Induction Example: Stylistic Choices

Math-300 Summer 2017

Below are four different presentations of a proof of the following theorem. Note that the logic behind each of them is exactly the same, and all are correct proofs. The differences between them are entirely stylistic, and you may choose to present your proofs in whichever form you prefer. This is by no means an exhaustive list. It's just meant to illustrate some of the possibilities.

Theorem 0.1. *For all $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2 \tag{1}$$

Proof. Base case: Observe that

$$\sum_{k=1}^{1} 2^k = 2^1 = 2 = 4 - 2 = 2^{1+1} - 2.$$

Hence, the result holds for $n = 1$.

Inductive Step: Now, suppose the result holds for some $n \in \mathbb{N}$. Then,

$$\begin{aligned}\sum_{k=1}^{n+1} 2^k &= \sum_{k=1}^{n} 2^k + 2^{n+1} \\ &= 2^{n+1} - 2 + 2^{n+1} \\ &= 2(2^{n+1}) - 2 \\ &= 2^{n+2} - 2.\end{aligned}$$

Hence, the result holds for $n + 1$ as well. Therefore, by the principle of mathematical induction, the result holds for all natural numbers. $\square$

Proof. Let P(n) be the statement that equation (1) holds for the natural number n.

Base case: Observe that $\sum_{k=1}^{1} 2^k = 2^1 = 2$ and that $2^{1+1} - 2 = 4 - 2 = 2$, so $P(1)$ is true.

Inductive step: Suppose $P(n)$ is true for some $n \in \mathbb{N}$. Then, we want to show that $P(n + 1)$ is also true.

$$
\begin{aligned}
\sum_{k=1}^{n+1} 2^k &= \sum_{k=1}^{n} 2^k + 2^{n+1} \\
&= 2^{n+1} - 2 + 2^{n+1} \\
&= 2(2^{n+1}) - 2 \\
&= 2^{n+2} - 2.
\end{aligned}
$$

So by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

$\square$

Proof. Let P(n) be the statement that equation (1) holds for the natural number n.

Base case: Observe that $\sum_{k=1}^{1} 2^k = 2^1 = 2$ and that $2^{1+1} - 2 = 4 - 2 = 2$, so $P(1)$ is true.

Inductive step: Suppose $P(N)$ is true for some $N \in \mathbb{N}$. Then, we want to show that $P(N+1)$ is also true.

$$
\begin{aligned}
\sum_{k=1}^{N+1} 2^k &= \sum_{k=1}^{N} 2^k + 2^{N+1} \\
&= 2^{N+1} - 2 + 2^{N+1} \\
&= 2(2^{N+1}) - 2 \\
&= 2^{N+2} - 2.
\end{aligned}
$$

Hence, $P(1)$ is true and $P(N) \Rightarrow P(N+1)$, so by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Proof. Base case: Observe that for $n = 1$, the left hand side of equation (1) is $\sum_{k=1}^{1} 2^k = 2^1 = 2$. Furthermore, the right hand side of equation (1) is $2^{1+1} - 2 = 4 - 2 = 2$. So, equation (1) is true for $n = 1$..

Inductive step: Suppose the result holds for some $n \in \mathbb{N}$. Then, observe that the left-hand side of equation (1) for $n+1$ is

$$
\begin{aligned}
\sum_{k=1}^{n+1} 2^k &= \sum_{k=1}^{n} 2^k + 2^{n+1} \\
&= 2^{n+1} - 2 + 2^{n+1} \\
&= 2(2^{n+1}) - 2 \\
&= 2^{n+2} - 2.
\end{aligned}
$$

Observe that the last line of the equality matches the right hand side of equation (1) for $n+1$. Hence, by the principle of mathematical induction, equation (1) is true for all natural numbers.

$\square$