riemann-Roch,

$$h^0(mL) = 1 - \frac{m}{2}LK_S.$$

But

$$0 = L^2 = L(H + \frac{p}{q}K_S) = LH + \frac{p}{q}LK_S.$$

L is nef and so $LH \geq 0$. If $LH = 0$, since $H^2 > 0$, by the index theorem, it would mean, since $L^2 = 0$ that L is trivial. This would mean

$$H = -\frac{p}{q}K_S$$

but by Case 1 we have ruled this out. So $LH > 0$ and so $LK_S < 0$. So $\dim |mL| > 1$ for large m. We abuse notation and call mL as L from now on. We have,

$$L = L' + L''$$

where L' is the moving part and L'' the fixed part of the system. For moving part we clearly have,

$$L'L' = 0.$$

Also we have

$$0 = L^2 = LL' + LL''$$

and since L is nef, each summand is non-negative and so each summand is zero. Since $LL' = 0$ and $L'L' = 0$ we get,

$$0 = LL' = L'L''.$$

Now split L' as

$$L' = D + R$$

Since $LL' = 0$ and L is nef it follows that

$$LL' = 0 = LD + LR$$

hence $LD = 0$. Since $L'L'' = 0$, it follows now that $L''D = 0$ and hence from $LD = 0$ that $L'D = 0$. Because

$$L' = D + R$$

we conclude that,

$$D^2 + DR = 0$$

Note that D, R are moving and so,

$$D^2 = 0$$

Further $HD > 0$ and because $LD = 0$ from the definition of L it follows that $DK_S < 0$. Thus by the adjunction formula, D has to be a rational curve. This ends the proof.

QED

Numerical Invariants for Complex Surfaces

In this section we will collect basic facts about Numerical Invariants for smooth Compact, Complex Surfaces. We shall obtain these invariants in the general case, that is no assumption will be made that the first Betti number is even, i.e the surface is Kähler. The topics are taken from the paper of Kodaira[2]. We begin with establishing notations.

Notation: We always set $P_1 = p_g$ to be the dimension of the space of holomorphic 2-forms. That is the dimension of $H^2(S, \mathcal{O})$.

We set $h^{1,0}(S)$ to be the dimension of holomorphic 1-forms. The irregularity q is the dimension of $H^1(S, \mathcal{O})$ and this can be also identified with the dimension of the anti-holomorphic 1-forms.

c_2 denotes the Euler characteristic and b_1 the first Betti number, b_2 the second Betti number. b_2^+ and b_2^- denote the dimension of the space of real 2-forms with positive and negative self-intersection respectively. The basic theorem for numerical invariants for surfaces can now be stated.

Theorem: Let S be a smooth, compact and complex surface. Then

(a) Let b_1 be odd, in particular S is **not** Kähler. Then $h^{1,0}(S) = q - 1$ and $h^{0,1}(S) = q$. Moreover $b_2^+ = 2p_g$. We also have,

$$c_1^2(K_S) + 8q + b_2^- = 10p_g + 8.$$

(b) Let b_1 be even and thus Kähler by the theorem of Yum-Tong Siu [8]. Then $b_1 = 2q$, $b_2^+ = 2p_g + 1$ and

$$c_1^2(K_S) + 8q + b_2^- = 10p_g + 9.$$

There are several important consequences of this theorem that we shall discuss later. We need some elementary lemmas to complete the proof.

Lemma 1: On a compact, smooth complex surface, any holomorphic 1-form is closed.

Proof: Let θ be any $(2,0)$ form. Then locally we can write

$$\theta = f dz \wedge dw.$$

Thus locally,

$$\theta \wedge \overline{\theta} = |f|^2 dV$$

where dV is the volume form. We conclude that for any $(2,0)$ form θ,

$$\int_S \theta \wedge \overline{\theta} \geq 0,$$

and the expression above vanishes if and only if $\theta \equiv 0$. Now for any ψ a holomorphic 1-form we set,

$$d\psi = \partial\psi = \theta, \; d = \partial + \overline{\partial}$$

where θ is necessarily a $(2,0)$ form and we used the fact that $\overline{\partial}\psi = 0$. We thus have by Stokes theorem,

$$\int_S d\psi \wedge d\overline{\psi} = \int_S d(\psi \wedge d\overline{\psi}) = \int_S \theta \wedge \overline{\theta} = 0.$$

But then this means $d\psi = \theta \equiv 0$. Thus ψ is closed.

Lemma 2: We have for **any** compact, complex surface,

$$b_2^+ \geq 2p_g.$$

Proof: Let $\{\theta_i\}_{i=1}^{p_g}$ denote the linearly independent $(2,0)$ forms on S. Let us define a Hermitian inner product on this finite dimensional vector space of $(2,0)$ forms by setting,

$$(\theta, \eta) = \int_S \theta \wedge \overline{\eta}.$$

This is an inner product in view of the observation at the beginning of Lemma 1 above. Using the Gram-Schmidt process we may then select an orthonormal basis $\{\eta_i\}_{i=1}^{p_g}$ of $(2,0)$ forms such that,

$$(\eta_i, \eta_j) = \delta_{ij}.$$

Now consider the real 2-forms,

$$\omega_j = \eta_j + \overline{\eta_j},\ j = 1, 2, \cdots, p_g$$

and

$$\rho_j = \imath\overline{\eta_j} - \imath\eta_j,\ \imath = \sqrt{-1},\ j = 1, 2, \cdots, p_g.$$

It is easily seen that, the forms $\omega_j,\ \rho_j$ are real and have positive self-intersection and by consideration of types $(2,0)$ and $(0,2)$ $\omega_j,\ \rho_j$ are linearly independent over the real numbers. Thus we have the conclusion that

$$b_2^+ \geq 2p_g.$$

Lemma 3: Let $\{\psi_j\}_{j=1}^{n}$ be a collection of holomorphic 1-forms that are linearly independent over the complex numbers. Then the collection of 1-forms $\psi_j,\ \overline{\psi_j}$ are d-cohomologically independent.

Proof: Assume there exists constants $\beta_j,\ \gamma_j$, such that

$$\sum_{j=1}^{n} \beta_j \psi_j + \sum_{j=1}^{n} \gamma_j \overline{\psi_j} = df. \tag{1}$$

We will now show all the constants must necessarily vanish. This is the meaning of d-cohomologically independent. From Lemma 1 the forms ψ_j are closed, thus locally we can write,(as ψ_j is a $(1,0)$ form)

$$df_j = \psi_j = \partial f_j.$$

Thus locally we have from (1),

$$\sum \beta_j f_j + \sum_j \gamma_j \overline{f_j} = f + c \tag{2}$$

where c is some constant. Next we observe that f is a harmonic function. Apply the operator $\overline{\partial}\partial$ to both sides of (2). On applying ∂ to (2) we get,

$$\partial f = \sum_j \beta_j \psi_j.$$

Now we apply $\overline{\partial}$ and use the fact that ψ_j is holomorphic. We conclude that,

$$\overline{\partial}\partial f \equiv 0.$$

Thus f is a harmonic function on the compact surface S. We conclude using the maximum principle that f has to be a constant. Thus from (1) again, we have,

$$\sum_j \beta_j \psi_j + \sum_j \gamma_j \overline{\psi_j} \equiv 0.$$

By type considerations we conclude,

$$\sum_j \beta_j \psi_j \equiv 0, \ \sum_j \gamma_j \overline{\psi_j} \equiv 0.$$

We thus conclude by the hypothesis of linear independence that β_j, γ_j all vanish. This concludes the proof.

Lemma 4: We always have,

$$b_1 \leq 2q.$$

Proof: We consider the sequence, ($\mathbf{C}$ is the vector space of complex numbers)

$$0 \to \mathbf{C} \to \mathcal{O} \to d\mathcal{O} \to 0.$$

By the long exact sequence of cohomology we get,

$$\to H^0(S, d\mathcal{O}) \to H^1(S, \mathbf{C}) \to H^1(S, \mathcal{O}) \to$$

Thus we have by elementary linear algebra,

$$b_1 = \dim H^1(S, \mathbf{C}) \leq \dim H^0(S, d\mathcal{O}) + \dim H^1(S, \mathcal{O}) = \dim H^0(S, d\mathcal{O}) + q. \tag{3}$$

But by Lemma 3,

$$2 \dim H^0(S, d\mathcal{O}) \leq b_1.$$

Thus plugging in the last inequality in (3) we get,

$$b_1 \leq \frac{b_1}{2} + q.$$

It follows that,

$$b_1 \leq 2q.$$

We now recall two fundamental facts.

Theorem(Max Noether): For S compact, complex,

$$\frac{c_1^2(K_S) + c_2(S)}{12} = \chi(\mathcal{O}) = 1 - q + p_g.$$

Theorem(Hirzebruch Index Theorem): For **any** compact, complex surface S,

$$b_2^+ - b_2^- = \frac{1}{3}(c_1^2(K_S) - 2c_2(S)).$$

We are now in a position to prove our main theorem. Using the theorem of Max Noether and Hirzebruch we eliminate $c_1^2(K_S)$ between the two. We get,

$$c_2 = 4\chi(\mathcal{O}) - (b_2^+ - b_2^-).$$

But

$$c_2 = 2 + b_2 - 2b_1, b_2 = b_2^+ + b_2^-.$$

So combining the last two identities,

$$2 - 2b_1 + b_2^+ + b_2^- = 4(1 - q + p_g) - b_2^+ + b_2^-.$$

Simplifying and re-arranging terms in the identity above we get,

$$(b_2^+ - 2p_g) + (2q - b_1) = 1$$

But the expressions in each bracket above is non-negative by Lemma 2 and Lemma 4 and also they are integral. Thus we either have,

$$b_2^+ = 2p_g + 1, \; b_1 = 2q$$

or

$$b_2^+ = 2p_g, \; b_1 = 2q - 1$$

In the second case since $q = h^{0,1}$ it follows that $h^{1,0} = q - 1$.

The remaining assertions of the theorem follow from the theorem of M. Noether. For example from the M. Noether theorem,

$$c_1^2(K_S) + c_2 = c_1^2(K_S) + 2 + b_2 - 2b_1 = 12(1 - q + p_g) \tag{4}$$

Now in case b_1 is even $b_1 = 2q$ and $b_2^+ = 2p_g + 1$. We also have in all cases $b_2 = b_2^+ + b_2^-$. Inserting this into (4) we get,

$$c_1^2(K_S) + 2 - 4q + b_2^+ + b_2^- = c_1^2(K_S) + 2 - 4q + 2p_g + 1 + b_2^- = 12 - 12q + 12p_g.$$

Simplifying we get the assertion of the theorem. We leave the case of b_1 odd to the reader.

There are several important consequences of this theorem, but we will be content to give one application.

Proposition: Assume that a compact complex surface has the Hodge diamond,

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ 0 & & b_2 & & 0 \\ & q & & q & \\ & & 1 & & \end{array}$$

Then the surface is projective algebraic.

Proof: We consider the exact sequence,

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0.$$

We have the long exact sequence of cohomology,

$$\to H^1(S, \mathcal{O}^\star) \to H^2(S, \mathbf{Z}) \to H^2(S, \mathcal{O}) = \{0\}.$$

The last equality follows because $p_g = \dim H^2(S, \mathcal{O}) = 0$. Next since b_1 is even, from our theorem

$$b_2^+ = 2p_g + 1 = 1$$

Thus there exists an element of $H^2(S, \mathbf{Z})$ with positive self-intersection. But the map from the Picard variety to $H^2(S, \mathbf{Z})$ is onto. Thus there is a line bundle L on S such that $c_1(L)^2 > 0$. This guarantees that S is projective algebraic. See [2] for instance (Theorem 8). It follows that under the assumption $c_1(L)^2 > 0$ we can construct two **algebraically** independent meromorphic functions in the function field of S and the projective algebraic property then follows by a theorem of W.-L. Chow and Kodaira [9].

Remarks: (a) Note we have established that when $b_1 = 2q$ then $b_2^+ = 2p_g + 1$. Thus the intersection form when restricted to $H^{1,1}(S)$ must have signature $(1, h^{1,1} - 1)$.

In the case when b_1 is odd, we know $b_2^+ = 2p_g$ and so the intersection form when restricted to $H^{1,1}$ must have the signature $(0, h^{1,1})$.

(b) If we blowup surfaces at points, we add a (-1) curve. Thus b_2 changes but b_2^+ does not. Furthermore q also does not change. Since we have obtained formulae connecting p_g to b_2^+ in both Kähler and non-Kähler situations, we conclude that for any compact, complex surface, p_g and q are also bi-rational invariants.

Surfaces that Carry No Meromorphic Functions

Our aim in this section is to understand and classify those compact, complex surfaces that carry no meromorphic functions. This classification will be based on the numerical invariants we have found in the previous section. We begin with a lemma which is really a variant of a lemma we proved in the section on Castelnuovo rationality.

Lemma: Assume that S carries no meromorphic function and is minimal. Assume $p_g \neq 0$. Then $c_1^2(K_S) = 0$.

Proof: Since $p_g \neq 0$, it follows that $K_S \geq 0$, that is it is an effective divisor. So we can write

$$K_S = \sum_i a_s C_i, a_i > 0,$$

and a_i natural numbers. Note if K_S is trivial the lemma is already proved. Thus the non-trivial situation to consider is the one above.

Next we show K_S is nef. If not, there is an effective divisor C which we can take to be one of the C_i above such that,

$$K_S \cdot C_i < 0.$$

Next we see right away that,

$$(K_S - a_i C_i) \cdot C_i = C_i \cdot \sum_{j \neq i} a_j C_j \geq 0.$$

But this means

$$K_S \cdot C_i - a_i C_i^2 \geq 0,$$

hence,

$$a_1 C_i^2 \leq K_S \cdot C_i < 0.$$

Thus by the adjunction formula, we obtain that $C_i^2 = -1$, and C_i a rational curve. Hence C_i is an exceptional curve. But this contradicts the hypothesis that S is minimal. Thus K_S is nef. But K_S is also effective. We conclude that

$$K_S^2 \geq 0.$$

But if we assume that $K_S^2 > 0$ we can use as before the result in [9] and conclude that S is projective algebraic and hence carries a meromorphic function. This is a contradiction. Hence,

$$K_S^2 = 0.$$

To proceed further, we need one more Proposition.

Proposition: If a surface S has no non-trivial meromorphic function, then $p_g \leq 1$ and $h^{1,0} \leq 2$. Note we make no assumption here that the first Betti number b_1 is odd or even.

Proof: Note that if $p_g \geq 2$ and if s_1, s_2 are two linearly independent sections of $H^2(S, \mathcal{O})$, then s_1/s_2 is a meromorphic function. Thus if S carries no meromorphic function, necessarily $p_g \leq 1$.

Next assume that $h^{1,0} \geq 3$. Let ϕ_1, ϕ_2, ϕ_3 be three linearly independent $(1,0)$ holomorphic forms. Then we consider the $(2,0)$ forms,

$$\omega_1 = \phi_1 \wedge \phi_2, \ \omega_2 = \phi_1 \wedge \phi_3.$$

If ω_1, ω_2 are linearly independent, then a meromorphic function can be constructed by considering

$$\frac{\omega_1}{\omega_2}.$$

Thus we may assume that ω_1 ω_2 are dependent. So there are non-zero constants, c_1, c_2 such that,

$$c_1\omega + c_2\omega_2 = \phi_1 \wedge (c_1\phi_2 + c_3\phi_3) \equiv 0.$$

But then this means that,

$$\phi_1 = f(c_2\phi_2 + c_3\phi_3)$$

where f is a non-trivial meromorphic function. f cannot be a constant, since then we would have ϕ_1, ϕ_2, ϕ_3 are dependent. Our proposition is proved.

We now split further discussion into two cases.

Proposition: Let b_1 the first Betti number be even for the surface S. Let S carry no meromorphic function. Then $h^{1,0} = 0$ in which case S is a K3 surface, that is the canonical bundle of S is trivial, or $h^{1,0} = 2$, in which case S is a complex torus. The case $h^{1,0} = 1$ is ruled out.

Proof: From the proposition at the beginning of this section, we know that the possible values of $h^{1,0} = 0, 1, 2$. We will rule out $h^{1,0} = 1$. Let ϕ be a basis vector for $H^{1,0}(S, \mathbf{C})$. Then we consider the Albanese map

$$\mathcal{A} : S \to \left(\int_{\gamma_1} \phi, \int_{\gamma_2} \phi \right)$$

where γ_1, γ_2 are the two cycles that generate the homology of S. It is here we are using the fact that b_1 is even and in particular $b_1 = 2$. Thus we have set up a map into the complex 1-dimensional torus. On this torus one can select the Weierstrass function $\wp(\cdot)$ as a meromorphic function. Then we set,

$$F(z, w) = \wp \circ \mathcal{A}(z, w).$$

The function F is now a meromorphic function on S. Thus we cannot have $h^{1,0}(S) = 1$.

We now consider $h^{1,0} = 0$. We will now show that the canonical bundle is trivial and this will thus satisfy the definition of the surface S being K3. See the section on K3 surfaces in these notes. Note we cannot have $p_g = 0$. For as proved above in the case b_1

is even if $p_g = 0$, the surface is projective algebraic and hence will have a meromorphic function. Thus $p_g = 1$, and in view of the Lemma above $c_1^2(K_S) = 0$. We now apply Riemann-Roch to the line bundle $2K_S$. We get using $K_S^2 = 0$

$$h^0(2K_S) + h^0(-K_S) \geq \frac{1}{2}(4K_S^2 - 2K_S^2) + 2 = 2.$$

Now $h^0(2K_S)$ cannot be greater than 2. For then we could take two linearly independent sections s_1, s_2 of $2K_S$ and form a meromorphic function by considering s_1/s_2. Thus we have

$$h^0(-K_S) = 1$$

That is $-K_S$ is effective, $-K_S \geq 0$. We already know $K_S \geq 0$, and so putting the two together, K_S is trivial. That is the surface has $q = 0$ and K_S is trivial, it is a K3 surface.

We now consider the remaining case $h^{1,0} = 2$. Let $\phi_1,\ \phi_2$ be two linearly independent holomorphic 1-forms. Then

$$\phi_1 \wedge \phi_2$$

cannot vanish identically. For if it did then,

$$\phi_1 = f\phi_2$$

where f is a meromorphic function which S does not carry. We now again resort to the Albanese map,

$$\mathcal{A} : S \to \left(\int_{\gamma_j} \phi_1, \int_{\gamma_j} \phi_2 \right),\ j = 1, 2, 3, 4.$$

We stop here and leave to the reader to follow the rest of the details in [10] to show that the Albanese map is biholomorphic onto the complex torus(Theorem 5.3 in [10]).

Caution: We emphasize here that K3 surfaces can carry meromorphic functions and also can have elliptic fibrations.

We now turn to the case of b_1 odd that is the non-Kähler case.

Proposition: Let b_1 be odd. Let S carry no meromorphic functions. Then $h^{1,0} = 0,\ q = 1$ and $p_g = 0$.

Proof: Again the only possibilities are $h^{1,0} = 0, 1, 2$. We will not show how to rule out $h^{1,0} = 1$. We refer the reader to Kodaira [2]. First we rule out $h^{1,0} = 2$. Now we can construct a $(2, 0)$ form using the basis vectors of $H^{1,0}(S)$ as before. Thus we conclude $p_g \neq 0$ and in particular $p_g = 1$ since S carries no meromorphic function. The fact that $p_g = 1 \neq 0$ also yields $c_1^2(K_S) = 0$. From our discussion on numerical invariants we know that $q = 3$ since we are in the odd b_1 case. We also know,

$$c_1^2(K_S) + 8q + b_2^- = 10p_g + 8 = 18$$

Thus,

$$0 + 24 + b_2^- = 18$$

We get $b_2^- = -6$. This is a contradiction.

Thus we are left with $h^{1,0} = 0$ and since b_1 is odd we have $q = 1$. We are only left to check that $p_g = 0$. By contradiction assume that $p_g = 1$. We have using the long exact sequence of cohomology,

$$0 \to H^1(S, \mathbf{Z}) \to H^1(S, \mathcal{O}) \to H^1(S, \mathcal{O}^\star) \to H^2(S, \mathbf{Z}) \to H^2(S, \mathcal{O}) \to$$

But $b_1 = 1$ and so we get from above,

$$0 \to \mathbf{Z} \to \mathbf{C} \to H^1(S, \mathcal{O}^\star) \to H^2(S, \mathbf{Z}) \to$$

We now consider line bundles L on S such that $c_1(L) = 0$. These are exactly those line bundles that are in the kernel of the map into $H^2(S, \mathbf{Z})$, i.e in the kernel of the Chern map c_1. Since we are dealing with exact sequences, we conclude that such line bundles are characterized by points in the cylinder,

$$\mathbf{C}/\mathbf{Z}.$$

In particular there are uncountably many points and thus uncountably many line bundles with vanishing Chern class. Let L be such a line bundle. From the Riemann-Roch theorem we conclude, using $q = 1$ and $p_g = 1$, and since $c_1(L) = 0$ we know $L^2 = K_S \cdot L = 0$ and so,

$$h^0(L) + h^0(K_S - L) \geq \frac{1}{2}(L^2 - K_S \cdot L) + 1 = 1.$$

Thus either L or $K_S - L$ has a section. Since we have uncountably many such L we conclude the cardinality of the set of linear systems on S is **uncountable**. But Theorem 5.1 in [10] states that if a surface S carries no meromorphic function then it contains only **finitely** many irreducible curves. Thus we can create only countably many linear systems out of these finitely many curves by taking integral linear combinations of this finite set of curves. Thus we have arrived at a contradiction and hence $p_g = 0$. This ends the proof.

The Surface of Enriques, A first Introduction

Castelnuovo wished very much to remove the hypothesis $P_2 = 0$ and replace it with just $p_g = P_1 = 0$. The first example that this cannot be done was provided by his student Federigo Enriques who constructed the now famous Enriques surface. It has the numerical invariants $q = 0$, $P_{2j} = 1, j \geq 1$ and $P_{2j+1} = 0, j \geq 0$. Notice that since $P_1 = 0$, by our previous remark, it is projective algebraic but it cannot be rational as it's plurigenera do not vanish. In fact it is a surface whose Kodaira dimension is 0. It has also an elliptic fibration and thus provides an example of a surface which is elliptic and whose Kodaira dimension $\kappa < 1$. The construction of this surface and many of its marvelous properties will take us far afield. We have sketched a construction in the notes for K3 surfaces. The construction we have followed is **not** the original construction of Enriques. See [3], [4] for more details.

We will end by proving in a general context why the Enriques surface has these numerical invariants. We have

Proposition: Assume that S is not a rational surface and has the Hodge diamond as displayed above. Then $P_2 = 1$ and $P_3 = 0$. In particular for the Enriques surface the statement made above is proved for the first three plurigenera.

Proof: In view of Castelnuevo's theorem and since S is not rational, we may assume that $P_2 \neq 0$.

Next we recall an elementary lemma in [4] Lemma VIII.1, part (c) which is stated slightly differently than what we use but the proof is the same. The Lemma states that if $P_m \neq 0$ and $P_n \neq 0$ then $P_d \neq 0$, where $d = \gcd(m, n)$.

Thus $P_3 = 0$. For if $P_3 \neq 0$, since $P_2 \neq 0$ it will follow $P_1 = p_g \neq 0$ and this contradicts our assumption. Now we apply Riemann-Roch with $L = 3K_S$, to get

$$h^0(3K_S) + h^0(-2K_S) \geq h^1(3K_S) + \chi(\mathcal{O}) + 3K_S^2 \geq 1.$$

As before we used Serre duality to show

$$h^2(3K_S) = h^0(-2K_s).$$

But $P_3 = h^0(3K_S) = 0$. So from Riemann-Roch we get,

$$h^0(-2K_S) \geq 1.$$

Thus $-2K_S$ has a section and thus,

$$-2K_S \geq 0,$$

and because $2K_S \geq 0$, it follows that $2K_S \equiv 0$. Thus 1 is the only global section of $2K_S$ and so $P_2 = 1$. This ends the proof.

The last order of business is to compute the numerical invariants of the Enriques surface. We will begin by proving

Lemma: For the Enriques surface, we have

$$K_S^2 = c_1^2(K_S) = 0.$$

Proof: We already know that K_S is effective and nef. Thus we already have

$$K_S^2 \geq 0.$$

So assume by contradiction that $K_S^2 > 0$. We apply Riemann-Roch to the line bundle NK_S to get,

$$h^0(NK_S) + h^0((1-N)K_S) \geq \frac{N(N-1)}{2} K_S^2 + \chi(\mathcal{O}),$$

where we have used Serre duality as before. Since $K_S^2 > 0$ the right side grows as N^2. Notice that for large N, we must have

$$h^0((1-N)K_S) = 0.$$

If not then we have a section and we can by moving in the linear system assert that,

$$(1-N)K_S \geq 0.$$

For large N, this means

$$K_S \leq 0.$$

This coupled with $K_S \geq 0$ since K_S is effective yields that, K_S is a trivial bundle. Thus we conclude $p_g = 1$ and this is a contradiction. Thus we conclude from Riemann-Roch that,

$$h^0(NK_S) \geq cN^2.$$

But this again contradicts that for Enriques surfaces, P_j is either 1 or 0 and cannot grow quadratic polynomially. Thus $K_S^2 = 0$. **QED**

Lemma: The Euler characteristic of the Enriques surface is 12.

Proof: This follows from the Max Noether formula, that is

$$\frac{c_1^2(K_S) + c_2(S)}{12} = \chi(\mathcal{O}).$$

Since $c_1^2(K_S) = 0$, and $\chi(\mathcal{O}) = 1$ it follows that $c_2(S) = 12$. This ends the proof.

Thus the Hodge diamond of the Enriques surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Enriques surface is a complex surface and so is a four dimensional manifold. A picture of a slice in three dimensions, showing the tetrahedron in the original construction of Enriques is available at the link:

http://www2.mathematik.uni-mainz.de/alggeom/docs/en/en34.big.lines.jpg

We discussed a little bit about Kodaira dimension 1 and why the fibers should be elliptic. Here I shall prove a slightly weaker result. That is I will assume there is a **single** meromorphic function and then show the fibers of this object have to be elliptic. I shall make some assumptions along the way so that we do not get stuck on technicalities.

Lemma: Assume that the transcendental dimension is exactly 1, that is the function field has only one linearly independent meromorphic function. Then the fibers of this function are necessarily elliptic curves.

Proof: We consider a map Φ from the surface S to CP^1 given by,

$$\Phi(z) = [f(z) : g(z)]$$

where the meromorphic function is $f(z)/g(z)$. We can remove indeterminacy by blowing up S at points of indeterminacy. Let us assume that we do not have to blow up for simplicity. Next by using Stein factorization, assume that the map Φ is to a smooth curve C with **connected** fibers. Let F be such a fiber. We want to show F is elliptic. For this we will apply the adjunction formula. First we claim, that

$$F \cdot F = 0.$$

Notice we can deform F to F' by setting

$$F = \Phi^{-1}(z_0),\ F' = \Phi^{-1}(z_0').$$

Then F, F' are disjoint. So $F \cdot F' = 0$. Thus our claim follows. Now we claim,

$$K_S \cdot F = 0.$$

Note we are already working under the assumption that K_S is nef from the remarks made earlier. So we have,

$$K_S \cdot F \geq 0.$$

So assume that,

$$K_S \cdot F > 0.$$

Then we consider for large n,

$$(K_S + nF)^2 = K_S \cdot K_S + nK_S \cdot F > 0.$$

But now we have a divisor $L = K_S + nF$ such that,

$$c_1^2(L) > 0.$$

But from what we discussed in class, this means that the surface S is projective algebraic and has **two** linearly independent meromorphic functions. This is a contradiction. Next we apply the adjunction formula.

$$c_1(K_F) = K_S \cdot F + F \cdot F.$$

The right side vanishes and $c_1(K_F) = 2g_F - 2 = 0$. So $g_F = 1$ which proves our claim. In practice we have the argument for the blownup surface and then we need to consider the fibers in the original surface and argue. This is a little bit of extra work which we do not display.

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Elementary Geometry of K3 Surfaces

The name K3 surfaces was given by Andre Weil in a whimsical report to the AFOSR(Air Force Office of Scientific Research) in 1954 the year the mountain in the Karakoram range of the Himalayas, K2 was climbed. He named these surfaces *ainsi nommes en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire*, (see AF 18 (603)-57).

They are the prime examples of Calabi-Yau surfaces and by the theorem of Yau, one can impose a Kähler metric on them with vanishing Ricci curvature. Thus they are important from the viewpoint of String Theory and Mirror symmetry etc. They are also deeply tied to Enriques surfaces being their universal covers with a finite fundamental group $\mathbf{Z}_2$. In this short note we will derive some elementary properties of these surfaces. One can consult the notes of David Morrison [1] for more details. The notes are freely available online.

Definition: A smooth compact complex surface S is called a K3 surface iff $q = 0$ and the canonical bundle K_S is trivial, i.e

$$K_S \equiv C.$$

Thus we have $c_1^2(K_S) = 0$ and hence by Noether,

$$\frac{c_1^2 + c_2}{12} = 2$$

Thus $c_2 = 24$ and so $h^{1,1} = 20$. Thus the Hodge diamond for K3 surfaces is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Note that because

$$b_2^+ = 2p_g + 1 = 3,$$

the intersection form when restricted to $H^{1,1}((S, C)$ has one positive eigenvalue and 19 negative eigenvalues. That is, it is of the form $(1, 19)$. In fact the intersection form is

$$\mathbf{E}_8 \oplus \mathbf{E}_8 \oplus A \oplus A \oplus A$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There can be algebraic and also **non-algebraic** K3 surfaces and it is easy to construct them. There are also K3 surfaces that have elliptic fibrations. These occur in a fundamental theorem proved by Kodaira(Theorem 13) in [3] on deformation of K3 surfaces which will be briefly mentioned below. Some examples.

Example 1. The quartic in CP^3.

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0.$$

This is by far the most important example in String Theory. We compute using adjunction, since we are in CP^3, we have(for H the hyperplane bundle in CP^3)

$$c_1(K_S) = -4H + S|_S = -4d + d^2 = 0$$

Off course this does not quite show K_S is trivial, but this is left as an exercise.

Example 2. The complete intersection of a cubic and a quadric in CP^4. We can induct on the adjunction formula. Let S' be the cubic say. Then

$$K_{S'} = -5H + S'.$$

Now we apply adjunction once again using the quadric. Then on the surface S the intersection of the cubic and quadric we have,

$$c_1(K_S) = -5 + 3 + 2 = 0.$$

Example 3. The intersection of three quadrics in CP^5.

Kummer Surface: We shall construct this surface and the Enriques surface together as their construction is similar. The Kummer surface is a K3 surface.

There were 4 fundamental problems as posed by Andreotti and Weil for K3 surfaces. These were:

(a) Every K3 surface is Kähler. This was solved by Yum-Tong Siu in 1983. See the proof by N. Buchdahl [2].

(b) Every K3 surface is the smooth deformation of the quartic of Example 1 above. This was solved by Kodaira, see Theorem 13 in [3].

(c) The period map is surjective. Solved partially by Jayant Shah and in generality by Todorov.

(d) A form of the global Torelli theorem holds. Solved by Piateskii-Shapiro and Shafarevitch.

K3 surfaces have a moduli space of 20 dimensions. This moduli space is foliated by countable strata each of 19 dimensions corresponding to the algebraic K3 surfaces. We can in principle see this in many cases. Let us compute the 20 dimensions for the moduli space.

Lemma: The moduli space for K3 surfaces is 20 dimensional and complete in the sense of Kodaira-Spencer.

Proof: We wish to compute based on the work in Deformation theory of Kodaira and Spencer, the dimension

$$h^1(S, \Theta)$$

where Θ is the tangent sheaf of holomorphic vector fields. We apply Serre duality, always remembering that K_S is trivial. We get,

$$h^1(S, \Theta) = h^1(S, \Lambda^{1,0}) = h^{1,1}(S, C) = 20.$$

Also note that,

$$h^2(S, \Theta) = h^0(S, \Lambda^{1,0}) = h^{1,0}(S, C) = 0.$$

Thus the moduli space is complete and has no obstructions.

Let us now assume that there is a curve C of genus g on the K3 surface. Let us also assume that $g \geq 2$. We have,

Proposition: (a) $C^2 = 2g - 2$

(b) $h^0(C) = g + 1$ and the linear system is base point free. Thus we always have a map

$$\Phi : S \to CP^g$$

for any K3 surface containing a curve of genus g.

We shall be able to say more in the sequel.

Proof: (a) This is the adjunction formula and we also use the fact that K_S is trivial. Thus,

$$2g - 2 = K_S C + C^2 = C^2.$$

Next to show (b) we use Riemann-Roch. We first note that

$$h^2(C) = h^0(K_S - C) = h^0(-C) = 0$$

Next we have,

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S(1) \to \mathcal{O}_C(1) \to 0.$$

Thus by the long exact sequence of cohomology we have, using $h^1(S, \mathcal{O}_S(1)) = q = 0$, and where $\mathbf{C}$ denotes the vector space of Complex numbers,

$$0 \to \mathbf{C} \to \mathbf{C} \to H^1(-C) \to \{0\}.$$

The second map is injective and so surjective and so $H^1(-C) = \{0\}$. Thus by Serre duality,

$$h^1(C) = h^1(K_S - C) = h^1(-C) = 0$$

Thus by Riemann-Roch,and part (a),

$$h^0(C) = \frac{1}{2}C^2 + 2 = g - 1 + 2 = g + 1.$$

We now show the sections are base point free. To show this fact we recall a fundamental fact about Riemann surfaces. That is the canonical map, Ψ on a Riemann surface C,

$$\Psi : C \to CP^{g-1}$$

is always base point free for $g \geq 2$ and always an embedding for $g \geq 3$ provided C is **not** hyperelliptic. Also recall **every** Riemann surface of genus $g = 2$ **is** always hyperelliptic. Now we restrict the sections we found for the surface S earlier to C and we find that it is the canonical embedding and therefore the map Φ has no base points. To see this we again recall K_S is trivial and consider,

$$0 \to \mathcal{O}_S(C) \to O_C(1) \to 0$$

Tensoring with K_S we get from the long exact sequence,

$$H^0(S, C) \to H^0(C, K_S + C|_C) \to H^1(S, \mathcal{O}) = \{0\}$$

But by adjunction formula $K_C = K_S + C|_C$ and so the map from sections of S to sections of the canonical map of K_C is onto and so $|C|$ cuts out on C the linear system of the canonical map.

Thus we also have the following

(c) If $g = 2$, since the Riemann surface is hyperelliptic and the canonical map is a 2:1 map to CP^1, the surface S is mapped onto CP^2 by a 2:1 map. $g + 1 = 3$ so there are 3 sections so CP^2.

(d) If $g \geq 3$ and the curve C is not hyperelliptic, then the map Φ is birational morphism into CP^g. Thus in particular such K3 surfaces are algebraic.

As we will see later specific choices of complex torii, which do not embed into complex projective space in view of their arithmetic properties will give rise to non-algebraic Kummer surfaces and ths non-algebraic K3 surfaces.

However let us show that the results above can be used to compute the moduli and we shall see this is 19 dimensional. That is

Proposition: (a) For every genus $g \geq 2$, there exists a K3 surface which contains a curve with genus g.

(b) The moduli space of K3 surfaces containing a curve of genus g is exactly 19.

We will not prove this in full generality and will certainly not prove (a). But we will show the computation for $g = 2, 3$ for (b). Note that we have shown the moduli space for K3 surfaces is 20 dimensional. For each genus g there is a 19 dimensional moduli space and this is a countable strata. Thus we begin to get some rough picture of the moduli space of K3 surfaces.

Proof: Let us first take $g = 2$. The canonical map maps to CP^1. We have observed this is a 2:1 map. So we apply Riemann-Hurwitz and get the number of branch points. We have,with $b_1 = 2$

$$2g_C - 2 = 2 = 2(-2) + \sum_i (b_1 - 1)$$

So $i = 6$, that is there are 6 branch points. Thus the K3 surface is the double cover of CP^2 branched in a sextic. Now we compute the dimension of homogeneous monomials in 3 variables(we are in CP^2) of degree 6 and find this is exactly 28. One of the parameters can be scaled away or divided out, so sextics depend on 27 parameters. But there is also a $PGL(3, C)$ action on CP^2. This corresponds to 8 parameters. Thus the number of free parameters, the moduli is $27 - 8 = 19$.

Let us do the same for $g = 3$ and non-hyperelliptic. Then the map Φ is a birational map into CP^3. The canonical map embeds into CP^2. By the adjunction formula, the degree of the curve we get in CP^2 is d given by,

$$g = \frac{(d-1)(d-2)}{2} = 3$$

So $d = 4$. This is our old friend the **quartic**. The curve of degree 4 is the intersection of a hyperplane say with the image of S in CP^3 and so the K3 surface S has been embedded as a degree 4 surface in CP^3. We can compute the moduli. We need to compute the dimension of homogeneous monomials in 4 variables in degree 4. The number of parameters is 35. We are homogeneous so there are really 34 parameters. We subtract the action of $PGL(4, C)$ on CP^3. This is 15 parameters. We get 34-15=19 parameters again.

The Kummer Surface: We now construct this surface. We take $W = C/Z + iZ$ and consider $U = W \times W$. We may view

$$U = S^1 \times S^1 \times S^1 \times S^1.$$

And on U we consider the involution

$$(\theta_1, \theta_2, \theta_3, \theta_4) \to (-\theta_1, -\theta_2, -\theta_3, -\theta_4).$$

Notice for each S^1 we have 2 fixed points, at $0, \pi$. Thus there are **16** points to the involution action. We blow up U at these 16 points. The resulting surface is smooth as we will see. We take the resulting surface and again mod out by the involution. The resulting surface is compact, smooth and a K3 surface called the Kummer surface. This is the origin of why Weil mentioned Kummer. To see why the blowup is smooth. Assume the blowup is performed at the origin. Then, for $(z, w) \in U$ we can take $z, w/z$ as local coordinates and then $z^2, w/z$ can be taken to be local coordinates near the blowup point and which is invariant under the involution. Set $w = tz$. Now the form

$$dz \wedge dw$$

is invariant under the involution. Set $w = tz$ and we get the 2-form above is,

$$zdz \wedge dt = d(z^2) \wedge dt.$$

Also on U the holomorphic 1-forms are dz, dw which are not invariant under the involution. Thus on the blownup surface there cannot be any holomorphic 1-forms. Likewise there is holomorphic 2-form that we displayed above. The divisor of this 2-form is concentrated on the (-1) curves of the blow-up. But there the form is smooth and so the canonical bundle of the blownup surface is trivial. We have our K3 surface. Note if the original torus cannot be embedded into projective space(there is the theorem of Lefschetz that uses the Riemann theta function to embed Jacobians into projective space) then the K3 surface we have constructed will not embed into projective space and such Kummer surfaces will not be algebraic. Thus there are K3 surfaces that are not algebraic, though all are Kähler.

Enriques Surface: The idea is similar to the construction above for Kummer surfaces. We start however with the K3 surface the quartic in CP^3. Note the two changes in sign which makes for some ease in computation.

$$X_0^4 + X_1^4 - X_2^4 - X_3^4 = 0.$$

We consider the map,

$$\sigma : [X_0 : X_1 : X_2 : X_3] \to [X_0, \imath X_1, -X_2, -\imath X_3]$$

σ has 4 fixed points at $[0,0,0,1], [0,0,0,1], [0,1,0,0]$ and $[1,0,0,0]$. σ^2 has two fixed lines

$$l_1 = (X_0 = X_2 = 0),\ l_2 = (X_1 = X_3 = 0).$$

These two lines intersect the quartic in 8 points, 4 points each. We blow up the quartic in these 8 points and then apply σ^2 again to the blownup surface. This is the Enriques surface. By using coordinates similar to the Kummer surface, one can show the Enriques surface is smooth. Since for the quartic $q = 0$(it being K3), it will follow that the blown up surface has no holomorphic 1-forms. We can show that $P_1 = 0$ and $P_2 = 1$. In fact it is enough to show $P_2 \neq 0$ to show $P_2 = 1$ by the discussion above. We shall be brief here. We know the quartic is a K3 surface. One can write down the generator of $H^2(S, \mathcal{O})$ for the quartic S as follows. We leave the details to the reader. Take the fixed point, $[1,0,1,0]$ and write our quartic here as

$$f(x,y,z) = 1 + x^4 - y^4 - z^4 = 0.$$

where,

$$x = X_1/X_0,\ y = X_2/X_0,\ z = X_3/X_0.$$

One can check then the generator is

$$\phi = \frac{dx \wedge dz}{\frac{\partial f}{\partial y}} = \frac{dx \wedge dz}{4y^3}.$$

Thus this form ϕ is not invariant under σ^2, but $\phi \otimes \phi$ is. This has to be rigorously checked in the blownup coordinates and we leave this to the reader. That is we check this in the coordinates,

$$x = x', z = z'x'$$

$$dx = dx',\ dz = z'dx' + x'dz',\ dx \wedge dz = x'dx' \wedge dz'.$$

The pullback of ϕ under the projection from the blowup is

$$\frac{dv \wedge dz}{2((1 + v^2 - z'^4 v^2)^{3/4})},\ v = x'^2.$$

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Elliptic Fibration of Compact, Complex Surfaces

We start with a basic definition:

Definition An elliptic surface is a smooth, compact, complex surface S equipped with a holomorphic and surjective map Π to a smooth, compact Riemann surface B,

$$\Pi : S \to B$$

such that the general fiber of Π is a smooth elliptic curve, that is the curve is a Riemann surface of genus 1. We will also assume in the sequel that all fibers of Π are **connected**. We make no restriction that S is algebraic.

Note by Bertini's theorem [1], which is a holomorphic version of Sard's theorem, we can conclude that Π has only finitely many singular values $z_0, z_1, \cdots z_n \in B$.

$$\Pi^{-1}(z_i)$$

is called a singular fiber. Kodaira classified the singular fibers in [2] and our goal is to show how this can be done. We will only show the major steps in the proof of the classification and leave a portion of the combinatorial arguments that go into the classification to the reader.

Remark: Let Π be a proper holomorphic and surjective map to a compact Riemann surface B.

$$\Pi : S \to B.$$

Then the Stein factorization theorem and Zariski Main theorem [1] states that there is a finite to one holomorphic map F and a proper holomorphic, surjective map G such that

$$\Pi = F \circ G$$

and

$$F : \tilde{B} \to B, \ G : S \to \tilde{B},$$

where $\tilde{B}$ is the normalization of B and where the fibers of G are **connected**. Thus the assumption in our definition that the fibers of Π are connected and B is smooth are not restrictions.

We now study the adjunction formula. More precisely we want a consequence of the adjunction formula for singular curves. We will forego all algebraic methods and use just Analysis. Given a curve C, we consider it's normalization or non-singular model $\tilde{C}$. More specifically we have a curve C on a surface S. For a sufficiently fine covering by open sets U_j on S we can represent C in each open set by an equation,

$$R_j(z, w) = 0. \tag{1}$$

We also have a map μ_j

$$\mu_j : \tilde{C} \to C \cap U_j,\ \mu_j(t) = (\mu_1(t), \mu_2(t))$$

and thus we have,

$$R_j(\mu_1(t), \mu_2(t)) = 0.$$

We can apply the Weierstrass preparation theorem and re-write (1) as,(we are assuming that $p_j = (0,0)$ is a point on C and U_j is centered at $p_j = (0,0)$)

$$w^n + a_{n-1}(z)w^{n-1} + \cdots a_0(z) = 0, \tag{2}$$

with $a_i(z)$ holomorphic. Then by the Newton-Puiseux theorem we can find the roots of (2) by a Puiseux series and thus take,

$$z = \mu_i(t) = t^m,\ w = \mu_2(t) = \phi(t).$$

Thus locally the curve C can be written via a Puiseux series as

$$w = \phi(\exp(\frac{2\pi\imath}{m} z^{1/m}),\ \imath = \sqrt{-1}.$$

Since we have,

$$R_i(t^m, \phi(t)) = 0$$

by taking differentials of the eqn. above we see that,

$$mt^{m-1}\frac{\partial R_i}{\partial z}(t^m, \phi(t))dt + \phi'(t)\frac{\partial R_i}{\partial w}(t^m, \phi(t))dt \equiv 0.$$

We re-write this as,

$$\frac{mt^{m-1}}{\frac{\partial R_i(t^m,\phi(t))}{\partial w}}dt = -\frac{\phi'(t)}{\frac{\partial R_i(t^m,\phi(t))}{\partial z}}dt = \sigma_{p_j}.$$

We may again re-write the 1-form σ_{p_j} as,

$$\sigma_{p_j} = t^{-c_j}(c_0 + c_1 t + c_2 t^2 + \cdots)dt. \tag{3}$$

Note the curve C is singular at p_j if and only if $c_j > 0$.

Definition: The divisor,

$$\sum_j c_j p_j,$$

is called the conductor.

We know want to use the Gauss curvature equation on the non-singular curve $\tilde{C}$ with singular curvature. For our purposes it is enough that $\tilde{C}$ is $CP^1 = S^2$. We may identify the points p_j with points on S^2 and abuse notation by calling the points on S^2 as p_j again. We thus wish to solve,

$$-\Delta_{S^2} u + 1 = K(x) e^{2u} + 2\pi \sum_j c_j \delta_j, \tag{4}$$

where δ_j is the Dirac delta function. The function $K(x)$ is the Gauss curvature of the curve C away from the finitely many singular points p_j. We integrate both sides of (4) over S^2 and use the Gauss-Bonnet theorem, we get,

$$4\pi = 2\pi(2 - 2g_C) + 2\pi \sum_j c_j, \tag{5}$$

where g_C is the virtual genus of C given by the adjunction formula,

$$2g_C - 2 = K_S \cdot C + C^2,$$

where K_S is the canonical bundle. We see easily from (5) that,

$$g_C = \frac{1}{2} \sum_j c_j \tag{6}$$

This is the adjunction formula we desire. Lastly we want to really see that the c_j in (6) are the **same** quantities as (3) above. To see this, notice we may use (3) and introduce a singular metric on $\tilde{C}$ using the 1-form σ_j and defining the metric as,

$$ds^2 = |t|^{-2c_j} |dt|^2 = e^{-2c_j \log |t|} |dt|^2 = e^{2u(t)} |dt|^2$$

Then we consider the Gauss curvature eqn. (4) and solve it locally in flat space, where we have to solve(we are only interested in the contributions to the singular part of the curvature and we are localizing near the Dirac mass so we can assume we are in flat space of R^2)

$$-\Delta u = 2\pi c_j \delta_j$$

Notice that $u(t) = -c_j \log |t|$ is indeed a solution to this equation in R^2, $t \in R^2$. Thus the coefficients of the conductor coming from (4) are indeed the same as that which appears in the adjunction formula for singular curves (6).

Lastly we want to compute an explicit example. Consider the curve

$$w^2 = z^3.$$

The Puisuex series is

$$w = z^{3/2}.$$

There is a cusp at $(0, 0)$ and the non-singular model is CP^1 and we have a parametrization

$$\mu(t) = (t^2, t^3) = (z, w).$$

We shall now compute the coefficient of the conductor. In particular we compute the 1-form σ_p, where $p = (0,0)$. Write the curve as

$$R(z,w) = w^2 - z^3 = 0$$

Then,

$$\sigma_p = \frac{2t}{\frac{\partial R(t^2,t^3)}{\partial w}} dt = t^{-2} dt.$$

Thus we see that the coefficient of the conductor $c_j = 2$ reflecting the second order contact of the cusp at $p = (0,0)$.

We now examine a singular fiber that we denote by Z_0. We may write

$$Z_0 = \sum_{i=1}^{k} n_i C_i, \tag{7}$$

where the curves C_i are irreducible, $n_i > 0$ and n_i natural numbers.

Definition: We say a singular fiber is multiple, if and only if $\gcd\{n_1, n_2, \cdots, n_k\} = m > 1$.

We emphasize that the lemma that follows is general, no assumption is made on the type of the curves C_i. The lemma is sometimes called Zariski's lemma. First we form a vector space V by taking linear combinations over the rationals of the curves C_i. We now consider the intersection form $Q(\cdot,\cdot)$ over V. We have,

Lemma 1: Assume S is minimal, and the algebraic dimension of the function field $K(S)$ is $0, 1$. Then,

(a) The intersection form Q is negative semi-definite.

(b) The kernel of Q is 1-dimensional and spanned by the fiber Z_0.

(c)If in (7), $k > 1$, then $C_i^2 = -2$ for all i and each C_i is **rational**. If $k = 1$, then $C_1^2 = 0$.

Proof: We first prove (a) by contradiction. Assume there is an element of V, call it D, such $D^2 > 0$. Then S is necessary projective algebraic. The algebraic index theorem holds(see the section on numerical invariants). Since Z_0 is a fiber we can move it, so $Z_0^2 = 0$ and also

$$Z_0 \cdot D = 0.$$

Thus by the algebraic index theorem since $Z_0^2 = 0$, we obtain that the singular fiber Z_0 is homologous to zero, contradicting (7). Thus there is no such D in V with $D^2 > 0$ and (a) is established.

We now establish (b). First note that because

$$Z_0 \cdot C_i = 0,$$

because we can move fibers, Z_0 is in the kernel of Q. Now let D be another element in the kernel of Q. Then by contradiction we may write,

$$D = R + \alpha Z_0,\ \alpha \in \mathbf{Q}.$$

We split R further,

$$R = P - N$$

where both P and N are divisors written using positive rational linear combinations of the C_i and the curves C_i occurring in P and N are disjoint. Since D is in the kernel and so is Z_0, it follows that

$$(D - \alpha Z_0)^2 = (P - N)^2 = 0. \tag{8}$$

Thus,

$$(P - N)^2 = P^2 + N^2 - 2P \cdot N = 0.$$

But from (a), $P^2 \leq 0,\ N^2 \leq 0$. Now we recall that the fiber Z_0 is **connected**. Thus $P \cdot N > 0$. Thus we conclude that

$$(P - N)^2 < 0,$$

contradicting (8). Thus R must be zero and we have (b).

If $k = 1$, then C_1 is in the kernel from (b) and so $C_1^2 = 0$. If $k > 1$, then $C_1, C_2, \cdots, C_k$ cannot be in the kernel of Q and so Q is negative definite on C_i and so right away,

$$C_i^2 < 0.$$

Obviously we cannot have $K_S \cdot C_i < 0$ for then by the adjunction formula, C_i is a (-1) curve and rational. S is minimal and cannot contain such curves. Under the assumption that the algebraic dimension is $0, 1$, we know that the canonical bundle is nef from the section on Rationality. Thus we must have $K_S \cdot C_i \geq 0$ for every curve C_i. But Z_0 is a fiber and so we also have

$$0 = K_S \cdot Z_0 = \sum_i n_i K_S \cdot C_i.$$

But $n_i > 0$ and $K_S \cdot C_i \geq 0$. Thus we conclude for every C_i,

$$K_S \cdot C_i = 0.$$

Alternatively, note that $K_S^2 = 0$, since K_S is nef and $K_S^2 > 0$ cannot happen, since the algebraic dimension is $0, 1$. If $K_S \cdot C_i > 0$ for some C_i, then for a large enough n, $(nK_S + C_i)^2 > 0$. Then the algebraic dimension of $K(S)$ will be 2, which contradicts our assumptions.

Thus by the adjunction formula,

$$K_S \cdot C_i + C_i^2 = C_i^2 = 2g_{C_i} - 2 < 0.$$

But then $g_{C_i} = 0$ and $C_i^2 = -2$.

We are now ready to begin the classification that mainly proceeds via a combinatorial argument based on (6) and the lemma above. As a consequence to Lemma 1, we have Lemma 2, which is the main identity for the combinatorial arguments that follow.

Lemma 2: Assume $k > 1$. Then we have,

$$n_iC_i^2 + \sum_{j \neq i} n_jC_j \cdot C_i = 0. \tag{9}$$

Proof:. We know that,

$$0 = Z_0 \cdot C_i = n_iC_i^2 + \sum_{j \neq i} n_jC_j \cdot C_i.$$

This proves the lemma.

Using (c) of Lemma 1 in Lemma 2, we re-write (9) in the form we will use,

$$2n_i = \sum_{j \neq i} n_jC_j \cdot C_i. \tag{10}$$

The modern way is to use (10) to associate a Dynkin diagram to the vector space V and do the combinatorial argument, see for example the presentation by Demazure [3]. This is just an elegant way to doing the bare hands combinatorial argument of Kodaira. We shall exhibit a few steps in this direction. We also direct the reader's attention to the Dynkin diagram associated to the cases we consider(see the page on Figures).

The Dynkin Diagram: Though we will not follow this path, we indicate how a Dynkin diagram is associated.

1. Each curve C_i is represented as a vertex on a graph.
2. Since our fiber is connected, each vertex is connected to the graph by a number of edges, and there can be self-loops connecting a vertex to itself.
3. We create edges based on the intersection number as follows. It is enough to write the intersection numbers of the curves C_i.

Now set,

$$C_i^2 = -2 + \{\text{number of self} - \text{loops at vertex i}\}$$

Since $C_i^2 = -2$, for $k > 1$, the self-loops only occur when there is just one vertex in the graph and there are no self-loops when the number of vertices exceeds 1.

We now prescribe the rule to assign edges in the Dynkin diagram.

$$C_i \cdot C_j = 0, \text{ if the two curves do not meet.}$$

That is no edge connects the vertex C_i to the vertex C_j if the intersection number of the curve C_i with C_j is zero.

If $C_i \cdot C_j = s$, we connect the vertex C_i and C_j with s edges.

The combinatorial lemma in Demazure [3] is that under the rules listed above for constructing the Dynkin diagram, there is a list for Dynkin diagrams that can appear and the classification of singular fibers is then complete.

Now let us do a piece of the combinatorial argument without recourse to the Dynkin diagram. For example we will see

Lemma 3: In the Dynkin diagram when $k = 2$ then **only** two edges connect the vertices. See Fig 2. When $k > 2$, if a vertex is connected to another one, then it is through a single edge. That is the intersection number can be only $0, 1$. In all cases the intersection number cannot be 3 or more.

Proof: Let us show as a first step, that the intersection number $C_i \cdot C_j \leq 2$. Assume that there is a pair for which $C_i \cdot C_j \geq 3$. Assume wlog that $n_i \leq n_j$. Then applying (10) we get,

$$2n_i = n_j C_i \cdot C_j + \sum_{s \neq j,i} n_s C_s \cdot C_i.$$

But the right side by hypothesis is $\geq 3n_j > 2n_i$. Thus the identity above cannot hold. Thus for all pairs $C_i \cdot C_j \leq 2$.

Case 1: Types mI_0, mI_1, II

Z_0 consists of a single curve C_1. We know $C_1^2 = 0$. We know that,

$$K_S \cdot C_1 = 0,$$

thus by adjunction $2g_C - 2 = 0$ and $g_C = 1$. Thus C is an elliptic curve, or by using the adjunction formula (6), a rational curve with conductor value $c = 2$. That is a rational curve with a cusp or node. See Dynkin diagram Fig 1. This ends the analysis. This is Kodaira's Type II fiber(one cusp), and if $Z_0 = mC_1$ and C_1 non-singular elliptic, type mI_0 and if C_1 rational and non-singular then type mI_1.

Case 2: Type mI_2, III.

We now show that if for a pair $C_1 \cdot C_2 = 2$, then $Z_0 = mC_1 + mC_2$. If the two curves meet at a single point p, then $m = 1$.

See Dynkin diagram 2. Let us now consider the case of **at least two** irreducible curves in Z_0. Assume wlog that n_1 is the smallest number in the decomposition (7). We apply (10). Then by the connected property, there exists j, such that $C_1 \cdot C_j \geq 1$. We have by (10),

$$2n_1 = n_j C_1 \cdot C_j + \sum_{s \neq 1, s \neq j} n_s C_s \cdot C_1.$$

But n_1 is the smallest number. If $C_1 \cdot C_j = 2$, then $n_1 = n_j$ and $C_1 \cdot C_s = 0$. We then apply (10) again with C_j and get,

$$2n_j = n_1 C_1 \cdot C_j + \sum_{s \neq j, s \neq 1} n_s C_s \cdot C_j.$$

Thus we conclude that $C_j \cdot C_s = 0$. Thus the curves C_1 and C_j do not meet the curves C_s which is a contradiction since the fibers are connected. Thus if one pair has a self-intersection of 2, then Z_0 simply consists of two rational curves C_1 and C_2 with $n_1 = n_2 = m$, that is

$$Z = mC_1 + mC_2. \tag{11}$$

Thus if there are 3 or more than three vertices there can be **at most** one edge connecting the vertices. If there are just two vertices, then they must be joined by a double edge, Fig 2. Two cases now arise. The first one where the curves C_1, C_2 meet in two points p_1, p_2 with $C_1 \cdot C_2 = 1$ at each point, and Z_0 is represented by (11) and is thus a fiber with multiplicity m. This is the case of Type mI_2 fibers.

The second possibility is that the curves $C_1,\ C_2$ meet at one point p with $C_1 \cdot C_2 = 2$. In this case when the curves meet at a single point p, we must have then $n_1 = n_2 = 1$ because in this case the fiber is simply connected and so by the lemma in the appendix it follows $n_1 = n_2 = 1$. Thus in this case

$$Z_0 = C_1 + C_2.$$

This is the Kodaira Type III fiber. See Fig 3. Incidentally note we have also **proved** Lemma 3.

So from now on we assume that $C_i \cdot C_j \leq 1$. Our goal is now to use (10) and use a combinatorial argument and reduce the situation further to graphs that are **not** cyclic.

Case 3: Type IV.

If three curves meet at a single point p, we show,

$$Z_0 = C_1 + C_2 + C_3.$$

Let us re-label the curves $C_1,\ C_2,\ C_3$ and assume $n_1 \leq n_2 \leq n_3$. Then we have from (10),

$$2n_1 = n_2 C_1 \cdot C_2 + n_3 C_1 \cdot C_3 + \sum_{s \neq 1,2,3} n_s C_1 \cdot C_s. \tag{12}$$

But since $C_1 \cdot C_2 = 1,\ C_1 \cdot C_3 = 1$, we must have $n_1 = n_2 = n_3$ and $C_s \cdot C_1 = 0$ if $s \neq 2, 3$. We now re-write (12) with C_2 on the lhs to get,

$$2n_2 = 2n_1 + 2n_3 + \sum_{s \neq 1,2,3} n_s C_2 \cdot C_s.$$

We already know $n_1 = n_2 = n_3 = m$ and so we again conclude that $C_2 \cdot C_s = 0,\ s \neq 1, 2, 3$. Thus the curves C_1, C_2, C_3 do not meet the rest violating the connectedness of the fiber. Next if the three curves meet at a single point, the fiber is simply connected. Thus by the lemma in the appendix $n_1 = n_2 = n_3 = 1$. This proves our claim.

Thus we are now reduced to considering graphs that have single edges, and any vertex is connected to only two others. We now explore the possibility of a cyclic chain. Fig 4.

Case 4: Type mI_b.

We assume that $C_i \cdot C_{i+1} = 1,\ i = 1, 2, \cdots, b-1$ and $C_1 \cdot C_b = 1$, where we can assume wlog $n_1 \leq n_s$ for $s = 1, 2, \cdots, b$. Then applying (10) again we have,

$$2n_1 = n_2 + n_b + \sum_{s \neq 1,2,b} n_s C_s \cdot C_1.$$

It follows that $n_1 = n_2 = n_b$ and $C_1 \cdot C_s = 0$, if $s \neq 1, 2, b$. We also have

$$2n_2 = n_1 + n_3 + \sum_{s = \not{1},2,3} n_s C_s \cdot C_2.$$

But $n_2 = n_1 \leq n_3$. So we again conclude that $n_1 = n_2 = n_3 = n_b$ and $C_s \cdot C_2 = 0,\ s \neq 1, 3$. Thus in the cyclic case,

$$Z_0 = \sum_s m C_s,\ m = n_1$$

and $C_i \cdot C_{i+1} = 1,\ C_{i-1} \cdot C_i = 1$ and the intersection numbers of C_i with the other curves is 0. This gives rise to Fig. 4.

Thus we can now assume that our graph is not cyclic. The remaining cases are Types $I_b^\star$, $II^\star$, $III^\star$, $IV^\star$, $I_0^\star$ and $I_1^\star$. We leave to the reader to see how to exhaust the combinatorial possibilities for the Dynkin diagrams either by following the approach of Demazure [3] or the approach of Kodaira [2].

APPENDIX

Lemma: Assume the fiber Z_0 is simply connected. Then the multiplicity m of the fiber is 1.

Proof: Let the fiber correspond to a singular value a_0 of the map Π. Let Δ denote a small disk on B centered at a_0. We will construct a holomorphic function h on $\Pi^{-1}(\Delta)$, such that,

$$h^m = \tau \circ \Pi$$

where τ is a map on Δ that assigns a local uniformizing parameter on B. If such a map exists, then for a non-singular value $a \in \Delta$ of Π we conclude that $\Pi^{-1}(a)$ must consist of m connected components. We are assuming that m divides all n_i where

$$Z_0 = \sum_i n_i C_i.$$

We cover Z_0 by open sets U_k such that if $U_k \cap U_l$ is non-empty then $Z_0 \cap U_k \cap U_l$ is non-empty. On each U_k we can find a holomorphic function h_k such that,

$$h_k^m = \tau \circ \Pi$$

We also have,

$$h_k = c_{kl} h_l,$$

with $c_{kl}^m = 1$. c_{kl} determine a co-cycle and since we are reasoning the assumption of simple-connectedness we have,

$$c_{kl} = c_k / c_l.$$

We simply set $h = c_k h_k$ to obtain our function h.

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[3] Demazure, M; A, B, C, D, E, F, etc, in *Seminaire sur les singularites de surfaces*, Springer Lecture Notes in Math., **777** (1982), 222-227.

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Fig 1.

(Dynkin)
$\tilde{A}_0$ (Type mI_0, mI_1)

Type II

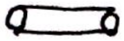
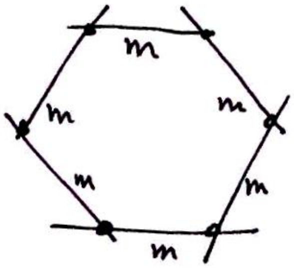


Fig. 2

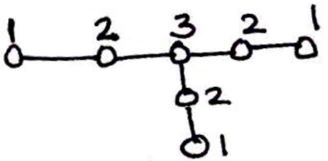
(Dynkin) (Type mI_2, Type III)
$\tilde{A}_1$

Type IV

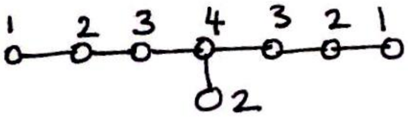


Type ${}_mI_b$ (here $b=6$).

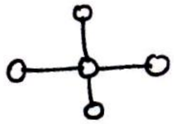
The above cases are fully discussed in these notes.



$\tilde{E}_6$ (7 smooth rational curves, Type IV*)



$\tilde{E}_7$ (8 smooth rational curves, Type III*)

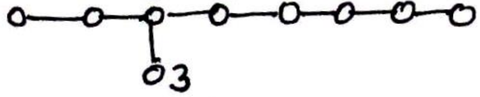


$\tilde{D}_4$ [illegible]

$\tilde{D}_N$, [illegible], N+1 vertices.

Type I_N^*, N+5 smooth [illegible] rational curves, with $N \geq 0$.

$\tilde{E}_8$



9 smooth rational curves, Type II*

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