These notes are designed to supplement Calculus: Early Transcendentals, the third edition, by Rogawski and Adams. Calculus 1 students tend to have a lot of trouble with related rates, so I've written these notes in an attempt to simplify the topic. The algorithm outlined in these notes is not the only way to go about these problems, nor is it a standard step-by-step process used by other instructors or Mathematicians. It is merely an outine of my own mental process when solving a related rates problem. My hope is that by writing down the steps I take explicitly, I can make the process more clear to students who are having trouble.

1 The Procedure

1.1 Step 1: Draw a Picture

When solving a related rates problem, we are almost always modeling some change happening in the world around us. As a result, it is often *extremely* helpful to draw a picture. If you do, you should always remember to label your diagram with variables (some of which may be filled in with constants in step 3).

This step is not always absolutely necessary, however. In fact, it may sometimes be impossible, for example if you are modeling some physical phenomenon that you are given the equation for (see example 3).

1.2 Step 2: Write Down an Equation (or two)

Eventually, we will be taking derivatives. Therefore, we need an equation to take a derivative of. If you've drawn a picture, the equation will likely come from that picture. In some cases, the problem gives you the equation outright. This is often the case if you are modeling a physical phenomenon (see example 3).

1.3 Step 3: Write Down What is Constant

While related rates problems are all about things changing (and the rates at which they change), there are likely values in the problem which remain constant. We want to plug these numbers into our equation before we take our derivative. Note that at this point, the only values you should be plugging in are constants. You do not plug in any values that are at a specific point in time. This is most easily explained in examples, so I'll repeat this point in example 1.

1.4 Step 4: Take a Derivative

Now that we have an equation with all of our constants plugged in, we can take a derivative. Don't forget that we are taking $\frac{d}{dt}$, so we have to use implicit differentiation.

1.5 Step 5: Plug in Values and Solve

The ultimate question to these types of problems is nearly always something along the lines of "What is the rate of _______ when x= _______?" This is where we plug in those values. We also plug in the values of the rates that we know. Sometimes, you may need to use a secondary equation or go back to the original equation in order to find all of the necessary values. Once we have all of those plugged in, all we need to do is solve for what we want.

Let's start doing some examples!

2 Example 1: The Standard "Sliding Ladder" Problem

Suppose a 10 foot ladder is leaning against a wall. The top of the ladder is sliding down the wall at a rate of 0.5 feet per second. At what rate is the foot of the ladder sliding away from the wall, when the top of the latter is 6 feet above the floor?

2.1 Step 1: Draw a Picture

Walls are always at right angles with the floor (we like structurally sound buildings). Since the ladder has to touch both the floor and the wall, it appears we have a right triangle.

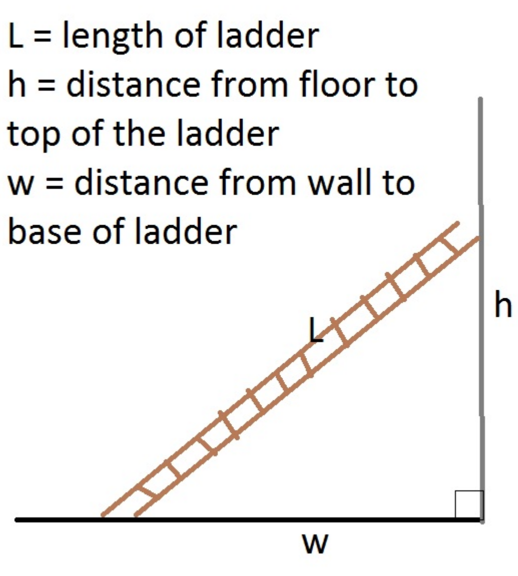

Figure 1

2.2 Step 2: Write Down an Equation or Two

Since walls and floors meet at right angles, we have a right triangle. Since this problem is asking about lengths of sides of our triangle, we use the Pythagorean Theorem.

$$w^2 + h^2 = L^2$$

2.3 Step 3: Write Down What is Constant

In this problem, we have a fixed ladder whose height is always 10. Therefore, L is constant and we can plug in $L = 10$ to our equation.

$$w^2 + h^2 = 10^2 = 100$$

Something to note here: the problem asks us to find a rate when $h = 6$. However, h is changing throughout the problem. So, we don't want to plug that in until *after* we take a derivative.

2.4 Step 4: Take a Derivative

We now take $\frac{d}{dt}$ of both sides of the equation we found in step 3, using implicit differentiation.

$$\frac{d}{dt}(w^2 + h^2) = \frac{d}{dt}(100)$$
$$2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 0$$

2.5 Step 5: Plug in Values and Solve

We want to solve for $\frac{dw}{dt}$ when $h = 6$. Looking at our equation, we need to know $\frac{dh}{dt}$ and w at this point as well. We are given $\frac{dh}{dt} = -0.5$ in the problem statement (where the negative is due to the fact that the height on the wall is shrinking). But where are we going to get w? Well, lets look at our original equation:

$$w^2 + h^2 = 100$$

Since we know what h is, we can use the above equation to find w.

$$w^2 + 6^2 = 100$$
$$w^2 = 64$$
$$w = 8$$

One little note to be aware of: generally, when we take a square root, we have to take the positive and negative roots. However, in this problem, w is a length. Therefore, it must always be positive. So, we only need the positive square root.

Now that we know all of the necessary values, we can find our final answer:

$$2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 0$$
$$2(8)\frac{dw}{dt} + 2(6)(-0.5) = 0$$
$$16\frac{dw}{dt} = 6$$
$$\frac{dw}{dt} = \frac{3}{8}$$

So, when the top of the ladder is 6 feet above the floor, the base of the ladder is sliding away from the wall at a rate of $\frac{3}{8}$ feet per second.

3 Example 2:

3.1 Problem

Suppose I have a conical tank, such as the one in figure 2 below. There is water flowing into the top of the tank at a rate of π cm^3/s. If the radius of the cone is three times the height, how fast is the height of the water increasing when there are 24π cm^3 of water in the tank?

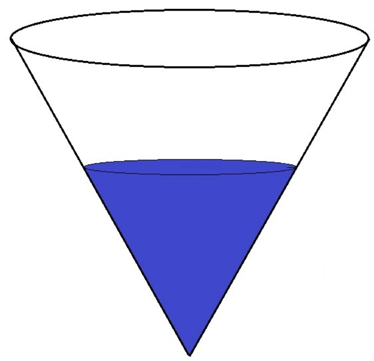

Figure 2

From this point on, I will not separate the problem into steps.

3.2 Solution

First, I begin with the figure given and add labels and variables, as shown in Figure 3 below.

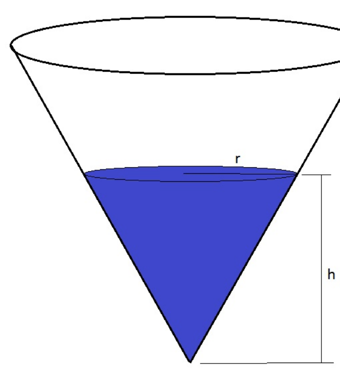

Figure 3

Then, I make the following definitions for my variables:

$$h = \text{the height of the water in the tank}$$
$$r = \text{the radius of the surface of the water in the tank}$$
$$V = \text{the volume of the water in the tank.}$$

From geometry, we know what the formula for volume of a cone is:

$$V = \pi r^2 \tfrac{h}{3}$$

We are also given in the problem that $r = 3h$. We can either plug this in now and then take a derivative, or take a derivative of both equations and then make our substitutions. Either option will get you to the correct answer. Note that if I leave both r and h in my equation for volume, I'll have to use product rule. So, I'll make the substitution now in order to make my computation easier.

$$\begin{aligned} V &= \pi r^2 \tfrac{h}{3} \\ &= \pi (3h)^2 \tfrac{h}{3} \\ &= 3\pi h^3 \end{aligned}$$

Now, we can take $\frac{d}{dt}$ of both sides.

$$\tfrac{dV}{dt} = 9\pi h^2 \tfrac{dh}{dt}.$$

The problem now wants to know what $\frac{dh}{dt}$ is when $V = 24\pi$. The problem told us that $\frac{dV}{dt} = \pi$, but we also need to know h. We can go back to our formula for volume, plug in $V = 24\pi$, and solve for h:

$$\begin{aligned} h &= \sqrt[3]{\tfrac{V}{3\pi}} \\ &= \sqrt[3]{\tfrac{24\pi}{3\pi}} \\ &= \sqrt[3]{8} \\ &= 2 \end{aligned}$$

We can now plug this in to our derivative and solve for our desired value.

$$\begin{aligned} \tfrac{dh}{dt} &= \tfrac{\frac{dV}{dt}}{9\pi h^2} \\ &= \tfrac{\pi}{9\pi 2^2} \\ &= \tfrac{1}{36} \end{aligned}$$

Therefore, the height of the water is increasing at a rate of $\frac{1}{36}$ cm/s when the volume is π cm^3.

4 Example 3: Ideal Gas Law

4.1 Problem

Recall from general chemistry the ideal gas law:

$$PV = nRT$$

where P stands for pressure (in pascals), V stands for volume (in m^3), n is the amount of substance (in moles), $R = 8.314$ is a constant known as Avogadro's constant, and T is temperature (in kelvins).

1. Suppose I have a solid box (whose volume does not change) and a fixed amount of some gas inside of that box. Suppose that when the temperature inside the box is $10K$, the pressure is $2Pa$.

   Now, I put the box over a Bunsen burner and increase the temperature at a rate of $3K/s$. Is the pressure increasing or decreasing? At what rate?

2. Now, suppose my box is made out of an elastic material and can therefore change in volume. When the temperature is $10K$, the pressure is $6Pa$ and the volume is $5m^3$. As the tempurature increases at a rate of $20K/s$, the pressure increases at a rate of $0.6Pa/s$. When the temperature reaches $30K$ and the pressure reaches $9Pa$, is the volume increasing or decreasing? At what rate?

4.2 Solution

(a) I first begin by rearranging the ideal gas law to reflect the information given:

$$P = \frac{nR}{V}T.$$

Since my volume doesnt change and I have a fixed amount of gas, I know that $\frac{nR}{V}$ is a constant:

$$\begin{aligned} P &= \frac{nR}{V}T \\ 2 &= \frac{nR}{V}10 \\ \frac{nR}{V} &= 5. \end{aligned}$$

Therefore, my equation becomes

$$P = 5T.$$

Taking $\frac{d}{dt}$ of both sides, I obtain

$$\frac{dP}{dt} = 5\frac{dT}{dt}.$$

Now, substituting in $\frac{dT}{dt} = 3$, I see that $\frac{dP}{dt} = 15$. Therefore, the pressure is increasing at a rate of $15Pa/s$.

(b) In this problem, volume is allowed to change, but I still have a fixed amount of gas in my container. So, nR is a constant:

$$PV = nRT$$
$$(6)(5) = (nR)(10)$$
$$30 = (nR)10$$
$$3 = nR.$$

Therefore, my equation is

$$PV = 3T.$$

I can now take $\frac{d}{dt}$ of both sides. Since both pressure and volume change over time, I must use product rule to take my derivative:

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 3\frac{dT}{dt}.$$

I want to know $\frac{dV}{dt}$ at a particular time. The problem gives me values for $\frac{dP}{dt}$, $\frac{dT}{dt}$, P, and T. In order to find V to plug into the derivataive, I must go back to my original equation:

$$\begin{aligned} V &= \frac{3T}{P} \\ &= \frac{3(30)}{9} \\ &= 10 \end{aligned}$$

Now, I can plug everything into the derivative and solve.

$$(9)\frac{dV}{dt} + (10)(0.6) = 3(20)$$
$$9\frac{dV}{dt} = 60 - 6$$
$$\frac{dV}{dt} = \frac{54}{9} = 6$$

Therefore, when the temperature reaches $30K$ and the pressure reaches $9Pa$, the volume is increasing at a rate of $6m^3/s$.