LINEAR ALGEBRA 350
SPRING 2020

01:640:350:H1 – MW4 01:40–3:00pm, Hill 009, Busch Campus

Instructor: Claire Burrin, claire.burrin@rutgers.edu, Hill Center 232
Office hour: Mondays, 11-12am, Hill 232

Textbook: Not required. (The standard textbook at Rutgers is *Linear Algebra*, Friedberg, Insel, and Spence, 4th Ed., Prentice Hall, 2003, if you wish to check it out for further references, and additional exercises.)

Prerequisites: CALC4, Math 250, Math 300

Final: May 12, 12:00-3:00pm

The course is heavily based on Math 250, and starting early in the course, we will be reviewing and applying the methods of Math 250. However, we will work axiomatically, starting from the abstract notions of vector space and of linear transformation. From this more abstract viewpoint, we’ll be developing linear algebra far beyond Math 250, with new insight and new applications. Although rewarding, the level of abstraction makes 350 a challenging class. You will be expected to follow and be able to explain the proofs discussed in class, as well as construct proofs of your own in the homework. For this reason, 300 is an important requirement for this class. As this is a 4-credits course, be prepared to spend 8-10 hours per week on the course material.

Finally, note that it is your responsibility to stay informed of any announcement, syllabus or policy adjustments made during class.

Tentative syllabus.

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<td>From geometry to linear algebra</td>
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<td>Jan 27, 29</td>
<td>Vector spaces, linear combinations</td>
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<td>Feb 3, 5</td>
<td>Base and dimension</td>
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<td>Feb 10, 12</td>
<td>Linear transformations and isomorphisms</td>
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<td>Feb 17, 19</td>
<td>Matrix representations and change of coordinates</td>
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<td>Feb 24, 26</td>
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<td>Mar 2, 4</td>
<td>Determinants and their properties</td>
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<td>Mar 9, 11</td>
<td>Eigenvalues, eigenvectors, diagonalizability</td>
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<td>Mar 16, 18</td>
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<td>Mar 23, 25</td>
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<td>Mar/Apr 30, 1</td>
<td>Jordans forms, minimal polynomial</td>
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<td>Apr 6, 8</td>
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<td>Apr 13, 15</td>
<td>Inner product spaces, orthonormal bases</td>
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<td>Apr 20, 22</td>
<td>Normal and self-adjoint operators</td>
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<td>Apr 27, 29</td>
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<td>May 4</td>
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**Evaluation.** The grade make-up is 20% Homework, 20% each midterm, 40% the final. There will be no quizzes. The homework assignments are all taken from the textbook. Your solutions are
expected to show some original effort; solutions that are directly copied from the internet will be sanctioned. Remember that your solutions are to be read – and graded! – by someone. Therefore, you should make sure to explain your reasoning as clearly as possible, and to write neatly. Anyone absent from an exam without medical proof will receive a zero score. Exceptional circumstances will require a letter from a Dean. Students are expected to behave in accordance with the Code of Academic Integrity [http://academicintegrity.rutgers.edu/academic-integrity-policy](http://academicintegrity.rutgers.edu/academic-integrity-policy). Cases of cheating will be immediately reported.

**Homework assignments.**

**Homework 1, due Feb 3**

1. Are the following statements true or false. Justify your answers.
   (a) In any vector space \( \lambda_1 v = \lambda_2 v \) implies that \( \lambda_1 = \lambda_2 \).
   (b) If \( f \) and \( g \) are polynomials of degree \( n \), then \( f + g \) is a polynomial of degree \( n \).
   (c) The empty set is a subspace of every vector space.
   (d) If \( V \) is a vector space other than the zero vector space, then \( V \) contains a subspace \( W \) such that \( W \neq V \).

2. Solve the following system of linear equations using Gaussian elimination.

\[
\begin{align*}
3x_1 - 7x_2 + 4x_3 &= 10 \\
x_1 - 2x_2 + x_3 &= 3 \\
2x_1 - x_2 - 2x_3 &= 6
\end{align*}
\]

Does the following system of linear equations have a solution?

\[
\begin{align*}
x_1 + 2x_2 + 2x_3 &= 2 \\
x_1 + 8x_3 + 5x_4 &= -6 \\
x_1 + x_2 + 5x_3 + 5x_4 &= 3
\end{align*}
\]

3. Equip \( V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{C}, i = 1, \ldots, n \} \) with the operations of coordinatewise addition and multiplication. Is \( V \) a vector space over \( \mathbb{C} \)? Is \( V \) a vector space over \( \mathbb{R} \)?

4. Let \( V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R} \} \). Define addition coordinatewise, and scalar multiplication for each \( \lambda \in \mathbb{R} \) by

\[
\lambda \cdot (a_1, a_2) = \begin{cases} 
(0, 0) & \text{if } \lambda = 0, \\
\left( \lambda a_1, \frac{a_2}{\lambda} \right) & \text{if } \lambda \neq 0.
\end{cases}
\]

Is \( V \) a vector space over \( \mathbb{R} \) with these operations? Justify your answer.

5. Is the set \( W = \{f(x) \in \mathcal{P}(\mathbb{F}) : f(x) = 0 \text{ or } f(x) \text{ has degree } n \} \) a subspace of \( \mathcal{P}(\mathbb{F}) \) if \( n > 1 \)? Justify your answer.

6. Let \( W_1 \) and \( W_2 \) be subspaces of a vector space \( V \). Prove that \( V \) is the direct sum of \( W_1 \) and \( W_2 \) if and only if each vector in \( V \) can be uniquely written as \( x_1 + x_2 \), where \( x_1 \in W_1 \) and \( x_2 \in W_2 \).

**Homework 2, due Feb 10**

1. An \( m \times n \) matrix \( A \) is called upper triangular if all entries lying below the diagonal entries are zero, that is, if \( a_{ij} = 0 \) whenever \( i > j \). Prove that, the upper triangular matrices form a subspace of \( M_{m,n}(\mathbb{F}) \).

2. Let \( u \) and \( v \) be distinct vectors in a vector space \( V \). Show that \( \{u, v\} \) is linearly dependent if and only if \( u \) or \( v \) is a multiple of the other.

3. Show that if \( S_1 \) and \( S_2 \) are subsets of a vector space \( V \), then \( \text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2) \), where \( + \) here denotes the sum of two subspaces, as defined in class.
(4) Is the following set linearly dependent or linearly independent?
\[
\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\} \subset M_{2,2}(\mathbb{R})
\]

(5) Is the following set a basis for \( P_2(\mathbb{R}) \)?
\[
\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}
\]

(6) Find bases for the following subspaces of \( F^5 \),
\[
W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 - a_3 - a_4 = 0\}
\]
and
\[
W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.
\]
What are the dimensions of \( W_1 \) and \( W_2 \)?