# POSITIVE MAPS ON OPERATOR ALGEBRAS AND QUANTUM MARKOV SEMIGROUPS

## An introductory course with application in quantum mechanics

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# 1 Fundamentals of the Theory of Banach Algebras

## 1.1 Basic definitions and notation

**1.1 DEFINITION** (Banach algebra)**.** A Banach algebra is an algebra $\mathscr{A}$ over the complex numbers equipped with a norm $\|\cdot\|$ under which it is complete as a metric space, and such that

$$\|AB\| \leq \|A\|\|B\| \quad \text{for all} \quad A, B \in \mathscr{A} \ . \tag{1.1}$$

**1.2 EXAMPLE.** For a locally compact Hausdorff space $X$, we write $\mathscr{C}_0(X)$ to denote the set of continuous complex valued functions on $X$ that vanish at infinity. We equip it with the supremum norm and the usual algebraic structure of pointwise addition and multiplication. Then $\mathscr{A} = \mathscr{C}_0(X)$ is a commutative Banach algebra. There is a multiplicative identity if and only if $X$ is compact.

**1.3 EXAMPLE.** Equip $\mathbb{R}^n$ with Lebesgue measure, and let $\mathscr{A}$ be the Banach space $L^1(\mathbb{R}^n)$ further equipped with the convolution product

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y \ .$$

Then $\mathscr{A}$ is a commutative Banach algebra that does not have an identity.

**1.4 EXAMPLE.** Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{A} = \mathscr{B}(\mathscr{H})$, the set of all continuous linear transformations from $\mathscr{H}$ to $\mathscr{H}$, equipped with the composition product and the operator norm

$$\begin{aligned} \|A\| &= \sup\{\ \|A\psi\|_{\mathscr{H}}\ :\ \psi \in \mathscr{H}, \|\psi\|_{\mathscr{H}} = 1\ \} \\ &= \sup\{\ \Re(\langle \varphi, a\psi\rangle_{\mathscr{H}})\ :\ \varphi, \psi \in \mathscr{H}, \|\varphi\|_{\mathscr{H}}, \|\psi\|_{\mathscr{H}} = 1\ \} \ , \end{aligned} \tag{1.2}$$

where $\|\cdot\|_{\mathscr{H}}$ is the norm on $\mathscr{H}$, and $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ is the inner product in $\mathscr{H}$. This is a canonical example of a non-commutative Banach algebra.

**1.5 EXAMPLE.** Let $\mathscr{A}$ be the algebra of $n \times n$ matrices. The *Frobenius norm*, or *Hilbert-Schmidt norm*, on $\mathscr{A}$ is the norm $\|\cdot\|_2$ given by $\|A\|_2 = \left(\sum_{i,j=1}^n |A_{i,j}|^2\right)^{1/2}$ where $a_{i,j}$ denotes the $i,j$th entry of $a$. By the Cauchy-Schwarz inequality, for all $A, B \in \mathscr{A}$,

$$\|AB\|_2 = \left(\sum_{i,j=1}^n \left|\sum_{k=1}^n A_{i,k}B_{k,j}\right|^2\right)^{1/2} \leq \left(\sum_{i,j=1}^n \left(\sum_{k=1}^n |A_{i,k}|^2\right)\left(\sum_{k=1}^n |B_{k,j}|^2\right)\right)^{1/2} = \|A\|_2\|B\|_2 \ ,$$

and thus (1.1) is satisfied. Note that the algebra of $n \times n$ matrices with the operator norm is the special case of Example 1.4 in which $\mathscr{H} = \mathbb{C}^n$, but the Frobenius norm is not the operator norm.

## 1.2 The spectrum and the resolvent set

A Banach algebra $\mathscr{A}$ that has a multiplicative identity 1 is said to be *unital*. Otherwise, $\mathscr{A}$ is *non-unital*.

Let $X$ be a locally compact Hausdorff space that is not compact. Then $\mathscr{A} = \mathscr{C}_0(X)$ equipped with the structures specified in Example 1.2 is a Banach algebra without an identity. Let $\widetilde{\mathscr{A}}$ be the larger algebra obtained by adjoining to $\mathscr{A}$ the constant functions, $\lambda 1$, $\lambda \in \mathbb{C}$. Then every $\widetilde{f} \in \widetilde{\mathscr{A}}$ has the form $\widetilde{f}(x) = \lambda + f(x)$ where $f \in \mathscr{C}_0(X)$. Then for $\lambda + f$ and $\mu + g$ in $\widetilde{\mathscr{A}}$,

$$(\lambda + f)(\mu + g) = \lambda\mu + (\lambda g + \mu f + fg) \ .$$

The constant function 1 is the multiplicative identity in $\widetilde{\mathscr{A}}$; we have adjoined an identity to $\mathscr{A}$. The procedure can be done in general.

**1.6 DEFINITION** (Canonical unital extension)**.** Let $\mathscr{A}$ be any Banach algebra, with or without a unit. Define $\widetilde{\mathscr{A}}$ to be $\mathbb{C} \oplus \mathscr{A}$ with the multiplication

$$(\lambda, A)(\mu, B) = (\lambda\mu, \lambda B + \mu A + AB) \ , \tag{1.3}$$

and the norm

$$\|(\lambda, A)\| = |\lambda| + \|A\| \ . \tag{1.4}$$

By the definitions,

$$\begin{aligned} \|(\lambda, A)(\mu, B)\| = \|(\lambda\mu, \lambda B + \mu A + AB)\| &= |\lambda\mu| + \|\lambda B + \mu A + AB\| \\ &\le |\lambda||\mu| + |\lambda|\|b\| + |\mu|\|a\| + \|a\|\|b\| \\ &= (|\lambda| + \|A\|)(|\mu| + \|B\|) = \|(\lambda, A)\|\|(\mu, B)\| \ . \end{aligned}$$

This shows that (1.1) is satisfied, and hence that $\widetilde{\mathscr{A}}$ is a Banach algebra. Now define $\mathbb{1} := (1, 0) \in \widetilde{\mathscr{A}}$. Then $(1, 0)(\lambda, A) = (\lambda, A)(1, 0) = (\lambda, A)$ so that $\mathbb{1}$ is the identity in $\widetilde{\mathscr{A}}$.

The map $A \mapsto (0, A)$ is an isometric embedding of $\mathscr{A}$ into $\widetilde{\mathscr{A}}$. None of these elements $(0, A)$ are invertible in $\widetilde{\mathscr{A}}$, even when $\mathscr{A}$ itself has an identity. Indeed, if $(\lambda, A)$ has an inverse $(\mu, b)$, then

$$(1, 0) = (\lambda, A)(\mu, B) = (\lambda\mu, \lambda B + \mu A + AB) \ ,$$

and this is impossible if $\lambda = 0$. However, it will be important in what follows that if $\mathscr{A}$ has a unit $\mathbb{1}$, then $\mathbb{1} - A$ is invertible in $\mathscr{A}$ if and only if $(1, -a)$ is invertible in $\widetilde{\mathscr{A}}$.

**1.7 PROPOSITION.** *Let $\mathscr{A}$ be a Banach algebra with unit* 1*. Then $1 + A$ is invertible in $\mathscr{A}$ if and only if there exists $B \in \mathscr{A}$ such that*

$$A + B + AB = A + B + BA = 0 \ . \tag{1.5}$$

*consequently, $1 + A$ is invertible in $\mathscr{A}$ if and only if $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$.*

*Proof.* Suppose that $1+A$ is invertible. Define $B := (1+A)^{-1} - 1$. Then $(1+A)B = 1-(1+A) = -A$, and hence $A + B + AB = 0$. The proof of $A + B + BA = 0$ is similar.

Now suppose that there exists $B \in \mathscr{A}$ such that (1.5) is true. Then

$$(1+B)(1+A) = 1 + B + A + AB = 1 \quad \text{and} \quad (1+A)(1+B) = 1 + B + A + BA = 1 \ .$$

This proves the first part.

Next, $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$ if and only if there exists $B \in \mathscr{A}$ such that $(1, A)(1, B) = (1, B)(1, A) = (1, 0)$, and by the definition of the product in $\widetilde{\mathscr{A}}$, this is the same as (1.5). □

**1.8 DEFINITION** (Spectrum and resolvent set)**.** Let $\mathscr{A}$ be a Banach algebra, and let $A \in \mathscr{A}$. If $\mathscr{A}$ has a unit, the *spectrum of* $A$ *in* $\mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ is defined to be the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - A$ is not invertible. If $\mathscr{A}$ does not have a unit, then $\sigma_{\mathscr{A}}(A)$ is defined to be the spectrum of $(0, A) \in \widetilde{\mathscr{A}}$. The resolvent set of $A$ in $\mathscr{A}$, $\rho_{\mathscr{A}}(A)$ is defined to be the complement of $\sigma_{\mathscr{A}}(A)$.

It is useful to have an intrinsic characterization of the spectrum for non-unital $\mathscr{A}$. By Proposition 1.7, for any given $A \in \mathscr{A}$, there is at most one $X \in \mathscr{X}$ such that

$$A + X + AX = A + X + XA = 0 \ , \tag{1.6}$$

and there is one solution exactly when $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$, and in that case $(1, X) = (1, A)^{-1}$ in $\widetilde{\mathscr{A}}$. This justifies the following definition:

**1.9 DEFINITION.** qregdef Let $\mathscr{A}$ be a non-unital Banach algebra. Define a map $\circ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ by

$$A \circ B = A + B + AB \ , \tag{1.7}$$

$A \in \mathscr{A}$ is em quasi-regular in case there exists $A' \in \mathscr{A}$, necessarily unique, such that $AA \circ A' = A' \circ A = 0$. in this case, $A'$ is called the *quasi inverse* of $A$. We reserve the prime to denote the quasi inverse of a quasi regular element of a non-unital Banach algebra.

**1.10 LEMMA.** *Let* $\mathscr{A}$ *be a non-unital Banach algebra. Let* $A, B \in \mathscr{A}$ *be quasi regular. Then* $A \circ B$ *is quasi regular and*

$$(A \circ B)' = B' \circ A' \ , \tag{1.8}$$

*and for* $\lambda \in \mathbb{C} \backslash \{0\}$,

$$\lambda \in \sigma_{\mathscr{A}}(A) \iff -\lambda^{-1} A \text{ is quasi regular.} \tag{1.9}$$

*In particular.*

$$\sigma_{\mathscr{A}}(A) = \{0\} \cup \{\lambda \in \mathbb{C} \backslash \{0\} \ : \ -\lambda^{-1} A \text{ is quasi regular } \} \ . \tag{1.10}$$

*Proof.* By Proposition 1.7, $A, B \in \mathscr{A}$ are quasi regular exactly when $(1, A), (1, B)$ are invertible in $\widetilde{\mathscr{A}}$, and in this case

$$((1, A)(1, B))^{-1} = (1, B)^{-1}(1, A)^{-1} = (1, B')(1, A') = 1 + B' \circ A' \ .$$

Therefore, $A \circ B$ is quasi regular, and since $(1, A)(1, B) = (1, A \circ B)$, $B' \circ A'$ is the pseudo inverse $A \circ B$. Next, for $\lambda \in \mathbb{C} \backslash \{0\}$, $(-\lambda, A)$ is invertible in $\widetilde{\mathscr{A}}$ if and only if $(1, -\lambda^{-1} A)$ is invertible in $\widetilde{\mathscr{A}}$, which is the case if and only if $-\lambda^{-1} A$ is quasi regular. □

Let $\mathscr{A}$ be a Banach algebra with a identity 1. Then we can still carry out the process of adjoining an identity to form $\widetilde{\mathscr{A}}$, and can regard each $A \in \mathscr{A}$ also as an element of $\widetilde{\mathscr{A}}$. Since no element of $\mathscr{A}$ is invertible in $\widetilde{\mathscr{A}}$, $0 \in \sigma_{\widetilde{\mathscr{A}}}(A)$ for all $A \in \mathscr{A}$. However, for $\lambda \neq 0$, $\lambda 1 - A$ is invertible if and only if $1 - A/\lambda$ is invertible. Likewise, $(\lambda - a)$ is invertible if and only if $(1, -a/\lambda)$ is invertible. Then by Proposition 1.7, $\lambda 1 - A$ is invertible in $\mathscr{A}$ if and only if $(1, 0) - (0, aA/\lambda)$ is invertible in $\widetilde{\mathscr{A}}$. This shows that for $\lambda \neq 0$, $\lambda \in \sigma_{\mathscr{A}}(A) \iff \lambda \in \sigma_{\widetilde{\mathscr{A}}}((0, A))$. We summarize:

$$\{0\} \cup \sigma_{\mathscr{A}}(A) = \sigma_{\widetilde{\mathscr{A}}}((0, A)) \ . \tag{1.11}$$

**1.11 LEMMA** (Spectral Mapping Lemma)**.** *Let $\mathscr{A}$ be a Banach algebra, and let $p$ be a polynomial. In case $\mathscr{A}$ has no identity, we suppose that $p$ has no constant term. Then for all $A \in \mathscr{A}$,*

$$p(\sigma_{\mathscr{A}}(A)) = \sigma_{\mathscr{A}}(p(A)) \ .$$

*Proof.* We may suppose that $p$ is not identically constant. We first suppose that $\mathscr{A}$ has an identity. Fix $\lambda \in \sigma_{\mathscr{A}}(A)$. We shall show that $p(\lambda)1 - p(A)$ is not invertible. The polynomial $p(\lambda) - p(z)$ has a root at $z = \lambda$, and hence

$$p(\lambda) - p(z) = (\lambda - z)q(z)$$

for some polynomial $q(z)$. Replacing $z$ by $A$,

$$p(\lambda)1 - p(A) = (\lambda - A)q(A) \ .$$

Were $p(\lambda)1 - p(A)$ invertible, we would have $1 = (\lambda - a)[q(A)(p(\lambda) - p(a))^{-1}]$, and then since polynomials in $A$ commute, $1 = [q(A)(p(\lambda) - p(A))^{-1}](\lambda - A)$. This would mean that $\lambda 1 - A$ is invertible, with contradicts our hypothesis that $\lambda \in \sigma_{\mathscr{A}}(a)$. Hence $p(\lambda) - p(A)$ is not invertible, and hence $p(\lambda) \in \sigma_{\mathscr{A}}(p(A))$. this shows that $p(\sigma_{\mathscr{A}}(A)) \subset \sigma_{\mathscr{A}}(p(A))$.

Next, fix $\mu \in \sigma_{\mathscr{A}}(p(A))$, and factor

$$\mu - p(z) = \alpha(\lambda_1 - z) \cdots (\lambda_n - z)$$

where $\alpha \neq 0$ and $n \geq 1$. For each $j$, $\mu = p(\lambda_j)$. We have

$$\mu 1 - p(A) = \alpha(\lambda_1 1 - A) \cdots (\lambda_n 1 - A)$$

and if each $\lambda_j 1 - A$ were invertible, then $\mu 1 - p(A)$ would be invertible, but this is not the case. Hence Hence for some $j$, $\lambda_j \in \sigma_{\mathscr{A}}(A)$, and $\mu = p(\lambda_j) \in \sigma_{\mathscr{A}}(p(A))$. This shows that $\sigma_{\mathscr{A}}(p(A)) \subset p(\sigma_{\mathscr{A}}(A))$, and completes th proof when $\mathscr{A}$ has an identity. The general case now follows by adjoining an identity and then appealing to (1.11). □

## 1.3 Properties of the inverse function

Now let $\mathscr{A}$ be a Banach algebra with an identity 1. Let $A \in \mathscr{A}$ be such that $\|1 - A\| = r < 1$. Then by the defining property (1.1), $\|(1 - A)^n\| \leq r^n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, define

$$S_n = \sum_{j=1}^{n} (1 - A)^j$$

where, as usual, we interpret $(1-a)^0 = 1$. Then for all $n > m$, by the triangle inequality and (1.1),

$$\|S_n - S_m\| \le \sum_{j=m+1}^{n} \|(1-A)^j\| \le \sum_{j=m+1}^{n} r^j = \frac{r^m - r^n}{r-1} \ .$$

Hence $\{S_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{A}$. Now for the first time we use the metric completeness of $\mathscr{A}$: There exists $B \in \mathscr{A}$ such that $\lim_{n\to\infty} \|B - S_n\| = 0$. But then

$$BA = \lim_{n\to\infty} S_n A = \lim_{n\to\infty} s_n(1-(1-A)) = \lim_{n\to\infty}(1-(1-A)^{n+1}) = 1 \ .$$

The same reasoning shows that $AB = 1$, and so $A$ is invertible. Let $\Omega$ denote the set of invertible elements in $\mathscr{A}$. This brings us to:

**1.12 LEMMA.** *Let $\mathscr{A}$ be a Banach algebra with a unit. Let $\Omega$ be the set of invertible elements of $\mathscr{A}$. Then $\Omega$ contains every $A \in \mathscr{A}$ such that $\|1-A\| < 1$, and in this case $A^{-1}$ is given by the convergent series*

$$A^{-1} = \sum_{j=0}^{\infty}(1-A)^j \ .$$

*Moreover, if $|\lambda| > \|A\|$, then $\lambda 1 - A$ is invertible, with*

$$\|(\lambda 1 - A)^{-1}\| \le \frac{1}{|\lambda| - \|A\|} \ . \tag{1.12}$$

*In particular, $\sigma_{\mathscr{A}}(A)$ is contained in the centered closed disk in $\mathbb{C}$ of radius $\|a\|$.*

*Proof.* It remains to prove the final part. If $|\lambda| > \|A\|$, then $\lambda 1 - A = \lambda(1 - \lambda^{-1}A)$ and

$$\|1 - (1-\lambda^{-1}A)\| = |\lambda|^{-1}\|A\| < 1 \ ,$$

so that $(1-\lambda^{-1}A)$ is invertible. □

At this point, we do not know in general that $\sigma_{\mathscr{A}}(A)$ is not empty, but we do know this of $\rho_{\mathscr{A}}(A)$. We now claim that $\Omega$ is open. This has the immediate consequence that $\rho_{\mathscr{A}}(A)$ is open, and hence that $\sigma_{\mathscr{A}}(A)$ is closed, though at this point the possibility that $\sigma_{\mathscr{A}}(a) = \emptyset$ has not yet ben eliminated.

Let $A_0 \in \Omega$ and $A \in \mathscr{A}$. Then $\|1 - AA_0^{-1}\| = \|(A_0 - A)A_0^{-1}\| \le \|A - A_0\|\|A_0^{-1}\|$ Therefore, for any $r \in (0,1)$,

$$\|A - A_0\| \le r\|A_0^{-1}\|^{-1} \quad \Rightarrow \quad \|1 - AA_0^{-1}\| \le r \quad \Rightarrow \quad AA_0^{-1} \in \Omega \ .$$

Since $\Omega$ is closed under multiplication, $A = (AA_0^{-1})A_0 \in \Omega$. This shows that for all $A_0 \in \Omega$, the open ball of radius $\|A_0^{-1}\|^{-1}$ an center $A_0$ is contained in $\Omega$. In particular, $\Omega$ is open.

The following simple identity will prove useful in what follows.

**1.13 LEMMA** (First resolvent identity)**.** *Let $\mathscr{A}$ be a Banach algebra with an identity 1. Let $\Omega$ be the set of invertible elements. For all $A, B \in \Omega$,*

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1} \ . \tag{1.13}$$

*Proof.* Simply expand the right hand side. □

Now recall that a function $F$ from a Banach space $X$ to itself is *Frechét differentiable* at $x_0 \in X$ in case there is a continuous linear transformation $L$ from $X$ to itself such that for all $x \in X$,

$$\|F(x_0 + x) - F(x_0) - Lx\| = o(\|x\|) \ ,$$

and in this case, $L$ is unique and is the *Frechét derivative* of $F$ at $x_0$. We now show that the inverse function $A \mapsto A^{-1}$ is Frechét differentiable at every $A_0 \in \mathscr{A}$, and that the derivative is the linear transformation

$$A \mapsto -A_0^{-1}AA_0^{-1} \ .$$

This is a simple consequence of an important identity that we record in a lemma:

**1.14 THEOREM.** *The map $A \mapsto A^{-1}$ is Frechét differentiable at all $A_0 \in \Omega$, and the inverse is that map $A \mapsto -A_0^{-1}AA_0^{-1}$.*

*Proof.* Let $A_0, A_0 + A \in \Omega$. Then

$$(A_0 + A)^{-1} - A_0^{-1} = -(A_0 + A)^{-1}AA_0^{-1} = -A_0^{-1}AA_0^{-1} + [A_0^{-1} - (A_0 + A)^{-1}]AA_0^{-1} \ .$$

By (1.1) once more and the continuity proved above,

$$\|[A_0^{-1} - (A_0 + A)^{-1}]AA_0^{-1}\| \leq \|A_0^{-1}\|\|A_0^{-1} - (A_0 + A)^{-1}\|\|A\| = o(\|A\|) \ .$$

Therefore

$$\|(A_0 + A)^{-1} - A_0^{-1} + A_0^{-1}AA_0^{-1}\| = o(\|A\|) \ .$$

□

We are now ready to show that for all $a$ in any Banach algebra, $\sigma_{\mathscr{A}}(A) \neq \emptyset$. Let $\varphi$ be any continuous linear functional on $\mathscr{A}$, regarded as a Banach space. Such functionals exist (and are plentiful) by the Hahn-Banach Theorem. Define a complex valued function $f$ in the resolvent set $\rho_{\mathscr{A}}(A)$ by

$$f(\zeta) = \varphi((\zeta 1 - A)^{-1}) \ .$$

Note that the resolvent set includes $\{\zeta \ : \ |\zeta| > \|A\|\}$, and that by (1.12),

$$\lim_{\zeta \to \infty} f(\zeta) = 0 \ . \tag{1.14}$$

Next, by the identity (1.13),

$$f(\zeta + \eta) - f(\zeta) = \eta\varphi[(((\zeta + \eta)1 - A)^{-1})(\zeta 1 - A)^{-1}] \ .$$

From this identity and the continuity of the inverse function, it follows that

$$\lim_{\eta \to 0} \frac{f(\zeta + \eta) - f(\zeta)}{\eta} = \varphi[(\zeta 1 - A)^{-2}] \ ,$$

which shows that $f$ is an analytic function on $\rho_{\mathscr{A}}(A)$.

If the resolvent set $\rho_{\mathscr{A}}(A)$ were all of $\mathbb{C}$, $f$ would be an entire analytic function, and on account of (1.14), $f$ would also be bounded. By Liouville's Theorem it would then be constant, and by (1.14), the constant would have to be zero. In particular, we would have $f(0) = 0$. Therefore, for every continuous linear functional $\varphi$ on $\mathscr{A}$, it would be the case that $\varphi(A^{-1}) = 0$. This contradicts the Hahn-Banach Theorem. We summarize:

**1.15 THEOREM.** *Let $\mathscr{A}$ be any Banach algebra with an identity* 1*. Then for all $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ is a nonempty closed set contained in the closed disc of radius $\|A\|$ centered at* 0 *in $\mathbb{C}$.*

It is now a simple matter to prove:

**1.16 THEOREM** (Gelfand-Mazur Theorem)**.** *Let $\mathscr{A}$ be a Banach algebra with identity* 1*. If $\mathscr{A}$ is a division algebra, then $\mathscr{A}$ is isomorphic to $\mathbb{C}$. More specifically, each element a of $\mathscr{A}$ satisfies $A = \lambda 1$ for some necessarily unique $\lambda \in \mathbb{C}$, and $A \mapsto \lambda$ is an isomorphism with $\mathbb{C}$.*

*Proof.* Suppose that $\mathscr{A}$ is a division algebra. By Theorem 1.15, there exists $\lambda \in \sigma_{\mathscr{A}}(A)$. Thus $\lambda 1 - A$ is not invertible. Since the only non-invertible element in a division algebra is 0, $A = \lambda 1$. □

**1.17 DEFINITION** (Spectral radius)**.** The spectral radius of an element $A$ of a Banach algebra $\mathscr{A}$ is

$$\nu(A) = \max\{\ |\lambda|\ :\ \lambda \in \sigma_{\mathscr{A}}(A)\ \}\ . \tag{1.15}$$

**1.18 THEOREM** (The Beurling-Gelfand Formula)**.** *The spectral radius of an element a of a Banach algebra $\mathscr{A}$ is given by*

$$\nu(A) = \lim_{n\to\infty} \|A^n\|^{1/n}\ . \tag{1.16}$$

*In particular, the limit exists.*

*Proof.* Suppose that $\lambda \in \sigma_{\mathscr{A}}(A)$. Then by the Spectral Mapping Lemma, $\lambda^n \in \sigma_{\mathscr{A}}(A^n)$, and then by Theorem 1.15 and (1.11) in case $\mathscr{A}$ lacks an identity, $|\lambda|^n \le \|A^n\|$. Taking the $n$th root, we obtain $\nu(A) \le \|A^n\|^{1/n}$ for all $n \in \mathbb{N}$.

It remains to show that

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \le \nu(A)\ . \tag{1.17}$$

To this end, pick $\lambda$ with $|\lambda| > \nu(A)$ so that $\lambda \in \rho_{\mathscr{A}}(A)$. Let $\varphi$ be any continuous linear functional on $\mathscr{A}$. Then, as in the proof of Theorem 1.15, the function $f$ defined by $f(\lambda) = \varphi((\lambda 1 - A)^{-1})$ is analytic on $\rho_{\mathscr{A}}(A)$. Define $\zeta = 1/\lambda$, $g(\zeta) = f(1/\lambda) = \zeta\varphi((1 - \zeta A)^{-1})$, which is analytic on the open disc about 0 with radius $1/\nu(A)$.

For $\zeta$ with $|\zeta| < \|A\|^{-1}$, $(1 - \zeta A)^{-1}$ has the convergent power series $(1 - \zeta A)^{-1} = \sum_{n=0}^{\infty} \zeta^n A^n$. Therefore, by the uniqueness of the power series representation, $g(z) = \sum_{n=0}^{\infty} \zeta^{n+1}\varphi(A^n)$ is a convergent power series for all $\zeta$ with $|\zeta| \le 1/\nu(A)$. It follows that for all such $\zeta$, $\lim_{n\to\infty} \zeta^{n+1}\varphi(A^n) = 0$. In particular, there exists a finite constant $C_\varphi$ such that

$$|\zeta^{n+1}\varphi(A^n)| \le C_\varphi \quad \text{for all} \quad n \in \mathbb{N}\ . \tag{1.18}$$

For each $n \in \mathbb{N}$ define a linear functional $\Lambda_n$ on $\mathscr{A}^*$, the Banach space dual of $\mathscr{A}$, by $\Lambda_n(\varphi) = \zeta^{n+1}\varphi(A^n)$. Then (1.19) says that

$$\sup_{n\in\mathbb{N}}\{\ |\Lambda_n(\varphi)|\} \le C_\varphi\ . \tag{1.19}$$

The Uniform Boundedness Principle then implies that there exists a finite constant $M$ such that $\|\Lambda_n\| \le M$ for all $n$, and hence for all $\varphi \in \mathscr{A}^*$ with $\|\varphi\| = 1$,

$$|\zeta|^{n+1}|\varphi(A^n)| \le M \quad \text{for all} \quad n \in \mathbb{N}$$

. The Hahn-Banach Theorem provides $\varphi \in \mathscr{A}^*$ with $\|\varphi\| = 1$ such that $\varphi(A^n) = \|A^n\|$. Hence we have $|\zeta|^{n+1}\|a^n\| \le M$. Taking the $n$th root of both sides, $|\zeta|\|A^n\|^{1/n} \le \left(\frac{M}{|\zeta|}\right)^{1/n}$. This proves that $|\zeta| \limsup_{n\to\infty} \|A^n\|^{1/n} \le 1$. However, $\zeta$ was any complex number with $|\zeta| < 1/\nu(A)$, this proves (1.17). □

We close this section with the following result that is trivial for commutative Banach algebras, and familiar for the algebra of $n \times n$ matrices.

**1.19 THEOREM** (Spectrum of $AB$ and $BA$)**.** *If $\mathscr{A}$ is a Banach algebra, then for all $A, B \in \mathscr{A}$,*

$$\{0\} \cup \sigma_{\mathscr{A}}(AB) = \{0\} \cup \sigma_{\mathscr{A}}(BA) \ . \tag{1.20}$$

*Proof.* Passing to $\widetilde{\mathscr{A}}$, we may suppose that $\mathscr{A}$ has an identity. For each $\lambda \ne 0$, we must show that $(\lambda 1 - AB)$ is invertible if and only iff $(\lambda 1 - BA)$ is invertible. Dividing through by $\lambda$, we may take $\lambda = 1$. Therefore, suppose that $(1 - AB)$ is invertible, and let $Z = (1 - AB)^{-1}$. Then

$$\begin{aligned}(1 - BA)(1 + BZA) &= 1 - BA + BZA - BABZA \\ &= 1 - BA + B(1 - AB)ZA = 1 - BA + BA = 1 \ .\end{aligned}$$

Likewise, $(1 + BZA)(1 - BA) = 1$, and so $(1 - BA)$ is invertible with inverse $(1 + BZA)$. □

**1.20 THEOREM** (Spectral Contraction Theorem)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a homomorphism. Then for all $a \in \mathscr{A}$,*

$$\sigma_{\mathscr{B}}(\pi(A)) \subset \{0\} \cup \sigma_{\mathscr{A}}(A) \ . \tag{1.21}$$

*Proof.* Adjoin identities to $\mathscr{A}$ and $\mathscr{B}$, and define $\widetilde{\pi} : \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ by $\widetilde{\pi}((1, A)) = (1, \pi(A))$. This is a homomorphism, and takes the identity in $\widetilde{\mathscr{A}}$ to the identity in $\widetilde{\mathscr{B}}$. Since adjoining an identity had no effect on non-zero spectrum, we may assume that $\mathscr{A}$ and $\mathscr{B}$ have identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ respectively, and that $\pi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$.

Now suppose that $\lambda \in \rho_{\mathscr{A}}(A)$. Then $1_{\mathscr{A}} = (\lambda 1_{\mathscr{A}} - A)(\lambda 1_{\mathscr{A}} - A)^{-1}$. Since $\pi$ is a homomorphism,

$$1_{\mathscr{B}} = \pi(1_{\mathscr{A}}) = (\lambda 1_{\mathscr{B}} - \pi(A))\pi((\lambda 1_{\mathscr{A}} - A)^{-1}) \ .$$

Thus $\pi((\lambda 1_{\mathscr{A}} - A)^{-1})$ is a right inverse of $\lambda 1_{\mathscr{B}} - \pi(A)$, and the same reasoning shows it is also a left inverse. Hence $\lambda \in \rho_{\mathscr{B}}(\pi(A))$. This shows that $\rho_{\mathscr{A}}(A) \subset \rho_{\mathscr{B}}(\pi(A)$, which is equivalent to the statement $\sigma_{\mathscr{B}}(\pi(A)) \subset \sigma_{\mathscr{A}}(A)$, and even shows that when $\mathscr{A}$ and $\mathscr{B}$ have identities and $\pi$ takes the identity in $\mathscr{A}$ to that in $\mathscr{B}$, it is not necessary to adjoin $\{0\}$ on the right side in (1.21) □

The next theorem gives a useful continuity property of the spectrum.

**1.21 THEOREM** (Newburgh's Theorem)**.** *Let $\mathscr{A}$ be a Banach algebra and $A \in \mathscr{A}$. Let $U$ be an open subset of $\mathbb{C}$ with $\sigma_{\mathscr{A}}(A) \subset U$. Then there exists a $\delta > 0$ such that if $\|B - A\| \le \delta$,*

$$\sigma_{\mathscr{A}}(B) \subset U \ .$$

*Proof.* Adjoining a unit if needed, we may assume that $\mathscr{A}$ is unital. First note that for all $B \in \mathscr{A}$ with $\|B - A\| < 1$, $\|B\| < \|A\| + 1$, and hence for all $\lambda \in \mathbb{C}$ with $\lambda \ge \|A\| + 1$, $\lambda \in \rho_{\mathscr{A}}(B)$. Hence when $\|B - A\| \le 1$, $\sigma_{\mathscr{A}}(B)$ is contained in the closed centered disc of radius $\|A\| + 1$.

Let $K = U^c \cap \{ \lambda \ : \ |\lambda| \le \|A\| + 1 \}$ which is a compact subset of $\rho_{\mathscr{A}}(A)$. It suffices to show that there is an $r > 0$ so that for all $\mu \in K$, $(\mu 1 - B)$ is invertible whenever $\|B - A\| < r$.

Let $\lambda \in K$. Then $\lambda \in \rho_{\mathscr{A}}(A)$, and for all $B \in \mathscr{A}$ and $\mu \in \mathbb{C}$ with

$$|\mu - \lambda| + \|B - A\| < \|(\lambda 1 - A)^{-1}\|^{-1} \quad \Rightarrow \quad \|(\mu 1 - B) - (\lambda 1 - A)\| \le \|(\lambda 1 - A)^{-1}\|^{-1} \ ,$$

and hence $\mu \in \rho_{\mathscr{A}}(B)$. For each $\lambda \in K$, define $U_\lambda = \{\mu \ : |\mu - \lambda| < \frac{1}{2}\|(\lambda 1 - A)^{-1}\|^{-1} \ \}$. Since $K$ is compact, there exists a finite sub-cover $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$. Define

$$r = \min\{\frac{1}{2}\|(\lambda_1 1 - A)^{-1}\|^{-1} \ , \dots , \ \frac{1}{2}\|(\lambda_n 1 - A)^{-1}\|^{-1} \ \} \ .$$

Then for any $b$ with $\|B - A\| < r$ and any $\mu \in K$, $\mu \in U_{\lambda_j}$ for some $j = 1, \dots, n$, and then

$$\|(\mu 1 - B) - (\lambda_j 1 - A)\| \le |\mu - \lambda_j| + \|B - A\| < \|(\lambda_j 1 - A)^{-1}\|^{-1} \ .$$

Therefore, $(1\mu - B)$ is invertible. Thus, for all $\mu \in K$, whenever $\|B - A\| < r$, $\mu \in \rho_{\mathscr{A}}(B)$. □

## 1.4 Characters and the Gelfand Transform

**1.22 DEFINITION** (Characters)**.** A *character* of a Banach algebra $\mathscr{A}$ is a non-zero algebraic homomorphism from $\mathscr{A}$ to $\mathbb{C}$. The set of characters of $\mathscr{A}$ is denoted $\Delta(\mathscr{A})$, and the set $\{0\} \cup \Delta(\mathscr{A})$ is denoted $\Delta'(\mathscr{A})$.

Though characters are defined with respect to the algebraic structure alone, they are necessarily continuous:

**1.23 LEMMA.** *If $\mathscr{A}$ is a Banach algebra and $\varphi$ is a character of $\mathscr{A}$, then for all $A \in \mathscr{A}$ $\varphi(A) \in \sigma_{\mathscr{A}}(A)$, and*

$$|\varphi(A)| \le \|A\| \ . \tag{1.22}$$

*Moreover, if $\mathscr{A}$ has an identity 1, then $\varphi(1) = 1$.*

*Proof.* Suppose first that $\mathscr{A}$ contains an identity 1. Since $\varphi(1) = \varphi(1^2) = (\varphi(1))^2$, $\varphi(1)$ solves $\zeta - \zeta^2 = 0$, so either $\varphi(1) = 0$ or $\varphi(1) = 1$. But if $\varphi(1) = 0$, then for all $A \in \mathscr{A}$, $\varphi(A) = \varphi(1A) = \varphi(1)\varphi(A) = 0$, and this is excluded by the definition. Hence $\varphi(1) = 1$.

Next, for any $A \in \mathscr{A}$, $\varphi(a)1 - A$ is not invertible, and hence $\varphi(A) \in \sigma_{\mathscr{A}}(A)$. To see this, note that $\varphi(\varphi(a)1 - A) = 0$, but if $\varphi(a)1 - A$ had even a right inverse $B$, we would have

$$1 = \varphi(1) = \varphi((\varphi(A)1 - A)B) = 0\varphi(B) = 0 \ .$$

Then since $\sigma_{\mathscr{A}}(A)$ is contained in the closed centered disc of radius $\|a\|$, (1.22) is proved.

Now suppose that $\mathscr{A}$ lacks a unit. Let $\widetilde{\mathscr{A}}$ be the algebra obtained by adjoining an identity, and let $\widetilde{\varphi}$ be the character on $\widetilde{\mathscr{A}}$ given by

$$\widetilde{\varphi}((\lambda, A)) = \lambda + \varphi(A) \ ,$$

which is easily seen to be a character. Since $\sigma_{\mathscr{A}}(A)) = \sigma_{\widetilde{\mathscr{A}}}((0, A))$ by definition, and $\widetilde{\varphi}((0, A)) = \varphi(A)$, it follows from the above that $\varphi(A) \in \sigma(A)$, and then that $\|\varphi(A)\| \le \|(0, A)\| = \|A\|$. □

Note that if $\varphi \in \Delta(\mathscr{A})$, then for all $A, B \in \mathscr{A}$,

$$\varphi(AB) = \varphi(A)\varphi(B) = \varphi(B)\varphi(A) = \varphi(BA) \ .$$

Consequently, any character $\varphi$ must satisfy $\varphi(AB - BA) = 0$ for all $A, B$. When the algebra $\mathscr{A}$ is not commutative, this can be a stringent constraint, and *there may not exist any characters at all.*

**1.24 EXAMPLE.** Let $\mathscr{A}$ be the algebra of $2 \times 2$ matrices. The Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ .$$

Then with $[A, B]$ denoting the commutator $AB - BA$,

$$[\sigma_1, \sigma_2] = i2\sigma_3 \ , \qquad [\sigma_2, \sigma_3] = i2\sigma_1 \quad \text{and} \quad [\sigma_3, \sigma_1] = i2\sigma_2 \ .$$

It follows that for any homomorphism $\varphi$ of $\mathscr{A}$ into $\mathbb{C}$, $\varphi(\sigma_j) = 0$ for $j = 1, 2, 3$. Next, the identity matrix $I$ satisfies $I = \sigma_1^2$, and so $\varphi(I) = (\varphi(\sigma_1))^2 = 0$. Thus, for all $(z_0, z_1, z_2, z_3) \in \mathbb{C}^4$,

$$\varphi(z_0 I + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3) = 0 \ .$$

Since every evidently $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is linearly independent and $\mathscr{A}$ is 4 dimensional, $\mathscr{A}$ is the span of $\{I, \sigma_1, \sigma_2, \sigma_3\}$, and hence $\varphi$ vanishes identically on $\mathscr{A}$. Thus, if $\mathscr{A}$ is the algebra of $2 \times 2$ matrices, $\Delta(\mathscr{A}) = \emptyset$ and $\Delta'(\mathscr{A})$ is the one-point space $\{0\}$.

Even when $\mathscr{A}$ is commutative, there may be no non-trivial characters. However, as we shall see in the next chapter, when $\mathscr{A}$ is a commutative $C^*$ algebra, characters are plentiful enough to justify our present considerations. In the rest of this chapter, commutativity of the algebras will not play any role in the proofs, and so we shall state the results without making any reference to commutativity. However, one should keep in mind that without commutativity, and even with commutativity alone, $\Delta(\mathscr{A})$ may be empty and $\Delta'(\mathscr{A})$ may be a one-point space, as in the previous example.

**1.25 DEFINITION** (Gelfand topology)**.** For a Banach algebra $\mathscr{A}$, the *Gelfand topology* on $\Delta'(\mathscr{A})$ is the relative weak-$*$ topology on $\Delta'(\mathscr{A})$ considered as a subset of $\mathscr{A}^*$, the Banach space dual to $\mathscr{A}$. That is, the Gelfand topology is the weakest topology on $\Delta'(\mathscr{A})$ that makes the functions $\varphi \mapsto \varphi(A)$ continuous for all $A \in \mathscr{A}$.

**1.26 LEMMA.** *Let $\mathscr{A}$ be a Banach algebra. Then $\Delta'(\mathscr{A})$, equipped with the Gelfand topology is a compact Hausdorff space. If $\mathscr{A}$ does not have an identity, then with the Gelfand topology, $\Delta(\mathscr{A})$ is a locally compact Hausdorff space, and $\Delta'(\mathscr{A})$ is its one-point compactification. If $\mathscr{A}$ has an identity, $\Delta(\mathscr{A})$ itself is compact and 0 is an isolated point in $\Delta'(\mathscr{A})$.*

*Proof.* Equip $\mathscr{A}^*$ with the weak-$*$ topology; i.e., the weakest topology making all of functions $\varphi \mapsto \varphi(A)$ continuous for all $A \in \mathscr{A}$. The Banach-Alaoglu Theorem asserts that the unit ball in $\mathscr{A}^*$ is compact in the weak-$*$ topology. For each $A, B \in \mathscr{A}$, define a function $f_{A,B}$ on $\mathscr{A}^*$ by

$$f_{A,B}(\varphi) = \varphi(AB) - \varphi(A)\varphi(B) \ .$$

This is evidently continuous for the weak-$*$ topology. Now note that

$$\Delta'(\mathscr{A}) = \bigcap_{A,B\in\mathscr{A}} \{\varphi \in \mathscr{A}^* \ : \ f_{A,B}(\varphi) = 0 \ \} \ .$$

This displays $\Delta'(\mathscr{A})$ as an intersection of closed sets. Hence $\Delta'(\mathscr{A})$ is a closed subset of the unit ball in $\mathscr{A}^*$, and hence is compact.

For $\varphi_1, \varphi_2 \in \Delta'(\mathscr{A})$ with $\varphi_1 \neq \varphi_2$, there exists $A \in \mathscr{A}$ such that $\varphi_1(A) \neq \varphi_2(A)$. Let $U_1$ and $U_2$ be disjoint open sets in $\mathbb{C}$ that contain $\varphi_1(A)$ and $\varphi_2(A)$ respectively. Then

$$\{\psi \in \Delta'(\mathscr{A}) \ : \ \psi(A) \in U_1 \ \} \quad \text{and} \quad \{\psi \in \Delta'(\mathscr{A}) \ : \ \psi(A) \in U_2 \ \}$$

are disjoint open sets in $\Delta'(\mathscr{A})$ that contain $\varphi_1$ and $\varphi_2$ respectively. In particular, for each $\varphi \in \Delta(\mathscr{A})$, there exist disjoint open neighborhoods $V_1$ of $\varphi$ and $V_2$ of 0, and then since $V_1 \subset V_2^c$, $V_2^c$ is a compact neighborhood of $\varphi$. Thus, $\Delta(\mathscr{A})$ is locally compact. If $\mathscr{A}$ has an identity 1, $\varphi(1) = 1$ for all $\varphi \in \Delta(\mathscr{A})$, while $0(1) = 0$. Consequently, the zero homomorphism is an isolated point of $\Delta'(\mathscr{A})$ in this case. □

**1.27 DEFINITION** (Gelfand transform)**.** Let $\mathscr{A}$ be a Banach algebra. The *Gelfand transform* is the map $\gamma$ from $\mathscr{A}$ to $\mathscr{C}(\Delta'(\mathscr{A}))$ given by

$$(\gamma(A))[\varphi] = \varphi(A) \ . \tag{1.23}$$

That is, $\gamma(A)$ is the function of evaluation at $A$, and it is continuous by the definition of the Gelfand topology.

**1.28 THEOREM.** *Let $\mathscr{A}$ be a Banach algebra. The Gelfand transform is a norm reducing homomorphism from $\mathscr{A}$ to $\mathscr{C}((\Delta'(\mathscr{A}))$. That is, the Gelfand transform is a homomorphism of algebras and for all $A \in \mathscr{A}$,*

$$\|\gamma(A)\|_{\mathscr{C}(\Delta'(\mathscr{A}))} \leq \|A\| \ .$$

*Proof.* The homomorphism property is evident since for all $A, B \in \mathscr{A}$ and all $\varphi \in \Delta'(\mathscr{A})$,

$$(\gamma(ab))[\varphi] = \varphi(AB) = \varphi(A)\varphi(B) = (\gamma(A))[\varphi](\gamma(B))[\varphi] \ .$$

Next, suppose that $\mathscr{A}$ has an identity 1. If $\varphi \in \Delta(\mathscr{A})$ and $A \in \mathscr{A}$, then

$$\varphi(\varphi(A)1 - A) = \varphi(A) - \varphi(A) = 0 \ ,$$

and so $\varphi(a)1 - A$ is not invertible. This means that $\varphi(A) \in \sigma_{\mathscr{A}}(A)$, and this is contained in the closed centered disc of radius $\nu(A) \leq \|A\|$.

If $\mathscr{A}$ lacks an identity, adjoin an identity to form $\widetilde{\mathscr{A}}$. For $\varphi \in \Delta(\mathscr{A})$, define $\widetilde{\varphi}$ on $\widetilde{\mathscr{A}}$ by

$$\widetilde{\varphi}(\lambda, A) = \lambda + \varphi(A) \ .$$

It is easy to check that $\widetilde{\varphi} \in \Delta(\widetilde{\mathscr{A}})$. Let $\mathbb{1} = (1,0)$ denote the identity in $\widetilde{\mathscr{A}}$. Then for all $A \in \mathscr{A}$,

$$\widetilde{\varphi}(\varphi(A)\mathbb{1} - (0,A)) = \varphi(A) - \varphi(A) = 0 \ ,$$

so that once again, we have that $\varphi(A) \in \sigma_{\mathscr{A}}(A)$. □

This result, as it stands, does not take us far at all. The problem is that at this level of generality, there may be no characters at all, and the transform may be a trivial homomorphism into a trivial algebra. As indicated above, characters can only be expected to be plentiful for commutative algebras. Even then, there may be too few characters for the Gelfand transform to be of much interest:

**1.29 EXAMPLE.** Let $A_0$ be the $n \times n$ matrix, $n > 1$, with $A_{i,j} = \begin{cases} 1 & j = i+1 \\ 0 & j \neq i+1 \end{cases}$, and note that $A_0^n = 0$, Let $\mathscr{A}$ denote that subalgebra of the $n \times n$ matrices that are polynomials in $A_0$. That is, every $A \in \mathscr{A}$ has the form

$$A = \sum_{j=0}^{n-1} p_j A_0^j \ . \tag{1.24}$$

This is a commutative algebra with an identity. Let $\varphi \in \Delta'(\mathscr{A})$. Then $0 = \varphi(A_0^n) = (\varphi(A_0))^n$ so that $\varphi(A_0) = 0$. Then for $A$ given by (1.24), $\varphi(A) = p_0\varphi(I) = p_0$. Thus, the only candidate for a character on $\mathscr{A}$ is the map $\varphi_0$ given by

$$\varphi_0\left(\sum_{j=0}^{n-1} p_j A_0^j\right) = p_0 \ ,$$

It is readily checked that this is indeed a homomorphism and it is non-zero. Hence $\Delta(\mathscr{A}) = \{\varphi_0\}$ and $\Delta'(\mathscr{A}) = \{\varphi_0\} \cup \{0\}$. Since $\Delta'(\mathscr{A})$ consists of two isolated points, we may identify $\mathscr{C}(\Delta'(\mathscr{A}))$ with $\mathbb{C}^2$ in the usual way, and then we may write the Gelfand transform as

$$\gamma\left(\sum_{j=0}^{n-1} p_j A_0^j\right) = (p_0, 0) \ ,$$

which is indeed a norm reducing homomorphism, but not very interesting.

Before leaving this example, we note that for elements of $\mathscr{A}$, the spectrum is as trivial as Theorem 1.15 allows: For all $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ consists of a single point: $\sigma_{\mathscr{A}}(A) = \{\varphi_0(A)\}$. This is true since when $A$ is given by (1.24), then $A$ is invertible if and only if $p_0 \neq 0$.

## 1.5 Characters and spectrum in commutative Banach algebras

We have seen in Example 1.29 that in a commutative Banach Algebra $\mathscr{A}$ with an identity, $\Delta'(\mathscr{A})$ may consist of only two isolated points. However, this happens only when for each $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ consists of a single point. We now show when $\mathscr{A}$ contains elements with a more interesting spectrum, then $\Delta'(\mathscr{A})$ is also more interesting.

The Hahn-Banach Theorem, which provides the existence of continuous linear functionals on a Banach space, may be viewed as a theorem asserting the existence of maximal closed subspaces containing a given subspace. In the Banach algebra setting, the kernel of a homomorphism of a Banach algebra $\mathscr{A}$ to $\mathbb{C}$ is not only a closed subspace, it is a closed *ideal*, as we now explain, and consideration of *maximal ideals* leads to a Banach algebra version of the Hahn-Banach Theorem for commutative Banach algebras. Much of what is introduced here is also useful without assuming the $\mathscr{A}$ is commutative. We therefore start in general, and shall be clear about the key point when commutativity enters.

**1.30 DEFINITION.** Let $\mathscr{A}$ be a Banach algebra. An *ideal* of $\mathscr{A}$ is a subspace of $\mathscr{A}$ such that for all $B \in \mathscr{J}$ and $A \in \mathscr{A}$, $BA \in \mathscr{J}$ and $AB \in \mathscr{J}$. An ideal of $\mathscr{A}$ is *proper* in case it is not equal to $\mathscr{A}$ itself. An ideal of $\mathscr{A}$ is a *closed* in case it is topologically closed as a subset of $\mathscr{A}$. If $\mathscr{J}$ is an ideal in $\mathscr{A}$, and $\overline{\mathscr{J}}$ is the norm closure of $\mathscr{J}$, then $\overline{\mathscr{J}}$ is also an ideal in $\mathscr{A}$. If $\mathscr{J}$ is an ideal, an element $U$ of $\mathscr{A}$ is called a *unit mod* $\mathscr{J}$ in case

$$AU - A \in \mathscr{J} \quad \text{and} \quad UA - A \in \mathscr{J} \quad \text{for all} \quad A \in \mathscr{A} \ . \tag{1.25}$$

An ideal $\mathscr{J}$ is called a *modular ideal* in case there exists a unit mod $\mathscr{J}$.

Given a Banach algebra $\mathscr{A}$ and an ideal $\mathscr{J}$, there is a natural equivalence relation $\sim$ on $\mathscr{A}$ given by

$$A \sim B \iff A - B \in \mathscr{J} \ .$$

Let $\{A\}$ and $\{B\}$ denote the equivalence classes of $A$ and $B$ respectively. Let $\tilde{A}$ and $\tilde{B}$ be any other representative of $\{A\}$ and $\{B\}$ respectively. Then for some $X, Y \in \mathscr{J}$, $\tilde{A} = A + X$ and $\tilde{B} = B + Y$. Then

$$\tilde{A}\tilde{B} = (A + X)(B + Y) = AB + (AY + XB + XY) \sim AB \ .$$

Even more simply one sees that $\tilde{A} + \tilde{B} \sim A + B$ and for all $\lambda \in \mathbb{C}$, $\lambda\tilde{A} \sim \lambda A$. Hence $\mathscr{A}/\mathscr{J}$, the set of equivalence classes in $\mathscr{A}$, equipped with the operations

$$\{A\}\{B\} = \{AB\} \quad \text{and} \quad \{A\} + \{B\} = \{A + B\} \quad \text{and} \quad \lambda\{A\} = \{\lambda A\}$$

is an algebra, and $A \mapsto \{A\}$ is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$.

Now introduce a norm on $\mathscr{A}/\mathscr{J}$ by

$$\|\{A\}\| = \inf\{ \ \|\tilde{A}\| \ : \ \tilde{A} \sim A \ \} = \inf\{ \ \|A - B\| \ : \ B \in \mathscr{J} \ \} \ .$$

Note that $\|\{A\}\| \leq \|A\|$. To see that

$$\|\{A\}\{B\}\| \leq \|\{A\}\|\|\{B\}\| \tag{1.26}$$

for all $\{A\}, \{B\} \in \mathscr{A}/\mathscr{J}$, let $0 < \epsilon < \min\{\|\{A\}\|, \|\{B\}\|\}$, and pick $\tilde{A} \in \{A\}$ and $\tilde{B} \in \{B\}$ so that $\{A\} > \|\tilde{A}\| - \epsilon$ and $\{B\} > \|\tilde{B}\| - \epsilon$. Then

$$\|\{A\}\{B\}\| = \|\{\tilde{A}\tilde{B}\}\| \leq \|\tilde{A}\tilde{A}\| \leq \|\tilde{A}\|\|\tilde{A}\| \leq (\|\{A\}\| + \epsilon)(\|\{A\}\| + \epsilon) \ .$$

Since $\epsilon$ can be taken arbitrarily small, (1.26) is proved.

Therefore, $\mathscr{A}/\mathscr{J}$ will be a Banach algebra with this norm provided it is complete in this norm. Consider a Cauchy sequence $\{\{A\}_n\}_{n\in\mathbb{N}}$ in $\mathscr{A}/\mathscr{J}$. A standard argument shows that this sequence always has a limit if $\mathscr{J}$ is closed. Thus, when $\mathscr{J}$ is a closed ideal, $\mathscr{A}/\mathscr{J}$ is a Banach algebra, and the map $a \mapsto \{A\}$ is a contractive homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$. This homomorphism is called the *natural homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$.*

It is possible for $\mathscr{A}/\mathscr{J}$ to have an identity even when $\mathscr{A}$ does not. Suppose that $\mathscr{J}$ is modular, and that $U$ is a unit mod $\mathscr{J}$. Then for all $A \in \mathscr{A}$, $\{U\}\{A\} = \{UA\} = \{A\}$ and $\{A\}\{U\} = \{AU\} = \{A\}$. Thus, $\{U\}$ is a multiplicative identity in $\mathscr{A}/\mathscr{J}$. Clearly if $\mathscr{A}$ has an identity 1, 1 is a unit mod $\mathscr{J}$.

There is a close connections between closed ideals and kernels of continuous homomorphisms of Banach algebras.

**1.31 PROPOSITION.** *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a continuous homomorphism. Then $\mathscr{J} = \ker(\mathscr{A})$ is a closed ideal in $\mathscr{A}$. Conversely, if $\mathscr{J}$ is a closed ideal in $\mathscr{A}$, then the map $A \mapsto \{A\}_{\mathscr{J}}$, sending $A$ to its equivalence class mod $\mathscr{J}$, is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$.*

*Proof.* Suppose that $\pi : \mathscr{A} \to \mathscr{B}$ be a continuous homomorphism. Then evidently $\mathscr{J} = \ker(\mathscr{A})$ is a closed by the continuity of $\pi$, and it is a subspace by the linearity of $\pi$. Next, for all $X \in \mathscr{J}$ and $A, B \in \mathscr{A}$, $\pi(AXB) = \pi(A)\pi(X)\pi(B) = \pi(A)0\pi(B) = 0$. Hence $AXB \in \ker(\pi)$, and so $\mathscr{J}$ is an ideal. The converse is clear from the construction of $\mathscr{A}/\mathscr{J}$ described above. □

Now consider a *commutative* Banach algebra $\mathscr{A}$. For any $X_0 \in \mathscr{A}$, we can define $\mathscr{J}(X_0)$ to be the subset of $\mathscr{A}$ given by

$$\mathscr{J}(X_0) = \{ \ X_0Y \ : \ Y \in \mathscr{A} \ \} \ . \tag{1.27}$$

Then for all $YX_0 \in \mathscr{J}(X_0)$ and all $A, B \in \mathscr{A}$, $AYX_0B = (AYB)X_0 \in \mathscr{J}(x_0)$, and evidently $\mathscr{A}$ is a subspace of $\mathscr{A}$. Hence $\mathscr{J}(X_0)$ is an ideal, and it is called *the ideal generated by* $X_0$.

In the non-commutative setting, one could consider the set $\{ \ YX_0Z \ : \ Y, Z \in \mathscr{A} \ \}$ which is closed under left and right multiplication by elements of $\mathscr{A}$. However, without some additional hypothesis on $X_0$, such as that $X_0$ commutes with all elements of $\mathscr{A}$, it need not be a subspace, and the closure of its span might be all of $\mathscr{A}$.

Let $\mathscr{A}$ be a commutative Banach algebra with an identity 1. Let $X_0$ be a non-invertible element of $\mathscr{A}$. Let $\mathscr{J}(X_0)$ be the ideal generated by $X_0$. Then no element of $\mathscr{J}(X_0)$ is invertible. Indeed, if $X_0Y$ were invertible, there would exist $Z \in \mathscr{A}$ such that $(X_0Y)Z = X_0(YZ) = 1$, and then (since $\mathscr{A}$ is commutative), $YZ$ would be an inverse of $X_0$, which is not possible. Hence, for all non-invertible $X_0$, $\mathscr{J}(X_0)$ consists entirely of non-invertible elements. Since the open unit ball about the identity consists of invertible elements, $\mathscr{J}(X_0)$ does not intersect the open unit ball about the identity 1. In particular, 1 does not belong to $\overline{\mathscr{J}(X_0)}$, the closure of $\mathscr{J}(X_0)$.

Now we come to a crucial construction of characters in a commutative Banach algebra:

**1.32 THEOREM.** *Let $\mathscr{A}$ be a commutative Banach algebra with identity 1. Then for all non-invertible $X_0 \in \mathscr{A}$, there exists a character $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(X_0) = 0$.*

*Proof.* Since $X_0$ is not invertible, $\mathscr{J}(X_0)$ is a proper ideal in $\mathscr{A}$, and in fact, as explained above, the open unit ball about 1 does not intersect $\mathscr{J}(X_0)$. Now consider any chain of proper ideals in $\mathscr{A}$, ordered by inclusion. Since no proper ideal contains the identity, the union of this chain is again a proper ideal. Hence by Zorn's Lemma, there exists a maximal proper ideal $\mathscr{M}$ containing $\mathscr{J}(X_0)$. Since no proper ideal can contain any invertible elements, this ideal does not intersect the open unit ball about 1. Hence its closure $\overline{\mathscr{M}}$ also contains $\mathscr{J}(x_0)$ and is proper. Since $\mathscr{M}$ is maximal among such ideals, $\mathscr{M} = \overline{\mathscr{M}}$. Hence in a commutative Banach algebra $\mathscr{A}$ with identity 1, for each non-invertible $X_0 \in \mathscr{A}$, there exists a closed proper ideal $\mathscr{M}$ that contains any ideal in $\mathscr{A}$ that contains $\mathscr{J}(X_0)$.

We now claim that the Banach algebra $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is a division algebra. Suppose not. Then it contains a non-zero, non-invertible element $\{Y_0\}_{\mathscr{M}}$. Let $\mathscr{N}$ be the closure of the ideal in $\mathscr{B}$ generated by $\{Y_0\}_{\mathscr{M}}$. Let $\pi_1$ be the natural homomorphism of $\mathscr{A}$ onto $\mathscr{B}$, and let $\pi_2$ be the natural homomorphism of $\mathscr{B}$ onto $\mathscr{B}/\mathscr{N}$. Then $\pi_2 \circ \pi_1$ is a homomorphism of $\mathscr{A}$ onto $\mathscr{B}/\mathscr{N}$. By Proposition 1.31, $\ker(\pi_2 \circ \pi_1)$ is a closed ideal that contains $\mathscr{M} = \ker(\pi_1)$. The containment is proper since $\pi_2 \circ \pi_1(Y_0) = 0$, but $Y_0 \notin \mathscr{M}$ since $\{Y_0\}_{\mathscr{M}} \neq 0$. Finally, $1 \notin \ker(\pi_1 \circ \pi_1)$ since $\{1\}_{\mathscr{M}}$ is a unit $\mathscr{B} = \mathscr{A}/\mathscr{M}$, and $\mathscr{N}$ does not contain any invertible elements, so $\pi_2(\{1\}_{\mathscr{M}}) = \pi_2(\pi_1(1)) \neq 0$. Thus, $\ker(\pi_2 \circ \pi_1)$ is a closed proper ideal that strictly contains $\mathscr{M}$, which is impossible. Hence the hypothesis that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ contains a non-zero, non-invertible element is false. This shows that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is a division algebra, and then the Gelfand-Mazur Theorem tells us that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is canonically isomorphic to $\mathbb{C}$. Hence $\pi_1$ may be regarded as a character of $\mathscr{A}$, and by construction $x_0 \in \mathscr{J}(x_0) \subset \mathscr{M} = \ker(\pi_1)$. □

This theorem has the following important consequence:

**1.33 COROLLARY.** *Let $A \in \mathscr{A}$, where $\mathscr{A}$ is a commutative Banach algebra. Let $\lambda \in \sigma_{\mathscr{A}}(a)$. Then there exists $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(A) = \lambda$. In particular, the spectral radius $\nu(A)$ of $A$ is given by*

$$\nu(A) = \sup\{\ |\varphi(A)|\ :\ \varphi \in \Delta(\mathscr{A})\ \}\ . \tag{1.28}$$

*Proof.* Adjoining an identity has no effect on the spectral radius, so we may assume that $\mathscr{A}$ has an identity 1. We have already seen that for all $\varphi \in \Delta(\mathscr{A})$, $\varphi(A) \in \sigma_{\mathscr{A}}(A)$. We now show that for *every* $\lambda \in \sigma_{\mathscr{A}}(A)$, there exists $\varphi \in \Delta(\mathscr{A})$ with $\varphi(A) = \lambda$.

Since $\lambda 1 - A$ is not invertible, by Theorem 1.32, there exists $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(\lambda 1 - A) = 0$. But $\varphi(\lambda 1 - A) = \lambda\varphi(1) - \varphi(A) = \lambda - \varphi(A)$. □

# 2 Fundamentals of the Theory of $C^*$ Algebras

## 2.1 $C^*$ algebras

**2.1 DEFINITION** (Banach $*$-algebra)**.** A Banach $*$-algebra is a Banach algebra $\mathscr{A}$ equipped with a map $* : \mathscr{A} \to \mathscr{A}$, the action of which is written as $a \mapsto a^*$, and which satisfies the properties

*(i)* The map $*$ is conjugate linear. That is, for all $A, B \in \mathscr{A}$ and all $z \in \mathbb{C}$, $(zA + B)^* = \overline{z}A^* + B^*$.

*(ii)* For all $A, B \in \mathscr{A}$, $(AB)^* = B^*A^*$ .

*(iii)* The $*$ map is an involution; for all $A \in \mathscr{A}$, $A^{**} = A$.

The map $A \mapsto A^*$ is called *the involution* in $\mathscr{A}$. The $C^*$ algebra $\mathscr{A}$ is *unital* in case it possesses a multiplicative identity.

**2.2 DEFINITION** ($C^*$-algebra)**.** A $C^*$ *algebra* is a Banach $*$ algebra for which the involution and the norm are related by the requirement that: *(iv)* For all $a \in \mathscr{A}$,

$$\|AA^*\| = \|A\|\|A^*\| \ . \tag{2.1}$$

The identity (2.1) is called *the $C^*$ algebra identity.*

Definition 2.2 comes from a seminal 1943 paper of Gelfand and Neumark [8]. The term $C^*$ algebra was introduced Segal [28] in 1947. Segal originally used the term [28, p. 75] to describe norm closed self adjoint subalgebras of the algebra of bounded linear operator on a Hilbert space; see Example 2.4 below. These later became known as "concrete $C^*$-algebras", and the *a priori* more general class of $C^*$ algebras specified in Definition 2.2 became known as "abstract $C^*$-algebras". The main result of [8] is that, under some additional assumptions (see below) can be stated as follows: Every unital abstract $C^*$ is isometric and $*$-isomorphic with a concrete $C^*$- algebra, The additional assumptions, and the requirement that $\mathscr{A}$ be unital, have since been shown to be unnecessary, and these days the distinction between abstract and concrete $C^*$ algebras is rarely emphasized.

In their paper, Gelfand and Neumark not only assumed the existence of a multiplicative identity in $\mathscr{A}$, they also imposed two other hypotheses that they conjectured to be consequences of *(i)* - *(iii)* and (2.1) which was their *(iv)*, but they were unable to show this. The additional postulates that they imposed were that

*(v)* The $*$ map is an isometry; for all $A \in \mathscr{A}$, $\|A^*\| = \|A\|$.

*(vi)* For all $A \in \mathscr{A}$, $1 + A^*A$ is invertible.

In the second footnote of their paper, they state their conjecture that *(i)* - *(iv)* imply *(v)* and *(vi)*, but note that the authors "have not succeeded in proof of this fact". They then note that postulates *(iv)* and *(v)* may be replaced by

$$\|A^*A\| = \|A\|^2 \tag{2.2}$$

for all $A \in \mathscr{A}$, since then $\|A\|^2 = \|A^*A\| \leq \|A^*\|\|A\|$, and hence $\|A\| \leq \|A^*\|$. Then by *(iii)*, the involution must be an isometry, so that (2.2) and *(iii)* imply *(v)*. Conversely, *(iv)* and *(v)* imply (2.2). It took 10 years for the conjecture of Gelfand and Neumark to be verified for *(vi)* in work of Fukamiya [5] and Kaplansky [27], and 17 years for *(v)* in work of Glimm and Kadison [6], and then only for unital $C^*$ algebras. Finally in 1967, Vowden [33] proved that every $C^*$ algebra $\mathscr{A}$ can be isometrically and $*$-isomorphically embedded in a unital $C^*$ algebra $\mathscr{A}_1$. As Vowden noted, it is an immediate consequence of this result that in any Banach $*$-algebra in which (2.1) is valid for all $A \in \mathscr{A}$, it is also the case that (2.2) is valid for all $A \in \mathscr{A}$

In 1946, Richart defined a *$B^*$-algebra* to be a Banach $*$-algebra $\mathscr{A}$ in which the norm and the involutions satisfy (2.2) for all $A \in \mathscr{A}$. By what Gelfand and Neumark observed in the second footnote, every $B^*$ algebra is $C^*$ algebra. In 1967, Vowden [33] completed the proof of the fact that every $C^*$ algebra is also a $B^*$ algebra, and the latter term is now rarely used.

In the more recent literature on the subject, many authors simply define a $C^*$-algebra to be a Banach $*$-algebra $\mathscr{A}$ in which the norm and the involution satisfy (2.2) for all $A \in \mathscr{A}$; see e.g., [1]. This has some advantages since, for instance, the isometry property of the involution is an immediate consequence of the other postulates. Here we shall have to work harder for this since we start from the original postulates *(i)* through *(iv)*, *(iv)* being (2.1), as originally proposed by Gelfand and Neumark, and present the proof of their conjecture on the redundancy of *(v)* and *(vi)* in the the general non-unital case. While this conjecture (in its non-unital form) took two dozen years and the contributions of a number of mathematicians to resolve, the insights through which this was achieved are quite simple, clean and clear. Like many simple, clean and clear ideas, they will turn out to have other uses, and this more than justifies our choice of starting point.

**2.3 EXAMPLE.** Let $\mathscr{A} = \mathscr{C}_0(X)$ with the structure specified in Example 1.2, together with the involution $*$ defined by $f^*(x) = \overline{f(x)}$ for all $x \in X$ and all $f \in \mathscr{A}$. Then one readily checks that $\mathscr{A}$ is a commutative $C^*$ algebra. A theorem of Gelfand and Neumark to be proved in this chapter says that up to isometric isomorphism, every commutative $C^*$ algebra is of this form.

**2.4 EXAMPLE.** Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{A} = \mathscr{B}(\mathscr{H})$ equipped with the structure specified in Example 1.4, together with the involution $*$ defined by taking $A^*$ to be the Hermitian adjoint of $A$. That is, $A^*$ is the unique operator in $\mathscr{B}(\mathscr{H})$ such that

$$\langle A^*\phi, \psi\rangle = \langle \phi, A\psi\rangle \tag{2.3}$$

for all $\phi, \psi \in \mathscr{H}$. Then properties *(i)*, *(ii)* and *(iii)* are evidently satisfied. To see that *(iv)* is satisfied, choose $\epsilon > 0$ and a unit vector $\psi \in \mathscr{H}$ such that $(1-\epsilon)\|A\| \leq \|A\psi\|$. Then

$$\langle \psi, A^*A\psi\rangle = \langle A\psi, A\psi\rangle = \|A\psi\|^2 \geq (1-\epsilon)^2\|A\|^2 \ .$$

Since $\epsilon > 0$ is arbitrary, it follows that $\|A^*A\| \geq \|A\|^2$. Since (2.3) says that $|\langle \psi, A^*\phi\rangle| = |\langle \phi, A\psi\rangle|$, it is evident that $\|A^*\| = \|A\|$ and hence $\|A^*A\| \geq \|A\|^2$ can be written as $\|A^*A\| \geq \|A^*\|\|A\|$. Since $\mathscr{B}(\mathscr{H})$ is a Banach algebra, we have $\|A^*A\| \leq \|A^*\|\|A\|$, and hence *(iv)* is proved.

Note that we have already prove the isometry property *(v)* in this example on our way to *(iv)*. It is also not hard to give a direct proof of *(vi)* in this context; we return to this point later. However, note that in our analysis of this example, we are making use of the vectors in the space on which the elements of $\mathscr{A}$ operate.. In the abstract setting, of Definition (2.2), the elements of $\mathscr{A}$ need not be operators on a Hilbert space, and such arguments are immediately available. However, a fundamental theorem also due to Gelfand and Neumark says that every $C^*$ algebra is isometrically isomorphic to a $C^*$ algebra of operators in a Hilbert space $\mathscr{H}$.

A difficulty that arises when studying $C^*$ non-unital $C^*$-algebras is that our standard construction for adjoining a unit does not in general yields a norm satisfying *(iv)*. However, we can aways embed a Banach $*$-algebra $\mathscr{A}$ in a unital Banach $*$-algebra $\widetilde{\mathscr{A}}$ by a small modification of our standard construction: Let $\mathscr{A}$ be a non-unital Banach $*$-algebra and let $\widetilde{\mathscr{A}} := \mathbb{C} \oplus \mathscr{A}$ with the product defined in (1.3) and the norm defined in (1.4). For $(\lambda, A) \in \widetilde{\mathscr{A}}$, define $(\lambda, A)^*(\overline{\lambda}, A^*)$. It is easy to see that $\widetilde{\mathscr{A}}$ satisfies *(i)*, *(ii)* and *(iii)* of Definition 2.1. However, *(iv)* will not in general be satisfied. When $\mathscr{A}$ is a Banach algebra, $\widetilde{\mathscr{A}}$ *always* denotes the unital Banach $*$-algebra obtained using this construction.

**2.5 LEMMA.** *Let $\mathscr{A}$ be a unital Banach $*$-algebra with unit $1$. Then $1^* = 1$, and $A$ is invertible if and only if $A^*$ is invertible.*

*Proof.* Evidently $1^* = 1^*1$, and then by *(ii)* and *(iii)*, $1 = 1^*1$. Next, suppose that $A$ is invertible and let $B$ be its inverse. Then $1 = AB = BA$. By *(ii)* and *(iii)*, and the first part of the proof, $1 = 1^* = B^*A^* = A^*B^*$. Thus, $A^*$ is invertible, and $B^*$ is the inverse. By *(iii)* once more, when $A^*$ is invertible, $A$ is invertible. $\square$

**2.6 LEMMA.** *Let $\mathscr{A}$ be a Banach $*$-algebra Then for all $A \in \mathscr{A}$,*

$$\sigma_{\mathscr{A}}(A^*) = \overline{\sigma_{\mathscr{A}}(A^*)} \ . \tag{2.4}$$

*In particular, $\nu(A) = \nu(A^*)$.*

*Proof.* Suppose $\mathscr{A}$ is non-unital. Let $\widetilde{\mathscr{A}}$ be unital Banach $*$-algebra obtain obtained from our standard construction. By definition, $\sigma_{\mathscr{A}}(A) = \sigma_{\widetilde{\mathscr{A}}}((0,A))$. By Lemma 2.5, $(\lambda, -A)$ is invertible if and only if $(\lambda, -A)^* = (\overline{\lambda}, -A^*)$ is invertible, and this proves (2.4). The unital case is even easier; adjoining a unit is not needed. The final statement follows form the definition of the spectral radius. □

**2.7 DEFINITION** (Self-adjoint and normal elements of a $C^*$ algebra)**.** Let $\mathscr{A}$ be a $C^*$ algebra. Then:

*(1)* $A \in \mathscr{A}$ is *self-adjoint* in case $A = A^*$.

*(2)* $A \in \mathscr{A}$ is *normal* in case $AA^* = A^*A$.

We write $\mathscr{A}_{\text{s.a.}}$ to denote the sets of self-adjoint elements in $\mathscr{A}$.

**2.8 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra. Then for all normal $A \in \mathscr{A}$,*

$$\|A\| = \|A^*\| = \nu(A) \ . \tag{2.5}$$

*Proof.* Suppose first that $A$ is self-adjoint. By *(iv)*, $\|A^2\| = \|A\|^2$, and by an obvious induction, for all $m \in \mathbb{N}$, $\|A^{2m}\| = \|A\|^{2m}$. By the Beurling-Gelfand Formula,

$$\nu(A) = \lim_{m\to\infty} (\|A^{2m}\|)^{1/2m} = \|A\|.$$

This proves (2.5) when $A$ is self-adjoint.

Now observe that when $A$ and $B$ commute, $(AB)^n = A^nB^n$ and hence $\|(AB)^n\|^{1/n} \leq \|A^n\|^{1/n}\|B^n\|^{1/n}$. By the Beurling-Gelfand Formula, $\nu(AB) \leq \nu(A)\nu(B)$. Also, by Lemma 2.6, $\nu(A^*) = \nu(A)$. Hence when $A$ is normal,

$$\|A\|^2 \geq \nu(A)^2 = \nu(A^*)\nu(A) \geq \nu(A^*A) = \|A^*A\| = \|A^*\|\|A\| \ , \tag{2.6}$$

where in the last two steps we have used (2.5) for the self-adjoint operator $A^*A$, and then *(iv)*. We conclude that when $A$ is normal, $\|A^*\| = \|A\|$, and then (2.6) implies that $\|A\|^2 \geq \nu(A)^2 \geq \|A\|^2$, which proves (2.5) for all normal $A$. □

**2.9 THEOREM** (In a $C^*$ algebra, self-adjointness implies real spectrum)**.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $A \in \mathscr{A}$ then if $A = A^*$, $\sigma_{\mathscr{A}}(A) \subset (\mathbb{R})$.*

*Proof.* If $\mathscr{A}$ has no unit, we adjoint a unit to obtain $\widetilde{\mathscr{A}}$ using our standard construction. Then $\widetilde{\mathscr{A}}$ is a Banach $*$-algebra, but not necessarily a $C^*$ algebra. We proceed by contradiction, and suppose that $A$ is a self-adjoint element of $\mathscr{A}$, and there exists some $\lambda \in \sigma_{\mathscr{A}}(a)$ such that $\lambda \notin \mathbb{R}$. Then taking an appropriate real multiple of $A$ (so the the multiple is still self adjoint), we may suppose that $e^{i\lambda} = 4$ for some $\lambda \in \sigma_{\mathscr{A}}(a)$.

For each $n \in \mathbb{N}$, define the polynomial $p_n(\zeta) = \sum_{j=0}^{n}(i\zeta)^j/j!$. By the Spectral Mapping Lemma, for each $n$, $p_n(\lambda) \in p_n(A)$. For $n > m$,

$$\|p_n(A) - p_m(A)\| \leq \sum_{j=m+1}^{n} \|A\|^j/j! \ ,$$

and hence by standard estimates on the exponential power series for numbers, $\{p_n(A)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\widetilde{\mathscr{A}}$. Therefore, there exist $U \in \widetilde{\mathscr{A}}$ with $U = \lim_{n\to\infty} p_n(A)$. Evidently, for all $n$, $(p_n(A))^* = p_n(-A)$, so that once again, by standard estimates for the exponential power series, $U^*U = UU^* = 1$, Now define $W = U - 1$, and note that $W$ is in $\mathscr{A}$. From $U^*U = UU^* = 1$ in $\widetilde{\mathscr{A}}$, we have

$$W + W^* + W^*W = W + W^* + WW^* = 0 \ , \tag{2.7}$$

and in particular, $W$ is normal. By (2.7) once more, $\|W^*W\| \leq \|W\| + \|W^*\|$, and then by Lemma 2.8, $\|W\|^2 \leq 2\|W\|$ nad hence $\|W\| \leq 2$. Then in $\widetilde{\mathscr{A}}$,

$$\lim_{n\to\infty}(p_n(\lambda)1 - p_n(A)) = e^{i\lambda}1 - 1 - W \ .$$

Since $e^{i\lambda}1 = 4$, and $\|W\| \leq 2$, $e^{i\lambda}1 - 1 - W$ is invertible. However, for each $n$ $p_n(\lambda)1 - p_n(A)$ is non-invertible. Since the set of invertible elements is open, it cannot be that a sequence of non-invertible elements converges to an invertible element. Thus, the hypothesis that $\mathscr{A}$ contains some self adjoint element $A$ with some $\lambda \in \sigma_{\mathscr{A}}(A)$ such that $\lambda \notin \mathbb{R}$ leads to contradiction. □

**2.10 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra. Then $\mathscr{A}_{\text{s.a.}}$, the set of self-adjoint elements of $\mathscr{A}$, is closed.*

*Proof.* Let $\{H_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathscr{A}_{\text{s.a.}}$ with $\lim_{n\to\infty} H_n = H + iK$, $H, K \in \mathscr{A}_{\text{s.a.}}$. Subtracting $H$ from each $H_n$, we may suppose that $H = 0$, and then, without loss of generality may suppose that $\|H_n\| \leq 1$ for all $n$, with the consequence that $\|K\| \leq 1$. By the Spectral Mapping Theorem,

$$\sigma_{\mathscr{A}}(H_n^2 - H_n^4) = \{\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathscr{A}}(H_n)\} \ .$$

Then by Lemma 2.5 and Theorem 2.9,

$$\|H_n^2 - H_n^4\| = \sup\{|\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathscr{A}}(H_n)\} \leq \sup\{|\lambda^2 : \lambda \in \sigma_{\mathscr{A}}(H_n)\} = \|H_n\|^2 \ ,$$

where we have used the fact that $\sigma_{\mathscr{A}}(H_n) \subset [-1, 1]$ by Lemma 2.5 once more. Taking the limit $n \to \infty$, we obtain $\|K^2 - K^4\| \leq \|K\|^2$. Choose $\lambda_1 \in \sigma_{\mathscr{A}}(K)$ so that $|\lambda_1| = \nu(K) = \|K\|$. Then since $\lambda_1$ is pure imaginary,

$$|\lambda_1|^2 + |\lambda_1|^4 = |\lambda_1^2 - \lambda_1^4| \leq |\lambda_1|^2 \ .$$

Therefore, $\|K\| = \nu(K) = |\lambda_1| = 0$. □

**2.11 COROLLARY.** *The involution in a $C^*$ algebra $\mathscr{A}$ is continuous.*

*Proof.* We apply the Open Mapping Theorem, which is also valid for conjugate linear maps. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that for some $B, C \in \mathscr{A}$, $\lim_{n\to\infty} A_n = B$ and $\lim_{n\to\infty} A_n^* = C$. We must show that $C^* = B$. Note that

$$\lim_{n\to\infty}\left(\frac{A_n + A_n^*}{2}\right) = \frac{B + C}{2} \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{A_n - A_n^*}{2i}\right) = \frac{B - C}{2i} \ .$$

By Lemma 2.10, both $\frac{B+C}{2}$ and $\frac{B-C}{2i}$ are self-adjoint, and hence

$$B + C = B^* + C^* \quad \text{and} \quad B - C = C^* - B^* \ ,$$

and thus $2B = (B + C) + (B - C) = (B^* + C^*) + (C^* - B^*) = 2C^*$, as was to be shown. □

## 2.2 Commutative $C^*$ algebras

**2.12 DEFINITION** (Hermitian character)**.** Let $\mathscr{A}$ be a $C^*$ algebra. A character $\varphi$ of $\mathscr{A}$ is *Hermitian* in case for all $A \in \mathscr{A}$,

$$\varphi(A^*) = (\varphi(A))^* \ .$$

**2.13 LEMMA.** *All characters of a $C^*$ algebra are Hermitian.*

*Proof.* For any $A \in \mathscr{A}$ define $X = \dfrac{1}{2}(A + A^*)$ and $Y = \dfrac{1}{2i}(A - A^*)$. Then $X$ and $Y$ are self-adjoint, and $A = X + iY$. For any character $\varphi$ of $\mathscr{A}$,

$$\varphi(A) = \varphi(A + iY) = \varphi(X) + i\varphi(Y) \quad \text{and} \quad \varphi(A^*) = \varphi(A - iY) = \varphi(X) - i\varphi(Y) \ .$$

By Theorem 2.9, $\varphi(X)$ and $\varphi(Y)$ are real, and hence $\varphi(A^*) = (\varphi(A))^*$. □

**2.14 THEOREM** (Commutative Gelfand-Neumark Theorem)**.** *Let $\mathscr{A}$ be a commutative $C^*$-algebra. Then the Gelfand transform is an isometric isomorphism of $\mathscr{A}$ onto $\mathscr{C}_0(\Delta(\mathscr{A}))$.*

*Proof.* By Lemma 2.13, for all $A \in \mathscr{A}$ and all $\varphi \in \Delta(\mathscr{A})$,

$$\gamma(a^*)[\varphi] = \varphi(A^*) = (\varphi(A))^* = \gamma(A)^*[\varphi]$$

since the involution in $\mathscr{C}_0(\Delta(\mathscr{A}))$ is pointwise complex conjugation.

Next, $|\gamma(A)[\varphi]| = |\varphi(A)|$, and by the easy Lemma 1.23, $\sup_{\varphi\in\Delta(\mathscr{A})} \{ \ |\varphi(A)| \ \} \leq \nu(A)$. By the deeper Corollary 1.33 of Theorem 1.32, $\sup_{\varphi\in\Delta(\mathscr{A})} \{ \ |\varphi(A)| \ \} \geq \nu(a)$. Combining these two inequalities with Theorem 2.5, and noting that in a commutative $C^*$ algebra, *every* element is normal,

$$\sup_{\varphi\in\Delta(\mathscr{A})} \{ \ |\gamma(A)[\varphi]| \ \} = \nu(A) = \|A\| \ ,$$

which proves that the Gelfand transform is an isometry, and hence is injective onto a subalgebra of $\gamma(\mathscr{A})$ of $\mathscr{C}_0(\Delta(\mathscr{A}))$. However, $\gamma(\mathscr{A})$ separates points, and does not vanish at any $\varphi \in \Delta(\mathscr{A})$, and is closed under complex conjugation. Hence by the Stone-Wierstrass Theorem, and the fact that $\gamma(\mathscr{A})$ is closed, $\gamma(\mathscr{A}) = \mathscr{C}_0(\Delta(\mathscr{A}))$. □

## 2.3 Spectral invariance and the Abstract Spectral Theorem

Let $\mathscr{A}$ be a Banach algebra with identity, and let $\mathscr{B}$ be a Banach subalgebra. It can happen that some $B \in \mathscr{B}$ is not invertible in $\mathscr{B}$, but is invertible in $\mathscr{A}$.

**2.15 EXAMPLE.** Let $D$ denote the closed unit disc in $\mathbb{C}$, and let $C$ denote its boundary, the unit circle. Let $\mathscr{A} = \mathscr{C}(C)$, the algebra of continuous functions on $C$. Let $\mathscr{B}$ denote the algebra of continuous functions on $D$ that are holomorphic in the interior of $D$. These functions are determined by their values on $C$, and their maximum absolute value is attained on $C$. Therefore, restriction to $C$ is an isometric embedding of $\mathscr{B}$ in $\mathscr{A}$, so we may regard $\mathscr{B}$ as a subalgebra of $\mathscr{A}$.

Let $B$ denote the function $f(e^{i\theta}) = e^{i\theta}$, which evidently belongs to $\mathscr{B}$. Then $1\lambda - B$ is invertible in $\mathscr{A}$ if and only if $\lambda \notin C$, in which case the inverse is the function $g(e^{i\theta}) = (\lambda - e^{i\theta})^{-1}$. However, for $\lambda$ in the interior of $D$, $\zeta \mapsto (\lambda - \zeta)^{-1}$ is not holomorphic in the interior of $D$, and so the inverse of $B$ in $\mathscr{A}$ does not belong to $\mathscr{B}$. That is $\sigma_{\mathscr{A}}(B) = C$, but $\sigma_{\mathscr{B}}(B) = D$.

Now we specialize to $C^*$ algebras, first introducing certain minimal subalgebras:

**2.16 DEFINITION.** Let $\mathscr{A}$ be a $C^*$ algebra, and $S$ a subset of $\mathscr{A}$. Then $C(S)$ is the smallest $C^*$ subalgebra of $\mathscr{A}$ that contains $S$.

**2.17 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\mathscr{B}$ be a $C^*$ subalgebra of $\mathscr{A}$. Suppose either that $\mathscr{A}$ is unital with unit* 1 *and that $1 \in \mathscr{B}$ or else that neither $\mathscr{A}$ nor $\mathscr{B}$ is unital. Then for all $B \in \mathscr{B}$,*

$$\sigma_{\mathscr{B}}(b) = \sigma_{\mathscr{A}}(b) \ . \tag{2.8}$$

*Proof.* It is evident that $\sigma_{\mathscr{A}}(B) \subset \sigma_{\mathscr{B}}(B)$, and we need only prove that $\sigma_{\mathscr{B}}(B) \subset \sigma_{\mathscr{A}}(B)$

Suppose first that $\mathscr{A}$ is unital with unit 1 and that $1 \in \mathscr{B}$. We first prove (2.8) for $B$ self adjoint: Suppose that $B$ is self adjoint and invertible in $\mathscr{A}$. By Theorem 2.9, $\sigma_{C(b)}(b) \subset \mathbb{R}$, and consequently, for all $n \in \mathbb{N}$, $B - (i/n)1$ is invertible in $C(\{1, B\})$. Since $\lim_{n\to\infty}(B - (i/n)1) = B$ in $\mathscr{A}$, and the inverse is continuous, $\lim_{n\to\infty}(B - (i/n)1)^{-1} = B^{-1}$ in $\mathscr{A}$. But since $(B - (i/n)1)^{-1} \in C(\{1, B\})$ for all $n$, and since $C(\{1, B\})$ is closed,

$$B^{-1} = \lim_{n\to\infty}(B - (i/n)1)^{-1} \in C(\{1, B\}) \ .$$

Hence, $B$ is invertible within $C(\{1, B\})$.

Now let $B$ be any invertible element of $\mathscr{A}$. Then $B^*$ and $B^*B$ are invertible in $\mathscr{A}$, and also belong to $C(b)$. Since $B^*B$ is self adjoint, what we have proved above says that $(B^*B)^{-1} \in C(b)$. Define $X = (B^*B)^{-1}B^* \in C(\{1, B\})$. Evidently $XB = 1$. Thus, $B$ has a left inverse in $C(\{1, B\})$. The same argument shows that $Y = B^*(BB^*)^{-1}$ is a well defined right inverse of $B$ in $C(\{1, B\})$, and then $X = X(BY) = (XB)Y = Y$ so $X = Y$ is the inverse of $B$ in $C(\{1, B\})$. In particular, for all $\lambda \in \mathbb{C}$, $\lambda 1 - B$ is invertible in $C(\{1, B\})$ if and only if it is invertible in $\mathscr{A}$. Thus, $\lambda 1 - B$ is invertible in $\mathscr{A}$ if and only if it is invertible in $C(\{1, B\})$, and this proves (2.8).

Next, consider the case in which neither $\mathscr{A}$ nor $\mathscr{B}$ are unital. Again, let $B \in \mathscr{B}$ be self adjoint. Then $\sigma_{\mathscr{B}}(B) \subset \mathbb{R}$. Let $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ be the Banach $*$ algebras obtained by using the standard construction to adjoin a unit; note that it is the same unit 1 that is adjoined to both $\mathscr{A}$ and $\mathscr{B}$. Although $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ need not be $C^*$ algebras, this does not matter. Suppose that $(1, B)$ is invertible in $\widetilde{\mathscr{A}}$. then since $\sigma_{\widetilde{\mathscr{B}}}((1, B))$ is real, $(1 + it, B)$ is invertible in $C(\{(1, 0), 1, B)\}$ for all $t \in \mathbb{R}$. Taking limits. we conclude as above that the inverse belongs to $C(\{(1, 0), 1, B)\}$.

Now let $(1, B)$ be any invertible element of $\widetilde{\mathscr{A}}$. Then $(1, B)^*$ and $(1, B)^*(1, B)$ are invertible in $\widetilde{\mathscr{A}}$, and also belong to $C(\{(1, 0), 1, B)\}$. By what we have proved above, $((1, B)^*(1, B))^{-1} \in C(\{(1, 0), 1, B)\}$. The argument proceeds from here just as in the case in the unital case considered at the beginning. □

**2.18 LEMMA.** *Let $\mathscr{A}$ be a unital $C^*$ algebra, and let $A \in \mathscr{A}$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta(C(\{1,A\}))$ onto $\sigma_{\mathscr{A}}(A)$. Let $\mathscr{A}$ be a non-unital $C^*$ algebra, and let $A \in \mathscr{A}$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta'(C(\{A\}))$ onto $\sigma_{\mathscr{A}}(A)$*

*Proof.* First suppose that $\mathscr{A}$ is unital. Since $A$ and $A^*$ commute, the closure of the linear span of $\{\ A^m(A^*)^n\ \ : m,n \geq 0\ \}$ is a $C^*$ algebra that contains 1 and $A$. Evidently, it is $C(\{1,A\})$. Hence if $\varphi \in \Delta(C(\{1,A\}))$, $\varphi$ is determined by its values on $A$ and $A^*$. In fact, since $\varphi$ is necessarily Hermitian, $\varphi$ is determined by its value on $A$. That is, for any $\varphi,\psi \in \Delta(C(\{1,A\}))$,

$$\varphi = \psi \qquad \Longleftrightarrow \qquad \varphi(A) = \psi(A)\ . \tag{2.9}$$

We have also seen that for all $\varphi \in \Delta(C(\{1,A\}))$, $\varphi(a) \in \sigma_{C(\{1,A\})}(A) = \sigma_{\mathscr{A}}(A)$, and for all $\lambda \in \sigma_{\mathscr{A}}(A) = \sigma_{C(\{1,A\})}(A)$, there is a $\varphi_\lambda \in \Delta(C(\{1,A\}))$ such that $\varphi_\lambda(A) = \lambda$. This shows that the map $\varphi \mapsto \varphi(A)$ is a one-to-one map of $\Delta(C(\{1,A\}))$ onto $\sigma_{\mathscr{A}}(A)$. This map is also continuous by the definition of the Gelfand topology, and continuous bijections between compact spaces are homeomorphisms.

The case in which $\mathscr{A}$ is non-unital is similar. Since $A$ and $A^*$ commute, the closure of the linear span of $\{\ A^m(A^*)^n\ \ : m,n \geq 0\ , m+n > 0\ \}$ is a $C^*$ algebra that contains $A$. Evidently, it is $C(\{A\})$. As before, (2.9) is valid, and $\varphi(A) = 0$ if and only if $\varphi$ is the zero character. Then, as before $\varphi \mapsto \varphi(A) \in \sigma_{C(\{A\})}(A)$ is a one to one continuous map from the compact set $\Delta'(C(\{A\}))$ to the compact set $\sigma_{C(\{A\})}(A) = \sigma_{\mathscr{A}}(A)$, and is a homeomorphism. □

We now come to the Abstract Spectral Theorem:

**2.19 THEOREM** (Abstract Spectral Theorem)**.** *Let $\mathscr{A}$ be a $C^*$ algebra with identity* 1*, and let $a \in \mathscr{A}$ be normal. Identifying $\mathscr{C}_{\Delta(C(a))}$ and $\mathscr{C}(\sigma_{\mathscr{A}}(a))$ through the homeomorphism provided by Lemma 2.18, we may regard the Gelfand transform as a homomorphism of $C(A)$ into $\mathscr{C}(\sigma_{\mathscr{A}}(A))$. Then the Gelfand transform $\gamma$ is an isometric isomorphism of $C(\{1,A\})$ onto $\mathscr{C}(\sigma_{\mathscr{A}}(A))$. For all non-negative integers $m,n$, $\gamma(a^m(a^*)^n)$ is the function on $\sigma_{\mathscr{A}}(a)$ given by*

$$\lambda \mapsto \lambda^m(\lambda^*)^n\ . \tag{2.10}$$

*In case $\mathscr{A}$ is non-unital, the homeomorphism provided by Lemma 2.18 identifies $C(\{1,A\})$ with $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$; i.e., the $C^*$ algebra of continuous functions $f$ on $sigma_{\mathscr{A}}(A)$ that vanish at* 0*, and the Gelfand transform $\gamma$ is an isometric isomorphism of $C(\{A\})$ onto $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$, and* (2.10) *is valid for all $m,n \geq 1$.*

*Proof.* The Commutative Gelfand-Neumark Theorem says that $\gamma$ is an isometric isomorphism, and if $\varphi \in \Delta(C(a))$,

$$\gamma(a^m(a^*)^m)[\varphi] = \varphi(a)^m((\varphi(a))^*)^n = \lambda^m(\lambda^*)^n$$

for $\lambda = \varphi(a)$ so that under the identification provided by Lemma 2.18, $\gamma(a^m(a^*)^m)$ is indeed given by (2.10). □

**2.20 DEFINITION.** For a unital $C^*$ algebra $\mathscr{A}$ and for any normal element $A$ of $\mathscr{A}$, and any $f \in \mathscr{C}(\sigma_{\mathscr{A}}(A))$, $f(A)$ is defined by $\gamma^{-1}(f)$; i.e., $f(A)$ is the inverse image of $f$ under the isometric isomorphism of $C(\{1,A\})$ onto $\mathscr{C}(\sigma_{\mathscr{A}}(A))$ that is provided by the Commutative Gelfand Neumark Theorem. Likewise, if $\mathscr{A}$ is not unital, and $f$ is a continuous function of $\sigma_{\mathscr{A}}(A)$ with $f(0) = 0$, $f(A)$ is defined by $\gamma^{-1}(f)$.

**2.21 THEOREM** (Spectral Mapping Theorem)**.** *Let $\mathscr{A}$ be a $C^*$ algebra, $A$ a normal element of $\mathscr{A}$, and $f \in \mathscr{C}(\sigma_{\mathscr{A}}(a))$. Then*

$$\sigma_{\mathscr{A}}(f(A)) = f(\sigma_{\mathscr{A}}(A)) \ .$$

*Proof.* First suppose that $\mathscr{A}$ is unital. For all $\mu \in \mathbb{C}$, the function $\lambda \mapsto \mu - f(\lambda)$ is invertible in $\mathscr{C}(\sigma_{\mathscr{A}}(A))$ if and only if $\mu$ does not belong to the range of $f$, which is $f(\sigma_{\mathscr{A}}(A))$. Then, using the isomorphism provided by the Commutative Gelfand Neumark Theorem, we see that $\mu 1 - f(A)$ is invertible in $C(\{1, A\})$ if and only if $\mu \notin f(\sigma_{\mathscr{A}}(A))$, and hence $\sigma_{C(\{1,A\})}(f(A)) = f(\sigma_{\mathscr{A}}())$. Finally, by Theorem 2.17, the spectrum of $f(A)$ is the same in all $C^*$ subalgebras of $\mathscr{A}$ that contain $f(A)$ and 1. In particular, $\sigma_{C(\{1,A\})}(f(a)) = \sigma_{\mathscr{A}}(f(a))$.

Their argument in the non-unital case is essentially the same, except we must use quasi inverses to characterize the spectrum. □

**2.22 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra with identity* 1*. Let $U \subset \mathbb{C}$ be open with $\overline{U}$ compact. Let $\mathscr{N}_U$ be given by*

$$\mathscr{N}_U = \{ \ A \in \mathscr{A} \ : AA^* = A^*A \quad \text{and} \quad \sigma_{\mathscr{A}}(A) \subset U \ \} \ . \tag{2.11}$$

*Then $\mathscr{N}_U$ is an open subset of the normal elements of $\mathscr{A}$. Moreover, let $f$ be a continuous complex valued function on $\overline{U}$, and for all $A \in \mathscr{N}_U$, define $f(A) \in \mathscr{A}$ using the Abstract Spectral Theorem. Then the map $A \mapsto f(A)$ is continuous on $\mathscr{N}_U$.*

*Proof.* The first assertion is an immediate consequence of Newburg's Theorem, Theorem 1.21. For the second, consider any sequence $\{p_n\}$ of polynomials converging uniformly to $f$ on $\overline{U}$. Then for all $A \in \mathscr{N}_U$,

$$\|p_n(A) = f(A)\| \le \sup_{\lambda \in \overline{U}} \{p_n(\lambda) - f(\lambda)|\}.$$

That is,

$$\lim_{n\to\infty} \left( \sup_{A \in \mathscr{N}_U} \{ \ \|p_n(A) - f(A)\| \ \} \right) = 0 \ .$$

Thus, the function $a \mapsto f(A)$ is the *uniform* limit of the continuous functions $a \mapsto p_n(A)$, □

## 2.4 Positivity in $C^*$ algebras

**2.23 DEFINITION.** Let $\mathscr{A}$ be a $C^*$ algebra. Then a self adjoint element $A \in \mathscr{A}$ is *positive* in case $\sigma_{\mathscr{A}}(a) \subset [0, \infty)$. The set of all positive elements of $\mathscr{A}$ is denoted $\mathscr{A}^+$.

If $A \in \mathscr{A}^+$, we may use the Abstract Spectral Theorem to define $\sqrt{A}$, and then $A = (\sqrt{A})^2 = (\sqrt{A})^*(\sqrt{A})$. One of the 1943 conjectures of Gelfand and Neumark was that for all $B \in \mathscr{A}$, $B^*B$ is positive. This conjecture was finally proved in 1952 and 1953 through the contributions of Fukamiya and Kaplansky, for the case of unital $C^*$ algebras. (Gelfand and Neumark only considered unitary $C^*$-algebras.) The history is interesting: Kaplansky had managed to prove that *if* the sum of two positive elements is necessarily positive, then $B^*B$ is necessarily positive. However, he was unable to show that $\mathscr{A}^+$ was closed under addition. He published nothing, but showed his proof to many people. When Fukamiya proved the closure of $\mathscr{A}^+$ under addition in 1952, the reviewer of Fukamiya's paper in Math Reviews know of Kaplansky's proof, and Kaplansky gave him permission to publish it in the review. Finally, in 1956, Bonsall removed the hypothesis that $\mathscr{A}$ be unital.

**2.24 THEOREM** (Fukamiya's Theorem)**.** *Let $\mathscr{A}$ be a unital $C^*$ algebra. Then $\mathscr{A}^+$ is a pointed convex cone. That is:*

*(1) For all $\lambda \in \mathbb{R}^+$, and all $A \in \mathscr{A}^+$, $\lambda A \in \mathscr{A}^+$, and for all $A, B \in \mathscr{A}^+$, $A + B \in \mathscr{A}^+$.*

*(2) $-\mathscr{A}^+ \cap \mathscr{A}^+ = \{0\}$.*

*(The first part says that $\mathscr{A}^+$ is a convex cone; the second part says that this cone is pointed.)*

*Proof.* Let $B_{\mathscr{A}}$ denote the closed unit ball in $\mathscr{A}$. Fukamiya observed that that $\mathscr{A}^+ \cap B_{\mathscr{A}}$ consists precisely of the self-adjoint elements $A$ with both $A$ and $1 - A$ are in $B_{\mathscr{A}}$. To see this suppose that $A \in \mathscr{A}^+ \cap B_{\mathscr{A}}$. Then since $a$ is self adjoint, $\nu(A) = \|A\| \leq 1$, and so $\sigma_{\mathscr{A}}(A) \subset [0, 1]$. By (an easy case of) the Spectral Mapping Lemma, $\sigma_{\mathscr{A}}(1 - A) \subset [0, 1]$, and hence $\|1 - A\| = \nu(1 - A) \leq 1$.

Conversely, suppose $A$ is self-adjoint and both $A$ and $1-A$ are in $B_{\mathscr{A}}$. Since $A$ is self adjoint and $\|a\| \leq 1$, $\sigma_{\mathscr{A}}(A) \subset [-1, 1]$. Then by the Spectral Mapping Lemma, $\sigma_{\mathscr{A}}(1 - A) \subset [0, 2]$. However, if $\|1 - A\| \leq 1$, then $\sigma_{\mathscr{A}}(1 - A) \subset [-1, 1]$, and altogether we have that $\sigma_{\mathscr{A}}(1 - A) \subset [0, 1]$, and then by the identity $A = 1 - (1 - A)$ and the Spectral Mapping Lemma once more, $\sigma_{\mathscr{A}}(A) \subset [0, 1]$, so that $A \in \mathscr{A}^+$. That is,

$$\mathscr{A}^+ \cap B_{\mathscr{A}} = \{\ a \in \mathscr{A}\ :\ A = A^* \quad \text{and} \quad A \in B_{\mathscr{A}} \cap (1 - B_{\mathscr{A}})\ \}\ . \tag{2.12}$$

Now let $A, B \in B_{\mathscr{A}}$. Then by Minkowski's inequality $\|(A + B)/2\| \in B_{\mathscr{A}}$ and

$$\left\| 1 - \frac{A + B}{2} \right\| \leq \frac{1}{2}(\|1 - B\| + \|1 - B\|)\ . \tag{2.13}$$

If furthermore, $A, B \in \mathscr{A}^+$, then we also have that $\|1 - A\| \leq 1$ and $\|1 - B\| \leq 1$, and then from (2.13), $\|1 - (A + B)/2\| \leq 1$. Thus, $(A + B)/2$ is self adjoint and belongs to both $B_{\mathscr{A}}$ and $1 - B_{\mathscr{A}}$. , and hence $(A + B)/2 \in \mathscr{A}^+$.

Since the closure of $\mathscr{A}^+$ under positive multiples is clear, it then clear that $\mathscr{A}^+$ is closed under sums. Finally, if $A \in \mathscr{A}^+$ and $-A \in \mathscr{A}^+$, then $\sigma_{\mathscr{A}}(A) \subset (-\infty, 0] \cap [0, \infty) = \{0\}$, so that $\|A\| = \nu(a) = 0$, and hence $A = 0$. □

**2.25 THEOREM** (Bonsall's Theorem)**.** *The condition in Fukamiya's Theorem that $\mathscr{A}$ be unital is superfluous.*

*Proof.* Let $\mathscr{A}$ be a non-unital $C^*$ algebra. It suffices to show that $\mathscr{A}^+$ is closed under addition, since the unital nature of $\mathscr{A}$ was used only in proving this fundamental part. Since $A \in \mathscr{A}^+$ is and only if $A$ is self adjoint and $(\lambda, -A)$ is invertible in $\widetilde{\mathscr{A}}$ for all $\lambda < 0$, $A \in \mathscr{A}^+$ if and only if $A$ is self adjoint and $tA$ is quasi regular for all $t > 0$. Now let $A, B \in \mathscr{A}^+$. Then $A + B$ is self adjoint, and to show that $A + B$ in $\mathscr{A}^+$, it suffices to show that for all $t > 0$, $t(A + B) = (tA + tB)$ is quasi regular. Hence it suffices to show that the sum of quasi regular elements in $\mathscr{A}^+$ quasi regular.

Let $A \in \mathscr{A}^+$. We first show that $A'$, the quasi inverse of $A$, satisfies $-A' \in \mathscr{A}^+$ and $\mathscr{A}'\| < 1$. Indeed, by the Abstract Spectral Theorem in $C(\{A\})$, $A' = f(A)$ where

$$f(t) = \frac{-t}{1 + t}\ ,$$

since in $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$, $t + f(t) + tf(t) = t + f(f) + f(t)t$. Since $\nu(A) \leq \|A\|$, $\|A'\| = \nu(A) = \|A\|/(1 + \|A\|) < 1$, and by the Spectral Mapping Theorem, $\sigma_{\mathscr{A}}(A') \subset (-\infty, 0]$.

Now let $A, B \in \mathscr{A}^+$. Then since $\|A'\|, \|B'\| < 1$, $\|A'B'\| < 1$ and hence $(1, -A'B')$ is invertible in $\widetilde{\mathscr{A}}$. That is, $-A'B'$ is quasi regular. Now a simple calculation shows that $A \circ (-A'B') \circ B = A + B$, and by Lemma 1.10, $A + B$ is quasi regular. □

**2.26 THEOREM** (Fukamiya-Kaplansky-Bonsall Theorem)**.** *Let $\mathscr{A}$ be a $C^*$ algebra. For all $A \in \mathscr{A}$, $A^*A \in \mathscr{A}^+$.*

*Proof.* We first show that if $A^*A \in -\mathscr{A}^+$, then $A^*A = 0$. Since $A^*A$ and $AA^*$ have the same spectrum, Fukamiya's Theorem says that $A^*A + AA^* \in -\mathscr{A}^+$. However, writing $A = X + iY$ with $Y$ and $Y$ self adjoint,

$$A^*A + AA^* = 2(X^2 + Y^2) \in \mathscr{A}^+$$

where once again we have used Fukamiya's Theorem, and the Spectral Mapping Lemma. Since $\mathscr{A}^+$ is a pointed cone, his means that $A^*A + AA^* = 0$. But then $A^*A = (A^*A + AA^*) - AA^* = -AA^* \in \mathscr{A}^+$. Again since $\mathscr{A}^+$ is pointed, this means that $A^*A = 0$, as claimed.

Define continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(t) = \max\{t, 0\}$ and $g(t) = t - f(t)$. Note that $f(t)g(t) = 0$ for all $t$. By the Abstract Spectral Theorem, if we define $Y = f(B^*B)$ and $Z = g(B^*B)$, then $YZ = 0$, and $Y + Z = B^*B$. Now define $W = BZ$, Then

$$W^*W = ZB^*BZ = Z(Y + Z)Z = Z^3 \ .$$

Since $\sigma_{\mathscr{A}}(Z) \subset (-\infty, 0]$, $Z^3 \in -\mathscr{A}^+$, and the first part of the proof says that $W^*W = 0$. Therefore, $Z = 0$, and so $B^*B = f(B^*B) \in \mathscr{A}^+$. □

**2.27 DEFINITION.** Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$ algebras. A linear transformation $\Phi : \mathscr{A} \to \mathscr{B}$ is *positive* in case $\Phi(A) \in \mathscr{B}^+$ for all $A \in \mathscr{A}^+$. Define $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ to be the set of positive maps linear transformations from $\mathscr{A}$ to $\mathscr{B}$. The term *positive map* is often used as a synonym for *positive linear transformation.*

**2.28 COROLLARY.** *Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$ algebras. Then $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is a pointed cone in $\mathscr{L}(\mathscr{A}, \mathscr{B})$, the space of linear transformations from $\mathscr{A}$ to $\mathscr{B}$.*

*Proof.* It is evident that $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is closed under multiplication by non-negative scalars, and it is closed under addition as an immediate consequence of Theorem 2.26. If $\Phi \in \mathscr{L}^+(\mathscr{A}, \mathscr{B})$ and $-\Phi \in \mathscr{L}^+(\mathscr{A}, \mathscr{B})$, then for all $A \in \mathscr{A}^+$, $\Phi(A) \in \mathscr{B}^+ \cap (-\mathscr{B}^+) = \{0\}$ by Theorem 2.24 and Theorem 2.25. Since every element of $\mathscr{A}$ is a linear combination of at most 4 elements of $\mathscr{A}^+$. $\Phi(A) = 0$ for all $A \in \mathscr{A}$. This proves that $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is a pointed cone. □

The next corollary of Theorem 2.26 gives us an import class of positive maps.

**2.29 COROLLARY.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $B \in \mathscr{A}$. Then the linear transformation $\Phi : \mathscr{A}$ to $\mathscr{A}$ given by $\Phi(A) := B^*AB$ is positive,*

*Proof.* Let $A \in \mathscr{A}^+$. Then since $f(t) := \sqrt{t}$ is continuous and vanishes at 0, we may use the Abstract Spectral Theorem to define $A^{1/2} \in \mathscr{A}^+$ with $A = A^{1/2}A^{1/2}$. Then

$$\Phi(A) = B^*A^{1.2}A^{1/2}B = (A^{1/2}B)^*(A^{1/2}B) \in \mathscr{A}^+ \ ,$$

where we used Theorem 2.26 in the final step. □

In the next section we shall make use of the following lemma:

**2.30 LEMMA** (Vowden's Lemma). *Let $\mathscr{A}$ be a $C^*$ algebra. For all $A, B \in \mathscr{A}^+$ such that $B - A \in \mathscr{A}^+$, $\|B\| \geq \|A\|$.*

When $\mathscr{A}$ has a unit, this is an easy consequence of the Fukamiya observed that that $\mathscr{A}^+ \cap B_{\mathscr{A}}$ consists precisely of the self-adjoint elements $X$ with both $X$ and $1 - X$ are in $B_{\mathscr{A}}$. : We may suppose without loss of generality that $\|B\| = 1$. Then $1 - B)$ is positive. By Theorem 2.26, $1 - A = (1 - B) + (B - A)$ is positive, and then by Spectral Mapping Theorem,

$$\sigma_{\mathscr{A}}(A) \in [1 - \|A\|, 1] \cap \mathbb{R}_+ \subset [0, 1] \ .$$

Therefore, $\|A\| \leq 1 = \|B\|$.

Vowden has given a proof of the same result without assuming the existence of an identity; his proof makes use of the Abstract Spectral Theorem and the main thermos of this section as follows. as follows:

*Proof of Lemma 2.30.* Define the function $f(t)$ on $[0, 1]$ by

$$f(t) = 1 - \sqrt{1 - t} \ . \tag{2.14}$$

Although the formula for $f$ makes explicit reference to 1, $f$ is continuous and $f(0) = 0$. Therefore, by the Abstract Spectral Theorem $f(B)$ is defined for any $B \in \mathscr{A}$ with $\sigma_{\mathscr{A}}(B) \subset [0, 1]$, even when $\mathscr{A}$ is not unital. Now define

$$C = f(B) \quad \text{and} \quad X = A - CA \ .$$

Then $X^*X = A^2 - ABA$ since $C^2 - 2C + B = 0$. Therefore,

$$A^2 - A^3 = X^*X + ABA - A^3 = X^*X + A(B - A)A \ .$$

Then $A(B - A)A \in \mathscr{A}^+$ by Corollary 2.29, and then $A^2 - A^3 \in \mathscr{A}^+$ by Theorem2.24 and Theorem 2.25. Since for $t > 1$, $t^2 - t^3 < 0$, it now follows from the Spectral Mapping Theorem that no $t > 1$ is in the spectrum of $A$, and hence $\|A\| \leq 1$. □

## 2.5 Adjoining a unit to a non-unital $C^*$ algebra

We will make use of the following approximation lemma. It has other uses, and so we state it a more general form that we need at present to avoid repeating ourselves later on.

**2.31 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\{A_1, \ldots, A_m\}$ be any finite subset of $\mathscr{A}$. Then there is a sequence $\{E_n\}_{n \in \mathbb{N}}$ such that $E_n \in \mathscr{A}^+$ and $\|E_n\| \leq 1$ for all $n \in \mathbb{N}$ such that for each $j = 1, \ldots, m$,*

$$\lim_{n \to \infty} \|A_j E_n - A_j\| = 0 \ . \tag{2.15}$$

*Furthermore:*

*(1) Let $\mathscr{J}$ be a closed ideal in $\mathscr{A}$, and suppose that $\{A_1, \ldots, A_m\} \subset \mathscr{J}$. Then we may take $\{E_n\}_{n \in \mathbb{N}}$ to have the additional property that $E_n \in \mathscr{J}$ for all $n \in \mathbb{N}$, and still have that (2.15) is valid for each $j = 1, \ldots, m$.*

*(2) If $\{A_1, \ldots, A_m\}$ is closed under the involution, we may choose $\{E_n\}_{n\in N}$ to have the additional property that both* (2.15) *and*

$$\lim_{n\to\infty} \|E_n A_j - A_j\| = 0 \tag{2.16}$$

*are valid for $j = 1, \ldots, m$.*

**2.32 REMARK.** *It is a simple consequence of part (1) of Theorem 2.31 that any closed ideal $\mathscr{J}$ in a c*-algebra is closed under the involution. We will make use of this in the next chapter, but will not make use of (1) in this chapter. Theorem 2.31 provides a kind of "approximate identity" in a $C^*$ algebra, and one can make precise sense of this notion. however, in applications, Theorem 2.31 can be applied directly without going through this (short) detour.*

*Proof of Theorem 2.31.* Define $X := \sum_{j=1}^m A_j^* A_j$. Then by Theorems 2.24, 2.25 and 2.26, $X \in \mathscr{A}^+$. Consider the sequence of continuous functions $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f_n(t) = \min\{nt, 1\}$. Note that

$$g_n(t) := t(1 - f_n(t))^2 = \begin{cases} t(1-nt)^2 & t \le 1/n \\ 0 & t > \frac{1}{n} \end{cases} .$$

Moreover, $g_n$ is continuous and $g_n(0) = 0$.

Evidently $\sup_{t\ge 0} |g_n(t)| \le 1/n$, and then. by the Abstract Spectral Theorem, $\|g_n(X)\| \le 1/n$. Note that for each $j = 1, \ldots, m$,

$$(A_j(f_n(X) - A_j)^*(A_j(f_n(X) - A_j) = f_n(X)A_j^*A_jf_n(X) + A_j^*A_j - A_j^*A_jf_n(X) - f_n(X)A_j^*A_j \ .$$

Summing on $j$, $\sum_{j=1}^m (A_j(f_n(X) - A_j)^*(A_j(f_n(X) - A_j) = g_n(X)$, and therefore,

$$\left\| \sum_{j=1}^m (A_j(f_n(X) - A_j)^*(A_j(f_n(X) - A_j) \right\| = \|g(X)\| \le \frac{1}{n} \ .$$

Since $g_n(X) \in \mathscr{A}^+$, and since for each $j = 1, \ldots, m$, $g_n(X) - (A_j(f_n(X) - A_j)^*(A_j(f_n(X) - A_j) \in \mathscr{A}^+$ (by Theorem 2.25 once more), Lemma 2.30 yields

$$\max_{j=1,\ldots,m} \left\{ \|(A_j(f_n(X) - A_j)^*(A_j(f_n(X) - A_j)\| \right\} \le \frac{1}{n} \ .$$

By the $C^*$ algebra identity, $\|(A_j(f_n(X) - A_j)^*(A_j(f_n(X) - J)\| = \|(f_n(X) - A_j)^*\|\|A_jf_n(X) - A\|$, and then by the continuity of the involution, there is a $c > 0$ such that for each $j = 1, \ldots, m$,

$$\frac{1}{n} \ge c\|A_jf_n(X) - A_j\|^2 \ .$$

Thus, $\lim_{n\to\infty} \|A_jf_n(X) - A_j\| = 0$. By the Abstract Spectral Theorem, $\|f_n(X)\| \le 1$ and $f_n(X) \in \mathscr{A}^+$ for all $n$. Hence we may take $E_n = f_n(X)$, and then we have proved that $\lim_{n\to\infty} \|A_jE_n - A_j\| = 0$. This prove the first statement. To prove *(1)*, simply note that if $\{A_1, \ldots, A_m\} \subset \mathscr{J}$, then $X\ in \mathscr{J}$, and $E_n = f(n(X)$ is the norm limit of polynomials in $\mathscr{J}$, and then since $\mathscr{J}$ is closed, $E_n \in \mathscr{J}$.

To prove*(2)*, suppose that $\{A_1, \ldots, A_m\}$ is closed under the involution. Then for each $j \in \{1, \ldots, m\}$, there is some $k \in \{1, \ldots, m\}$ so that $A_j^* = A_k$. Then, $\|E_nA_j - A_j\| = \|E_nA_k^* - A_k^*\| = \|(A_kE_n - A_k)^*\|$ and since $\lim_{n\to\infty} \|A_kE_n - A_k\| = 0$ and since the involution is continuous, $\lim_{n\to\infty} \|(A_kE_n - A_k)^*\| = 0$. It follows that $\lim_{n\to\infty} \|E_nA_j - A_j\| = 0$. □

**2.33 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra. Then for all $A \in \mathscr{A}$,*

$$\sup_{\|B\|\le 1}\{\|AB\|\} = \|A\| = \sup_{\|B\|\le 1}\{\|BA\|\} \tag{2.17}$$

*Proof.* It is evident that for $\|B\| \le 1$, $\|AB\| \le \|A\|$ and $\|BA\| \le \|A\|$. Then, for $A \neq 0$, choosing $B = \|A^*\|^{-1}A^*$, $\|B\| = 1$ and

$$\|AB\| = \frac{\|AA^*\|}{\|A^*\|} = \frac{\|A\|\|A^*\|}{\|A^*\|} = \|A\| \ ,$$

which shows that $\|A\| = \sup_{\|B\|\le 1}\{\|AB\|\}$. This proves the first identity, and the proof of the second is essentially the same. □

Now suppose that $\mathscr{A}$ is not unital, and let $\mathscr{B}(\mathscr{A})$ be the Banach space of bounded linear transformations on $\mathscr{A}$. We embed $\mathbb{C} \oplus \mathscr{A}$ into $\mathscr{B}(\mathscr{A})$ as follows: For $(\lambda, A) \in \mathbb{C} \oplus \mathscr{A}$, define $T_{(\lambda,A)} \in \mathscr{B}(\mathscr{A})$ by

$$T_{(\lambda,A)}B = \lambda B + AB$$

for all $B \in \mathscr{A}$. To see that $(\lambda, A) \mapsto T_{(\lambda,A)}$ is one-to-one, suppose that for some $(\lambda, A) \in \mathbb{C} \oplus \mathscr{A}$, $T_{(\lambda,A)} = 0$. If $\lambda = 0$, this is impossible unless also $A = 0$ since if $T_{(0,A)} = 0$, then $AB = 0$ for all $B$ and then by Lemma 2.33, $A = 0$. However, if $T_{(\lambda,A)} = 0$ and $\lambda \neq 0$, replacing $A$ by $-\lambda^{-1}A$, we have that $T_{(-1,A)} = 0$, but then for all $B \in \mathscr{A}$, $B = AB$. Taking $B = A^*$, $A^* = AA^*$, and hence $A$ is self-adjoint. But then $B^* = B^*A$ for all $B \in \mathscr{A}$, and hence we have that $B = AB = BA$ for all $B \in \mathscr{A}$. This is impossible since $\mathscr{A}$ lacks a multiplicative identity. Hence $(\lambda, A) \mapsto T_{(\lambda,A)}$ is one-to-one. from $\mathbb{C}\oplus\mathscr{A}$ into $\mathscr{B}(\mathscr{A})$. Now define a norm $|\!|\!|\cdot|\!|\!|$ on $\mathbb{C}\oplus\mathscr{A}$ by $|\!|\!|(\lambda, A)|\!|\!| = \|T_{(\lambda,A)}\|_{\mathscr{B}(\mathscr{A})}$. That is,

$$|\!|\!|(\lambda, A)|\!|\!| = \sup_{\|B\|\le 1}\{\|\lambda B + AB\|\} \ . \tag{2.18}$$

By Lemma 2.33.

$$|\!|\!|(0, A)|\!|\!| = \sup_{\|B\|\le 1}\{\|AB\|\} = \|A\| \ , \tag{2.19}$$

and so $A \mapsto T_{(0,A)}$ is an isometry form $\mathscr{A}$ in $\mathscr{B}(\mathscr{A})$. In particular, the image of $\mathscr{A}$ under this embedding is closed. The image of $\mathbb{C}\oplus\mathscr{A}$ under this embedding is the sum of the image of $\mathscr{A}$ and a one dimensional subspace. Since the sum of a closed subspace and a finite dimensional subspace is always closed, the image of $\mathbb{C}\oplus\mathscr{A}$ under this embedding is closed in $\mathscr{B}(\mathscr{A})$, and hence complete since $\mathscr{B}(\mathscr{A})$ is a Banach space. It follows that $\mathbb{C}\oplus\mathscr{A}$ is complete in the norm $|\!|\!|\cdot|\!|\!|$. Since $\mathscr{B}(\mathscr{A})$ is a Banach algebra, $\mathbb{C}\oplus\mathscr{A}$ is a Banach algebra in this norm; it inherits the multiplicative property from $\mathscr{B}(\mathscr{A})$.

**2.34 LEMMA.** *Let $\mathscr{A}$ be a non-unital $C^*$ algebra. Then $\mathbb{C}\oplus\mathscr{A}$, equipped with the standard multiplication define in* (1.3)*, and the norm $|\!|\!|\cdot|\!|\!|$ define in* (2.18) *is a Banach algebra. If we equip $\mathbb{C}\oplus\mathscr{A}$ with the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$, then it becomes a Banach $*$ algebra, and the map $A \mapsto (0, A)$ is an isometric $*$-isomorphism of $\mathscr{A}$ into this Banach $*$-algebra.*

**2.35 LEMMA.** *In the notation introduced above, we have the alternate formula*

$$|\!|\!|(\lambda, A)|\!|\!| = \sup_{\|C\|\le 1}\{\|\lambda C + CA\|\} \ . \tag{2.20}$$

*Proof.* Choose $\epsilon > 0$, and choose $B \in \mathscr{A}$, $\|B\| \leq 1$ such that

$$(1-\epsilon)|\!|\!|(\lambda, A)|\!|\!| \leq \|\lambda B + AB\| \ .$$

Then by the second identity in (2.17), there exists $C \in \mathscr{A}$, $\|C\| \leq 1$, so that

$$(1-\epsilon)\|\lambda B + AB\| \leq \|(\lambda C + CA)B\| \leq \|\lambda C + CA\| \ .$$

Since $\epsilon > 0$ is arbitrary, $\sup_{\|C\|\leq 1}\{\|\lambda C + CA\|\} \geq |\!|\!|(\lambda, A)|\!|\!| = \sup_{\|B\|\leq 1}\{\|\lambda B + AB\|\}$. The same sort of reasoning then shows that

$$\sup_{\|B\|\leq 1}\{\|\lambda B + AB\|\} \leq \sup_{\|C\|\leq 1}\{\|\lambda C + CA\|\} \ .$$

□

We are now ready to prove:

**2.36 THEOREM** (Vowden's Theorem)**.** *Let $\mathscr{A}$ be a non-unital $C^*$ algebra. Then $\mathbb{C}\oplus\mathscr{A}$, equipped with the standard multiplication define in* (1.3)*, the norm $|\!|\!| \cdot |\!|\!|$ define in* (2.18) *and the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$, is a unital $C^*$ algebra, and the map $A \mapsto (0, A)$ is an isometric $*$-isomorphism of $\mathscr{A}$ into this unital $C^*$-algebra. In particular, every $C^*$ algebra $\mathscr{A}$ is isometrically and $*$-isomorphically embedded in in unital $C^*$ algebra.*

*Proof.* We need only prove that $C^*$ algebra identity $|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| = |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!|$, and since we already know that $|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \leq |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!|$, it remains to prove that

$$|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \geq |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!| \ . \tag{2.21}$$

Choose $\epsilon > 0$. Then, by Lemma 2.35, we may choose $B, C \in \mathscr{A}$ with $\|B\|, \|C\| \leq 1$ such that

$$|\!|\!|(\lambda, A)^*|\!|\!| \leq (1+\epsilon)\|\overline{\lambda}B + BA^*\| \quad \text{and} \quad |\!|\!|(\lambda, A)|\!|\!| \leq (1+\epsilon)\|\lambda C + AC\| \ .$$

Then by Theorem 2.31 applied to $\{B, C\}$, for every $\delta > 0$, there exists $E \in \mathscr{A}^+$, with $\|E\| \leq 1$ such that

$$\|BE - B\|, \|EC - C\| \leq \delta \ .$$

For appropriate $\delta$ we then have

$$|\!|\!|(\lambda, A)|\!|\!| \leq (1+\epsilon)^2\|(\lambda E + AE)C\| \leq (1+\epsilon)^2\|\lambda E + AE\| \ .$$

and

$$|\!|\!|(\lambda, A)^*|\!|\!| \leq (1+\epsilon)^2\|B(\overline{\lambda}E + EA^*)\| \leq (1+\epsilon)^2\|(\lambda E + AE)^*\| \ .$$

Altogether we have

$$|\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!| \leq (1+\epsilon)^4\|(\lambda E + AE)^*\|\|\lambda E + AE\| \ . \tag{2.22}$$

Then by the $C^*$ norm identity,

$$\begin{aligned}
\|(\lambda E + AE)^*\|\|\lambda E + AE\| &= \|(\lambda E + AE)^*(\lambda E + AE)\| \\
&= \|(\lambda E + EA^*)^*(\lambda E + AE)\| \\
&= \|E(|\lambda|^2 E + (\overline{\lambda}A + \lambda A^* + A^*A)E)\| \\
&\leq \||\lambda|^2 E + (\overline{\lambda}A + \lambda A^* + A^*A)E\| \\
&\leq |\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \ .
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, combining this with (2.22) yields the result. □

Vowden's Theorem has a number of consequences. Our first application of it, in the next section, will be the one that motivated Vowden – to finally prove that the involution is an isometry in full generality. This has been done earlier by Glimm and Kadison in the unital case, and Vowden's theorem immediately extends this to the general case.

## 2.6 The Russo-Dye Theorem and the Isometry Property

**2.37 DEFINITION** (Unitary)**.** Let $\mathscr{A}$ be a unital $C^*$ algebra with multiplicative identity 1. Then $U \in \mathscr{A}$ is unitary in case

$$U^*U = UU^* = I \ . \tag{2.23}$$

Note in particular that every unitary $U$ is normal. Therefore, by Lemma 2.8, $\|U^*\| = \|U\|$, and by the $C^*$ algebra norm identity, $1 = \|1\| = \|U^*U\| = \|U^*\|\|U\| = \|U\|^2$. Therefore, $\|U\| = 1$ for all unitaries.

Noe let $A \in B_{\mathscr{A}}$. Define $(A^*A)^{1/2}$ using the Abstract Spectral Theorem, If $A$ is invertible, so are $A^*$ and $A^*A$, and then $(A^*A)^{-1}(A^*A)^{1/2}$ is the inverse of $(A^*A)^{1/2}$ which we denote by $(A^*A)^{-1/2}$. For such $A$, we then define

$$|A| := (A^*A)^{1/2} \qquad \text{and} \qquad V := A(A^*A)^{-1/2} \ .$$

Then $V^*V = (A^*A)^{-1/2}A^*A(A^*A)^{-1/2} = 1$. Since $V$ is the product of invertible elements of $\mathscr{A}$, $V$ is invertible, and hence its left inverse $V^*$ is also its inverse. That is $V^*V = VV^* = 1$. In particular, $V$ is unitary. This proves the greater part of the following lemma:

**2.38 LEMMA.** *Let $\mathscr{A}$ be a unital $C^*$ algebra, and let $A \in \mathscr{A}$. If $A$ is invertible, then there is a unitary $V$ and a positive operator $H$ such that $A = VH$. Moreover, $H$ and $V$ are uniquely determined, and are given by $H = |A| := (A^*A)^{1/2}$ and $V = A|A|^{-1}$.*

*Proof.* It remains to prove the uniqueness. Suppose that $A = WK$ where $K \in \mathscr{A}^+$ and $W$ is unitary. Then $A^*A = K^2$, and hence $K = (A^*A)^{1/2} = |A|$, and then $W = V$ follows immediately. □

The factorization $A = VH$ provided by Lemma 2.38 is called the *polar factorization of* $A$ .

**2.39 LEMMA.** *Let $\mathscr{A}$ be a unital $C^*$ algebra. Then for all invertible $A \in \mathscr{A}$, $\|A^*\| = \|A\|$.*

*Proof.* Let $A$ be invertible in $\mathscr{A}$, and let $A = VH$ be its polar factorization. Then

$$\|A\| = \|VH\| \leq \|v\|\|H\| = \|H\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = (\|A^*\|\|A\|)^{1/2} \ .$$

It follows that $\|A\| \leq \|A^*\|$. Since $A^*$ is also invertible, the same reasoning yields $\|A^*\| \leq \|A\|$. □

**2.40 LEMMA** (Gardner's Lemma)**.** *Let Let $\mathscr{A}$ be a unital $C^*$ algebra. For all invertible $A$ in $B_{\mathscr{A}}$, the unit ball of $A$, there exits two unitaries $U_1$ and $U_2$ such that*

$$A = \frac{1}{2}(U_1 + U_2) \ . \tag{2.24}$$

*Proof.* Let $A = VH$ be the polar factorization of $A$. Then since $A$ is invertible $\|H\| = \|(A^*A)^{1/2}\| = \|(A^*A)^{\|}1/2 = \|A\| \leq 1$ by Lemma 2.39. By the Spectral Mapping Theorem, $\sigma_{\mathscr{A}}(A^2) \subset [0,1]$. Then we may define, using the Abstract Spectral Theorem,

$$W := H + i\sqrt{1 - H^2}\ .$$

Then

$$W^*W = WW^* = H^2 + (1 - H^2) = 1\ ,$$

and hence $W$ and $W^*$ are unitary. Evidently, $H = \frac{1}{2}(W + W^*)$ and then with $U_1 = VW$ and $U_2 = VW^*$, $U_1$ and $U_2$ are unitary and (2.24) is satisfied. □

**2.41 THEOREM** (Kadison-Pederson)**.** *Let $n \in \mathbb{N}$, $n \geq 2$. If $A \in \mathscr{A}$ satisfies $\|A\| \leq 1 - \frac{2}{n}$, then there are unitaries $\{U_1, \dots, U_n\}$ such that*

$$A = \frac{1}{n}\sum_{j=1}^{n} U_j\ . \tag{2.25}$$

*Proof.* Let $B \in \mathscr{A}$, $\|B\| \leq 1$ and let $V$ be unitary. Then $\|V^*B\| \leq \|V*\|\|B\| < 1$. Therefore $1 + V^*B$ is invertible, with the consequence that $V + B$ is invertible. That is, for all $A$ with $\|B\| < 1$ and al unitaries $V$, $V + B$ is invertible. Then $\frac{1}{2}(V + B)$ is an invertible element of $B_{\mathscr{A}}$, and by Gardener's Lemma, there are unitaries $U_1$ and $U_2$ such that $\frac{1}{2}(V + B) = \frac{1}{2}(U_1 + U_2)$. That is, given any $B \in \mathscr{A}$ with $\|B\| < 1$, and any unitary $V$, there are two other unitaries $U_1$ and $U_2$ such that

$$B + V = U_1 + U_2. \tag{2.26}$$

We now claim that for any $n > 2$, there are unitaries $U_1, \dots, U$ such that

$$(n-2)B = \sum_{j=1}^{n} U_j \tag{2.27}$$

To see this, note that if we define $U_3 = -V$, then (2.26) gives us the $n = 3$ case of (2.27). We proceed by induction. Suppose that for some $n \geq 3$, (2.27) is valid. Then for any unitary $V$, using both (2.26) and (2.27)

$$(n-1)B = B + (n-2)B = B + \sum_{j=1}^{n} U_j = \sum_{j=1}^{n-1} U_1 + B + U_n = \sum_{j=1}^{n-1} U_1 + V_1 + V_2$$

with $U_1, \dots, U_n, V_1, V_2$ unitary. Changing the notation, we have the validity of (2.27) for $n + 1$, completing the inductive proof of (2.27), which can be written as

$$\left(1 - \frac{2}{n}\right) B = \frac{1}{n}\sum_{j=1}^{n} U_j\ .$$

Then for any $A \in \mathscr{A}$ with $\|A\| < (1 - 2/n)$, define $B := (1 - 2/n)^{-1}A$ so that $\|B\| < 1$. Then (2.27) is valid and in terms of $A$, it is (2.25). □

**2.42 COROLLARY** (Russo-Dye Theorem)**.** *Let $\mathscr{A}$ be a unital $C^*$ algebra. Then $B_{\mathscr{A}}$, the closed unit ball of $\mathscr{A}$, is the norm closure of the convex hull of the unitary elements of $B_{\mathscr{A}}$.*

**2.43 THEOREM** (Isometry property of the involution)**.** *Let $\mathscr{A}$ be a $C^*$ algebra, not necessarily unital. Then for all $A \in \mathscr{A}$, $\|A^*\| = \|A\|$.*

*Proof.* First suppose that $\mathscr{A}$ is unital. For $A \in \mathscr{A}$ with $\|A^*\|\|A\| = 1$ if $\|A\| \neq \|A^*\|$, then either $\|A\| < 1$, or $\|A^*\| < 1$. Replacing $A$ by $A^*$ if needed, we may suppose that $\|A\| < 1$ and then $\|A^*\| > 1$. Pick $n \in \mathbb{N}$, so that $\|A\| \leq (1 - 2/n)$. Then there are unitaries $U_1, \dots, U_n$ such that $A = \frac{1}{n}\sum_{j=1}^n U_j$. Then

$$\|A^*\| = \left\|\frac{1}{n}\sum_{j=1}^n U_j^*\right\| \leq \frac{1}{n}\sum_{j=1}^n \|U_j^n\| = 1\ .$$

This contradiction, and a simple scaling argument, proves the result in this case.

If $\mathscr{A}$ is not unital, Vowden's Theorem says that $\mathscr{A}$ is a $C^*$ subalgebra of a unital $C^*$-algebra $\mathscr{A}_1$. By what we just proves, the involution is an isometry in $\mathscr{A}_1$, and hence in $\mathscr{A}$. □

With the isometry property finally in hand, we may prove that the norm in a $C^*$ algebra $\mathscr{A}$ is uniquely determined by the $*$-algebra itself.

**2.44 THEOREM** (Uniqueness of the norm)**.** *Let $\mathscr{A}$ be a $C^*$ algebra. Then for all $A \in \mathscr{A}$,*

$$\|A\|^2 = \nu(A^*A)\ .$$

*Proof.* By the $C^*$ algebra identity, Theorem 2.43 and then Lemma 2.8, $\|A^*A\| = \|A^*\|\|A\| = \|A\|^2$. □

## 2.7 Homeomorphisms of $C^*$ algebras

**2.45 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\mathscr{J}$ be a norm-closed ideal in $\mathscr{A}$. Then $\mathscr{J}$ is closed under the involution.*

*Proof of Theorem 2.45.* Let $A \in \mathscr{J}$. By part *(1)* of Theorem 2.31, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of positive elements of $\mathscr{J}$, with $\|E_n\| \leq 1$ for all $n$, such that $\lim_{n\to\infty} \|AE_n - A\| = 0$. Since the involution is an isometry, $\|E_nA^* - A^*\| = \|AE_n - A\|$, and $E_nA^* \in \mathscr{J}$, $\lim_{n\to\infty} \|E_nE^* - E^*\| = 0$. Then since $\mathscr{J}$ is closed, $E^* \in \mathscr{J}$. □

Now let $\mathscr{A}$ be a $C^*$ algebra, and let $\mathscr{J}$ be a closed ideal in $\mathscr{J}$. As usual, let $\{A\}$ denote equivalence class of $a$ mod $\mathscr{J}$, and let $\|\{A\}\|\|$ denote the quotient norm of $\{a\}$; that is,

$$\|\{A\}\|\| = \inf\{\ \|A - B\|\ :\ B \in \mathscr{J}\ \}\ .$$

Then $\mathscr{A}/\mathscr{J}$ is a Banach algebra with the quotient space norm. By Theorem 2.45, $A - B \in \mathscr{J} \iff A^* - A^* \in \mathscr{J}$, and therefore we may define an involution on on $\mathscr{J}/\mathscr{A}$ by

$$\{A\}^* = \{A^*\}\ .$$

Evidently this involution is an isometry, and so for all $A \in \mathscr{A}$, $\|\{A\}^*\{A\}\| \leq \|\{A\}\|^2$. To show that $\mathscr{A}/\mathscr{J}$ is a $C^*$ algebra with this involution, we need only show that for all $A \in \mathscr{A}$,

$$\|\{A\}\|^2 \leq \|\{A\}^*\{A\}\|\ . \tag{2.28}$$

It will be convenient to have a unit. If $\mathscr{A}$ is not unital, let $\mathscr{A}_1$ be the $C^*$ algebra obtained by adjoining a unit in the canonical manner. Then $\mathscr{J}$, identified with its canonical embedding into $\mathscr{A}_1$, is still a closed ideal in $\mathscr{A}_1$, and for all $(\lambda, a), (\mu, B)$ in $\mathscr{A}_1$, $(\lambda, A) \sim (\mu, B)$ if and only if $(\lambda - \mu, A - B) \in \mathscr{J}$. Evidently this is the case if and only if $\lambda = \mu$ and $A \sim Bin\mathscr{A}$. In particular for all $A \in \mathscr{A}$, the equivalence class of $(0, A)$ in $\mathscr{A}_1$ is precidely the set of $(0, B)$ with $B \sim A$ in $\mathscr{A}_1$, and $\|\{(0, A)\}\| = \|\{A\}\|$. Hence it suffices to prove (2.28) in unital algebras.

**2.46 LEMMA.** *Let $\mathscr{A}$ be a unital $C^*$ algebra, and let $\mathscr{J}$ be a closed ideal in $\mathscr{A}$. For all $A \in \mathscr{A}$, the quotient norm of $\{A\}$ is given by*

$$\|\{A\}\| = \inf\{\ \|A - AE\|\ :\ E \in \mathscr{J} \quad \text{and} \quad E \in \mathscr{A}^+ \cap B_{\mathscr{A}}\ \}\ . \tag{2.29}$$

*Proof.* Whenever $E \in \mathscr{J}$, $A \sim (A - AE)$ so that $\|\{A\}\|$ is no greater than the right hand side of (2.29). To prove the equality, pick $\epsilon > 0$ and $B \in \mathscr{J}$ so that $\|\{A\}\| \geq \|A - B\| - \epsilon$. By Theorem 2.31 applied to $\{B\}$, we can choose $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ so $\|B - BE\| < \epsilon$.

Then by (2.12), $\|1 - E\| \leq 1$, and so

$$\begin{aligned}\|A - B\| \geq \|A - B\|\|1 - E\| \geq \|(A - B)(1 - E)\| = \|(A - AE) - (B - BE)\| \geq \\ \|A - AE\| - \|B - BE\| \geq \|A - AE\| - \epsilon\ .\end{aligned}$$

Hence $\|\{A\}\| \geq \|A - AE\| - 2\epsilon$, and since $\epsilon > 0$ is arbitrary, (2.29) is proved. □

Now to prove (2.28) for unital $\mathscr{A}$, pick $\epsilon > 0$ and $E = E^* \in \mathscr{J}$ with $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ so that

$$\|a^*a(1 - E)\| \leq \|\{A\}^*\{A\}\| + \epsilon = \|\{A\}\|^2 + \epsilon\ .$$

Then

$$\|\{A\}\|^2 \leq \|A(1 - E)\|^2 = \|(1 - E)A^*A(1 - E)\| \leq \|(1 - E)\|\|A^*A(1 - E)\| \leq \|A^*A(1 - E)\|\ .$$

where the last in equality is valid since by (2.12), $\|1 - E\| \leq 1$. Altogether, $\|\{A\}\|^2 \leq \|\{A\}^*\{A\}\| + \epsilon$, and since $\epsilon > 0$ is arbitrary, (2.28) is proved. We have shown:

**2.47 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\mathscr{J}$ be a norm closed ideal in $\mathscr{A}$, then $\mathscr{J}$ is closed under the involution, and the definition $\{A\}^* = \{A^*\}$ defines an involution on $\mathscr{A}/\mathscr{J}$ so that, equipped with the quotient norm, $\mathscr{A}/\mathscr{J}$ is a $C^*$ algebra.*

**2.48 LEMMA.** *Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$ algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Then $\pi$ is a contraction; i.e., $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathscr{A}$. If moreover $\pi$ is one-to-one, $\pi$ is an isometry.*

*Proof.* For all $A \in \mathscr{A}$, by the Spectral Contraction Theorem, $\nu(\pi(A)^*\pi(A)) = \nu(\pi(A^*A) \leq \nu(A^*A)$. Since for self adjoint elements of a $C^*$ algebra, the norm is the spectral radius, $\|\pi(A)^*\pi(A)\| \leq \|A^*A\| = \|A^*\|\|A\| = \|A\|^2$. Thus, $\|\pi(A)\|^2 \leq \|A\|^2$, and this proves that $\pi$ is a contraction.

Notice that if $\nu(\pi(A^*A)) = \nu(A^*A)$, the argument gives $\|\pi(A)\| = \|A\|$. Hence it remains to show that if $\pi$ is one-to-one, $\pi$ cannot decrease the spectral radius of any self adjoint element of $\mathscr{A}$.

Indeed, let $A = A^* \in \mathscr{A}$, and suppose that $\nu(\pi(A)) < \nu(A)$. Then there is a non-zero continuous bounded function $f$ supported on $[-\nu(A), \nu(A)]$ that vanishes identically on $[-\nu(\pi(A)), \nu(\pi(A))]$. since $f$ may be approximated by polynomials, $\pi(f(A)) = f(\pi(A))$. However, since $f$ vanishes identically on the spectrum of $\pi(A)$, $f(\pi(A)) = 0$. Thus, $f(A)$ is in the kernel of $\pi$, which is a contradiction. Hence, when $\pi$ is one-to-one, it preserves the spectral radius of self adjoint elements.
□

We summarize with the following theorem:

**2.49 THEOREM** (Homomorphisms of $C^*$ algebras)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$ algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Then $\pi$ is a contraction, $\pi(\mathscr{A})$ is a $C^*$-subalgebra of $\mathscr{B}$, and $\pi$ induces an isometric isomorphism of $\mathscr{A}/\ker(\pi)$ onto $\pi(\mathscr{A})$.*

# 3 The strong, the weak and the $\sigma$-weak operator topologies

## 3.1 Topologies on $\mathscr{B}(\mathscr{H})$

Three topologies other than the norm topology are fundamental to what follows. After recalling some basic facts, we introduce the first two of these. Let $\mathscr{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and norm $\| \cdot \|_{\mathscr{H}}$, and let $\mathscr{B}(\mathscr{H})$, as usual, denote the $C^*$-algebra of bounded linear operators on $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, $\text{ran}(A)$ denote the closure of the range of $A$, and $\ker(A)$ denotes the null space of $A$. If $\mathscr{K}$ is a subspace of $\mathscr{H}$, $\mathscr{K}^\perp$ is the orthogonal complement of $\mathscr{K}$, which is a closed subspace of $\mathscr{H}$.

**3.1 LEMMA** (Polarization identify)**.** *For all $A \in \mathscr{B}(\mathscr{H})$ and all $\zeta$ and $\xi$ in $\mathscr{H}$,*

$$\begin{aligned}\langle \zeta, A\xi \rangle_{\mathscr{H}} = &\frac{1}{4}\left[\langle (\zeta+\xi), A(\zeta+\xi)\rangle_{\mathscr{H}} - \langle (\zeta-\xi), A(\zeta-\xi)\rangle_{\mathscr{H}}\right] \\ &- \frac{i}{4}\left[\langle (\zeta+i\xi), A(\zeta+i\xi)\rangle_{\mathscr{H}} - \langle (\zeta-i\xi), A(\zeta-i\xi)\rangle_{\mathscr{H}}\right] \ .\end{aligned}$$

*Proof.* We compute

$$\frac{1}{4}\left[\langle (\zeta+\xi), A(\zeta+\xi)\rangle_{\mathscr{H}} - \langle (\zeta-\xi), A(\zeta-\xi)\rangle_{\mathscr{H}}\right] = \frac{1}{2}\left[\langle \zeta, A\xi\rangle_{\mathscr{H}} + \langle \xi, A\zeta\rangle_{\mathscr{H}}\right]$$

and

$$\frac{1}{4}\left[\langle (\zeta+i\xi), A(\zeta+i\xi)\rangle_{\mathscr{H}} - \langle (\zeta-i\xi), A(\zeta-i\xi)\rangle_{\mathscr{H}}\right] = \frac{i}{2}\left[\langle \zeta, A\xi\rangle_{\mathscr{H}} - \langle \xi, A\zeta\rangle_{\mathscr{H}}\right] \ .$$

□

**3.2 DEFINITION** (Strong and weak operator topologies)**.** The *strong operator topology* on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi \in \mathscr{H}$, the function $A \mapsto A\xi$ from $\mathscr{B}(\mathscr{H})$ to $\mathscr{H}$ is continuous with the usual norm topology on $\mathscr{H}$. The *weak operator topology* on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi, \zeta \in \mathscr{H}$, the function $A \mapsto \langle \zeta, A\xi \rangle_{\mathscr{H}}$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathbb{C}$.

It follows from the definitions that a basic set of neighborhoods of 0 for the strong operator topology is given by the sets

$$U_{\epsilon,\xi_1,\dots,\xi_n} = \{ A \in \mathscr{B}(\mathscr{H}) \ : \ \|A\xi_j\|_{\mathscr{H}} < \epsilon \quad \text{for} \quad j = 1, \dots, n \ \} \tag{3.1}$$

where $\epsilon > 0$ and $\xi_1, \dots, x_n \in \mathscr{H}$. Likewise, it follows that a basic set of neighborhoods of 0 for the weak operator topology is given by the sets

$$V_{\epsilon,\xi_1,\dots,\xi_n} = \{ A \in \mathscr{B}(\mathscr{H}) \ : \ |\langle \xi_j, A\xi_j \rangle_{\mathscr{H}}| < \epsilon \quad \text{for} \quad j = 1, \dots, n \ \} \tag{3.2}$$

$\epsilon > 0$ and $\xi_1, \dots, xi_n \in \mathscr{H}$. Both topologies are evidently Hausdorff. Since for all $A \in \mathscr{B}(\mathscr{H})$ and all $\xi \in \mathscr{H}$, $|\langle xi, A\xi \rangle| = |\langle xi, A^*\xi \rangle|$, $V_{\epsilon,\xi_1,\dots,\xi_n} = V^*_{\epsilon,\xi_1,\dots,\xi_n}$, and hence $A \mapsto A^*$ is continuous in the weak operator topology.

**3.3 LEMMA.** *The weak operator topology on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi, \zeta \in \mathscr{H}$, the function $A \mapsto \langle \zeta, A\xi \rangle_{\mathscr{H}}$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathbb{C}$.*

*Proof.* This follows immediately from the polarization identity. □

Since for each $\xi \in \mathscr{H}$, $A \mapsto A\xi$ is continuous in the norm topology on $\mathscr{B}(\mathscr{H})$, the norm topology is at least as strong as the strong operator topology. Similarly, for all $\xi \in \mathscr{H}$, the function $A \mapsto \langle \xi, A\xi \rangle_{\mathscr{H}}$ is continuous in the strong operator topology, and hence the strong operator topology is at least as strong as the weak operator topology.

The following proposition shows that the norm topology is *strictly stronger* than the strong operator topology, which is in turn *strictly stronger* than the weak operator topology.

**3.4 PROPOSITION** (Continuity of the norm and adjoint)**.** *Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Then:*

*(1) The function $A \mapsto \|A\|$ from $\mathscr{B}(\mathscr{H})$ to $\mathbb{R}_+$ is continuous in the norm topology, but is only lower semicontinuous in the strong and weak operator topologies.*

*(2) The function $A \mapsto A^*$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ in the norm and the weak operator topologies, but not in the strong operator topology.*

*Proof.* Let $\{\zeta_j\}$ be an orthonormal sequence in $\mathscr{H}$. For $n \in \mathbb{N}$, let $P_n$ denote the orthogonal projection onto the span of $\{\zeta_1, \dots, \zeta_n\}$. For all $\xi \in \mathscr{H}$, $\lim_{n\to\infty} \|P_n\xi\| = 0$ by Bessel's inequality, so that $\lim_{n\to\infty} P_n = 0$ in the strong operator topology. Since for $n \neq m$, $\|P_n - P_m\| = 1$, the sequence $\{P_n\}$ is not even Cauchy in the norm topology. Hence the norm is discontinuous in the strong operator topology, and hence also in the weak operator topology.

To see that the norm is lower semicontinuous in these topologies, it suffices to show that the sub-level sets $\{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$ are closed for each $t > 0$. Fix $t > 0$ and suppose that $B$ is in the strong operator topology closure of $\{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$. Then for each unit vector $\xi \in \mathscr{H}$, and each $n \in \mathbb{N}$ there is an $A_n \in \{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$ such that $B - A_n \in U_{1/n,\xi}$, which means that $\|(B - A_n)\xi\| < 1/n$, and hence $\|B\xi\| \leq \|A_n\| + 1/n \leq t + 1/n$. Since $n$ is arbitrary, $\|B\xi\| \leq t$. Then since $\xi$ is an arbitrary unit vector in $\mathscr{H}$, $\|B\| \leq t$. This proves the closure in the strong operator topology, and a very similar argument proves the closure for the weak operator topology.

For the second part, the continuity of the involution is obvious in the norm topology and we have seen it is true in the weak operator topology. To see that the involution is not continuous for the strong operator topology when $\mathscr{H}$ is infinite dimensional, note that every such Hilbert space contains a copy of $\ell_2$, the Hilbert space of all square summable functions from $\mathbb{N}$ to $\mathbb{C}$, we may suppose without loss of generality that $\mathscr{H} = \ell_2$. Define the shift operator $a \in \mathscr{B}(\mathscr{H})$ by

$$(A\zeta)_j = \begin{cases} \zeta_{j-1} & j \geq 2 \\ 0 & j = 1 \end{cases}$$

Evidently, for all $\zeta$, $\|A\zeta\|_{\mathscr{H}} = \|\zeta\|_{\mathscr{H}}$. The adjoint is given by $(A^*\zeta)_j = \zeta_{j+1}$ for all $j \in \mathbb{N}$. Therefore, $\|A^*\zeta\|^2_{\mathscr{H}} = \sum_{j=2}^{\infty} |\zeta_j|^2 = \|\zeta\|^2 - |\zeta_1|^2$. It follows that for all $\zeta \in \mathscr{H}$,

$$\lim_{n\to\infty} \|(A^n)^*\zeta\|_{\mathscr{H}} = 0 \quad \text{while} \quad \|A^n\zeta\|_{\mathscr{H}} = \|\zeta\|_{\mathscr{H}} \ .$$

Hence the sequence $\{(A^n)^*\}$ converges to zero in the strong operator topology, but the sequence $\{A^n\}$ does not. Since $\{A^n\} = \{(A^n)^{**}\}$ this shows that the involution is not continuous in the strong operator topology. □

As far as sequences are concerned, a sequence $\{A_n\}$ in $\mathscr{B}(\mathscr{H})$ converges to $A \in \mathscr{B}(\mathscr{H})$ in the strong operator topology if and only if for all $\xi \in \mathscr{H}$, $\lim_{n\to\infty} A_n\xi = A\xi$, and likewise, converges to $A \in \mathscr{B}(\mathscr{H})$ in the weak operator topology if and only if for all $\zeta, \xi \in \mathscr{H}$, $\lim_{n\to\infty}\langle \zeta, A_n\xi\rangle_{\mathscr{H}} = \langle A\xi\rangle_{\mathscr{H}}$. A sequence $\{A_n\}$ in $\mathscr{B}(\mathscr{H})$ is a *Cauchy sequence for the weak operator topology* in case for every basic open neighborhood $V_{\epsilon,\xi_1,\ldots,\xi_n}$ of 0, $a_m - a_\ell \in V_{\epsilon,\xi_1,\ldots,\xi_n}$ for all but finitely many $\ell, m$. Cauchy sequences for the strong operator topology are defined analogously.

**3.5 THEOREM.** *Let $\{A_n\}$ be a Cauchy sequence for the weak operator topology. Then $\{\|A_n\|\}$ is a bounded sequence, and there exists an $A \in \mathscr{B}(\mathscr{H})$ with $\|A\| \le sup_{n\in\mathbb{N}}\{\|A_n\|\}$ and such that $\lim_{n\to\infty} A_n = A$ in the weak operator topology. Moreover, the analogous statement for the strong operator topology is also true.*

*Proof.* Let $\{A_n\}$ be a Cauchy sequence for the weak operator topology. We first show that $\{\|A_n\|\}$ is a bounded sequence. To see this, note that for each $\xi \in \mathscr{H}$, $\{\langle \xi, A_n\xi\rangle_{\mathscr{H}}\}$ is a Cauchy sequence in $\mathbb{C}$, and hence convergent and bounded. Thus, if we define the sets $C_m \subset \mathscr{H}$ by

$$C_m = \{\xi \in \mathscr{H} \ : \ \sup_{n\in\mathbb{N}} |\langle \xi, A_n\xi\rangle_{\mathscr{H}}| \le m \ \}$$

we have that $\cup_{m\in\mathbb{N}} C_m = \mathscr{H}$. By Baire's Theorem, for at least one $m \in \mathbb{N}$, $C_m$ contains an open set, and then, using the polarization identity once more, it is clear that $\{\|A_n\|\}$ is a bounded sequence.

Now let $L = \sup_{n\in\mathbb{N}}\{\|A_n\|\}$, and for all $\zeta, \xi \in \mathscr{H}$, define $q(\zeta,\xi) = \lim_{n\to\infty}\langle\zeta, A_n\xi\rangle_{\mathscr{H}}$, which exists since, by the polarization identity, the sequence on the right is Cauchy in $\mathbb{C}$. It is easy to see that $\zeta, \xi \mapsto q(\zeta,\xi)$ is a sesquilinear form on $\mathscr{H}$, with $|q(\zeta,\xi)| \le L\|\zeta\|_{\mathscr{H}}\|\xi\|_{\mathscr{H}}$. For each $\xi \in \mathscr{H}$, the map $\zeta \mapsto q(\zeta,\xi)$ is a conjugate linear functional on $\mathscr{H}$, and hence by the Riesz Representation Theorem, there is a uniquely determined vector $\eta_\xi \in \mathscr{H}$ such that $q(\zeta,\xi) = \langle \zeta, \eta_\xi\rangle_{\mathscr{H}}$ for all $\zeta \in \mathscr{H}$, and $\|\eta_\xi\|_{\mathscr{H}} \le r\|\xi\|_{\mathscr{H}}$. Since $q$ is sesquilinear, the map $\xi \mapsto \eta_\xi$ is linear, and thus there exists $A \in \mathscr{B}(\mathscr{H})$ such that $\|A\| \le L$ and $\eta_\xi = A\xi$ for all $\xi \in \mathscr{H}$. It now follows that for each $\zeta, \xi \in \mathscr{H}$, $\lim_{n\to\infty}\langle\zeta, A_n\xi\rangle_{\mathscr{H}} = \langle \zeta, A\xi\rangle_{\mathscr{H}}$, and hence that $\lim_{n\to\infty} A_n = A$ in the weak operator topology. The corresponding proof for the strong operator topology is easier, and is left as an exercise. □

**3.6 THEOREM** (Continuous linear functions for the strong operator topology)**.** *Let $\mathscr{H}$ be a Hilbert space, and let $\varphi$ be a linear functional on $\mathscr{B}(\mathscr{H})$ that is continuous in the strong operator topology. Then there exists $n \in \mathbb{N}$ and two sets of vectors $\{\zeta_1, \ldots, \zeta_n\}$ and $\{\xi_1, \ldots, \xi_n\}$ such that for all $A \in \mathscr{B}(\mathscr{H})$,*

$$\varphi(A) = \sum_{j=1}^{n} \langle \zeta_j, A\xi_j\rangle_{\mathscr{H}} \ . \tag{3.3}$$

*Evidently, every such linear functional is weakly continuous, and hence every strongly continuous linear functional is weakly continuous. Consequently, a convex subset of $\mathscr{B}(\mathscr{H})$ is strongly closed if and only if it is weakly closed.*

*Proof.* If $\varphi$ is strongly continuous, then $\varphi^{-1}(\{\lambda \ :\ |\lambda| < 1\})$ contain a neighborhood of 0 in $\mathscr{B}(\mathscr{H})$. Thus, there exists an $\epsilon > 0$ and a set of $n$ vectors $\xi_1, \dots, \xi_n$, which without loss of generality we may assume to be orthonormal, such that if $\|A\xi_j\| < \epsilon$ for $j = 1, \dots, n$, $|\varphi(A)| < 1$. Note that if $A\xi_j = 0$ for $j = 1, \dots, n$, then $t > 0$, $\|tA\xi_j\| < \epsilon$ for $j = 1, \dots, n$, and consequently $t|\varphi(A)| < 1$. It follows that

$$A\xi_j = 0 \quad \text{for} \quad j = 1, \dots, n \quad \Rightarrow \quad \varphi(A) = 0 \ . \tag{3.4}$$

For any $A \in \mathscr{B}(\mathscr{H})$, define $\widehat{A}$ by $\widehat{A} = \sum_{j=1}^{n} A\xi_j \langle \xi_j, \cdot \rangle_{\mathscr{H}}$. Evidently $(A - \widehat{A})\xi_j = 0$ for $j = 1, \dots, n$, and hence by (3.4),

$$\varphi(A) = \varphi(\widehat{A}) = \sum_{j=1}^{n} \varphi[(A\xi_j)\langle \xi_j, \cdot \rangle_{\mathscr{H}}] \ . \tag{3.5}$$

For each fixed $j$, and any $\eta \in \mathscr{H}$, consider the rank-one operator $\eta\langle \xi_j, \cdot \rangle_{\mathscr{H}}$. Then $\eta \mapsto \varphi(\eta\langle \xi_j, \cdot \rangle_{\mathscr{H}})$ is a bounded linear functional on $\mathscr{H}$, and therefore by the Riesz Representation Theorem, there is a vector $\zeta_j \in \mathscr{H}$ such that $\langle \zeta_j, \eta \rangle_{\mathscr{H}} = \varphi(\eta\langle \xi_j, \cdot \rangle_{\mathscr{H}})$ for all $\eta \in \mathscr{H}$. Combining this with (3.5) yields (3.3). The final statement is a standard application of the Hahn-Banach Theorem. □

### 3.2 The Baire functional calculus

If $\mathscr{H}$ and $\mathscr{K}$ are two Hilbert spaces, a *unitary transformation* $U$ from $\mathscr{H}$ to $\mathscr{K}$ is a linear bijection from $\mathscr{H}$ onto $\mathscr{K}$ such that for all $\xi \in \mathscr{H}$, $\|U\xi\|_{\mathscr{K}} = \|\xi\|_{\mathscr{H}}$. Evidently, the inverse of $U$ is also unitary and is equal to $U^*$. In this case the map

$$A \mapsto UAU^*$$

is an isometric $*$-isomorphism of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$.

Let $A$ be a self adjoint operator on $\mathscr{H}$, and let $\eta$ be a non-zero vector in $\mathscr{H}$. Define

$$\mathscr{H}_\eta = \overline{\text{Span}(\{A^n\eta\}_{n\geq 0})} \ . \tag{3.6}$$

That is, $\mathscr{H}_\eta$ is the norm closure of the span of the polynomials in $\mathscr{A}$, and indeed of the polynomials with rational coefficients - a countable set. Hence $\mathscr{H}_\eta$ is separable even if $\mathscr{H}$ is not.

Evidently, $\mathscr{H}_\eta$ is invariant under $A$, and hence under every operator in $\mathscr{C}(\{1, A\})$. Thus, the restriction of $f(A) \in \mathscr{C}(\{1, A\})$ to $\mathscr{H}_\eta$ gives a cyclic irreducible representation of $\mathscr{C}(\{1, A\})$ on $\mathscr{H}_\eta$, and the image of this representation is a $C^*$ subalgebra of $\mathscr{B}(\mathscr{H})$. We are interested in describing the operators that one gets by taking the closure of this $C^*$ subalgebra in the strong operator topology. Recall the the spectrum of $A$ in $\mathscr{C}(\{1, A\})$ is the same as the spectrum of $A$ in $\mathscr{B}(\mathscr{H})$, and in this section we simply write $\sigma(A)$ to denote this spectrum.

Let $(X, \mathcal{F}, \mu)$ be a measure space, and consider the Hilbert space $\mathscr{K} := L^2(X, \mathcal{F}, \mu)$. an operator $T \in \mathscr{B}(\mathscr{K})$ is a *multiplication operator* in case for some function $f \in L^\infty(X, \mathcal{F}, \mu)$, $T\xi(x) = f(x)\xi(x)$ for all $\xi \in \mathscr{K}$. Note that in this case $\|T\| = \|f\|_\infty$. The next theorem shows that every self adjoint operator is unitarily equivalent to a multiplication operator, and a very simple on at that.

**3.7 THEOREM** (The Spectral Theorem in $\mathscr{B}(\mathscr{H})$). *Let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint. Let $\eta$ be any non-zero vector in $\mathscr{H}$. Let $\mathscr{H}_\eta$ be defined by (3.6). Then there is a Borel measure $\mu_\eta$ of total mass $\|\eta\|^2_{\mathscr{H}}$ such that the map $f \mapsto f(A)\eta$, $f \in \mathscr{C}(\sigma)$ satisfies*

$$\|f(A)\eta\|^2_{\mathscr{H}} = \int_{\sigma(A)} |f(\lambda)|^2 \mathrm{d}\mu_\eta(\lambda) \ . \tag{3.7}$$

*Let $\mathscr{K}_\eta$ denote the Hilbert space $L^2(\sigma(A), \mathcal{B}(\sigma(A)), \mu_\eta)$. The isometry $f \mapsto f(A)\eta$ extends to a unitary transformation $U_\eta$ mapping $\mathscr{K}_\eta$ onto $\mathscr{H}_\eta$. Define $\widehat{A}$ to be the operator on $\mathscr{K}_\eta$ given by $\widehat{A} = U_\eta A U_\eta^*$. Then for all $\psi \in \mathscr{K}_\eta$, $\mathscr{C}(\sigma)$ satisfies*

$$\widehat{A}\psi(\lambda) = \lambda\psi(\lambda) \ . \tag{3.8}$$

*For a bounded Borel function $f$ on $\sigma(A)$, define the operator $f(\widehat{A})$ on $\mathscr{K}_\eta$ by*

$$f(\widehat{A})\psi(\lambda) = f(\lambda)\psi(\lambda) \tag{3.9}$$

*for all $\phi \in \mathscr{K}_\eta$. Then when $f \in \mathscr{C}(\sigma(A))$, $U_\eta f(\widehat{A}) U_\eta^* = f(A)$, where $f(A)$ is given by the Abstract Spectral Theorem.*

*Proof.* Define a linear functional $\mu_\eta$ on $\mathscr{C}(\sigma(A))$ through

$$\mu_\eta(f) = \langle \eta, f(A)\eta \rangle_{\mathscr{H}} \ . \tag{3.10}$$

Then $\mu$ is a positive linear functional with $\mu_\eta(1) = \|\eta\|^2_{\mathscr{H}}$. By the Reisz-Markoff Theorem, there is a positive Borel measure on $\sigma(A)$ of total mass $\|\eta\|^2_{\mathscr{H}}$, also denoted by $\mu_\eta$, so that for all $f \in \mathscr{C}(\sigma(A))$,

$$\mu_\eta(f) = \int_{\sigma(a)} f \mathrm{d}\mu_\eta \ . \tag{3.11}$$

Combining (3.10) and (3.11), we conclude that for all $f \in \mathscr{C}(\sigma(A))$, $\langle \eta, f(A)\eta \rangle_{\mathscr{H}} = \displaystyle\int_{\sigma(A)} f \mathrm{d}\mu_\eta$

Define an operator $U_\eta : \mathscr{C}(\sigma(A)) \to \mathscr{H}_\eta$ by $U_\eta : f \mapsto f(A)$ where $f(A)$ is defined using the Abstract Spectral Theorem. Then, using the $*$-homomorphism property of $f \mapsto f(A)$,

$$\langle U\eta f, U_\eta f \rangle = \langle f(A)\eta, f(A)\eta \rangle = \langle \eta, |f|^2(A)\eta \rangle = \int_{\sigma(A)} |f(\lambda)|^2 \mathrm{d}\mu_\eta \ .$$

Then since $U_\eta : p \to p(A)$ for any polynomial $p$, $U_\eta$ is an isometry from a dense set in $\mathscr{K}_\eta$. Hence $U_\eta$ extends to a unitary operator from $\mathscr{H}_\eta$ to $\mathscr{K}_\eta$, and $U_\eta A U_\eta^*(\lambda) p(A)\eta = \lambda p(\lambda)$ for all polynomials $p$, and since the polynomials are dense in $\mathscr{K}_\eta$, this proves (3.8). Next, the vector of the form $g(A)\eta$, $g \in \mathscr{C}(\sigma(A))$ are dense in $\mathscr{H}_\eta$. For $f \in \mathscr{C}(\sigma(A))$, $f(A)g(A)\eta = fg(A)\eta$, and so

$$U_\eta^* f(A) U_\eta U_\eta^* g(A)\eta = U_\eta^* fg(A)\eta \ ,$$

and hence $U_\eta^* f(A) U_\eta : g \mapsto fg$ in $\mathscr{K}_\eta$. By density, $U_\eta^* f(A) U_\eta = f(\widehat{A})$ which is equivalent to $U_\eta f(\widehat{A}) U_\eta^* = f(A)$. □

Recall that the *Baire functions* on a topological space $(X, \mathcal{O})$ are the smallest class of (complex valued) functions that is closed under the operation of taking pointwise limits of sequences, and which contains all of the continuous functions . The *Baire class 1 functions* are the functions $f$ on $X$ such that for some sequence $\{f_n\}$ of continuous functions on $X$, $f(x) = \lim_{n\to\infty} f_n(x)$ for *all* $x \in X$. The *Baire class 1 functions* are the functions $f$ on $X$ such that for some sequence $\{f_n\}$ of Baire class 1 functions on $X$, $f(x) = \lim_{n\to\infty} f_n(x)$ for *all* $x \in X$, and so forth. The full class of Baire functions $X$ is obtained by transfinite induction, but we shall not need this. For our purposes, Baire class 1 functions suffice.

Let $A$ be a self adjoint operator in $\mathscr{B}(\mathscr{H})$ Note that the Baire class 1 functions form a $*$-algebra on $\sigma(A)$ with the usual operations. Let $f$ be any uniformly bounded Baire class 1 function on $\sigma(A)$; say $|f(\lambda)| \le K$ for all $\lambda \in \sigma(A)$. Since $f$ is Borel measurable, the operator on $f(\widehat{A})$ my be defined by (3.9).

Let $\{f_n\}$ be a sequence of continuous functions on $\sigma(A)$ converging pointwise to $f$. Without loss of generality, we may suppose that for all $n$, $|f_n(\lambda)| \le K$ for all $\lambda \in \sigma(A)$. Then

$$\lim_{n\to\infty} \int_{\sigma(A)} |f_n(\lambda) - f(\lambda)|^2 \mathrm{d}\mu_\eta = 0$$

by the Lebesgue Dominated Converge Theorem, and so

$$0 = \lim_{n\to\infty} \|U_\eta^* f_n - U_\eta^* f\|_{\mathscr{H}} = \lim_{n\to\infty} \|f_n(A)\eta - U_\eta^* f\|_{\mathscr{H}} \ .$$

In particular, for all non-zero $\eta \in \mathscr{H}$,

$$\lim_{n\to\infty} f_n(A)\eta = U_\eta^* f \ . \tag{3.12}$$

In particular, the limit exists in the norm topology on $\mathscr{H}$, and and the limit depends only on $f$, $\eta$ and $A$, and not on the approximating sequence $\{f_n\}$.

**3.8 DEFINITION.** Let $A$ be a self adjoint operator in $\mathscr{B}(\mathscr{H})$, and let $f$ be a uniformly bounded Baire class 1 function on $\sigma(A)$. For all non-zero $\eta \in \mathscr{H}$, define

$$f(A)\eta := U_\eta^* f \ , \tag{3.13}$$

and for $\eta = 0$, set $f(A)\eta = 0$.

**3.9 THEOREM.** *For all uniformly bounded Baire class 1 functions $f$ on $\sigma(A)$, the map*

$$\eta \mapsto f(A)\eta$$

*defined in* (3.13) *is a bounded linear transformation on $\mathscr{H}$ that is in the strong operator closure of $\mathscr{C}(\{1, A\})$. The map $f \mapsto f(A)$ from the $*$-algebra of uniformly bounded Baire class 1 function on $\sigma(A)$ into $\mathscr{B}(\mathscr{H})$ is a norm-reducing $*$-homomorphism, and moreover it is order preserving: If $f$ and $g$ are real values with $f(\lambda) \ge g(\lambda)$ for all $\lambda \in \sigma(A)$, then $f(A) - g(A) \ge 0$ in $\mathscr{B}(\mathscr{H})$.*

*Proof.* Let $\{f_n\}$ be a sequence of continuous functions on $\sigma(A)$ converging pointwise to $f$, and such that $|f_n(\lambda)| \le K$ for all $n$. (We may take $K$ to be the least upper bound of the values of $|f|$.) It is evident that if $|f(\lambda)| \le K$ for all $\lambda \in \sigma(A)$, then $\|U_\eta^* f\| \le K\|\eta\|$; likewise for each $n$, $\|f_n(A)\| \le K$.

Then as we have seen above, for each non-zero $\eta \in \mathscr{H}$, $\{f_n(A)\eta\}$ is a Cauchy sequence in $\mathscr{H}$, and this means that $\{f_n(A)\}$ a Cauchy sequence in the strong operator topology. By Theorem 3.5, it has a limit on $\mathscr{B}(\mathscr{H})$ that we denote by $f(A)$, and by (3.12), this limit is $f(A)$ is given by $f(A)\eta = U_\eta^* f$ for $\eta \neq 0$, and $f(A)0 = 0$. This shows that $f(A)$ is in the strong operator closure of $\mathscr{C}(\{1, A\})$, and completes the proof of the first part.

Next, let $f, g$ be uniformly bounded Baire class 1 functions $f$ on $\sigma(A)$. Let $\{f_n\}$, $\{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to $f$ and $g$ respectively. Then since $\lim_{n\to\infty} f_n g_n(\lambda = fg(\lambda)$ for all $\lambda$, and since $f_n(A)g_n(A) = f_n g_n(A)$ for all $n$, $f(A)g(A) = fg(A)$. Next, let $g$ and $h$ be the real and imaginary parts, respectively, of a uniformly bounded Baire class 1 function, so that $f(\lambda) = g(\lambda) + ih(\lambda)$ for all $\lambda \in \sigma(A)$. Let $\{h_n\}$, $\{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to $h$ and $g$ respectively. Then $f(A) = g(A) + ih(A) = \lim_{n\to\infty}(h_n(A) + ig_n(A))$, and $\overline{f}(A) = g(A) - ih(A) = \lim_{n\to\infty}(h_n(A) - ig_n(A))$. Evidently, $\overline{f}(A) = *f(A))^*$. The linearity of $f \mapsto f(A)$ is even simpler, and is left as an exercise. Hence $f \mapsto f(A)$ is a $*$-homomorphism from the space of Baire class 1 functions $f$ on $\sigma(A)$, equipped with the usual operations, to $\mathscr{B}(\mathscr{H})$.

Finally, if $f$ and $g$ are real Baire class 1 functions $f$ on $\sigma(A)$, and $f - g \geq 0$ on $\sigma(A)$, define $h = \sqrt{f-g}$, and note that $h$ is a real Baire class 1 functions $f$ on $\sigma(A)$. By what we have just proved, $f(A) - g(A) = (h(A))^* = (h(A))^* h(A) \geq 0$. □

The $*$-homomorphism provided by Theorem 3.9 need not be an isomorphism. The following example is useful elsewhere: Let $\lambda_0 \in \sigma(A)$ and consider the function $1_{\lambda_0}$ given by $1_{\lambda_0}(\lambda) = 1$ for $\lambda = \lambda_0$ and zero otherwise. Then for all $\lambda$, $\lambda_0 1_{\lambda_0}(\lambda) = \lambda 1_{\lambda_0}(\lambda)$. By the $*$-homomorphism property,

$$\lambda_0 1_{\lambda_0}(A) = A 1_{\lambda_0}(A) \ .$$

It follows that any non-zero vector in the range of $1_{\lambda_0}(A)$ is an eigenvector of $A$ with eigenvalue $\lambda_0$, and conversely any such eigenvector $\xi$ is in the range of $1_{\lambda_0}(A)$ as one sees by considering a continuous approximation $\{f_n\}$ to $1_{\lambda_0}$. Now consider $\mathscr{H} = L^2([0,1])$ with respect to Lebesgue measure. Let $A$ be the multiplication operator $A\psi(t) = t\psi(t)$. Then it is easy to see that $A$ is self adjoint and $\sigma(A) = [0,1]$, but $A$ has no eigenvectors at all. Hence for each $\lambda_0 \in [0,1]$, $1_{\lambda_0}(A)$ is the zero operator.

Next, let $E$ be an interval of the form $(a,b)$, $[a,b]$, $(a,b]$ or $[a,b)$. Then $1_E$ is easily seen to be a Baire class 1 function on $\mathbb{R}$, and hence on $\sigma(A)$ for any self adjoint $A$. Hence $1_E(A)$ is well defined and belongs to the strong operator topology closure of $\mathscr{C}(\{1, A\})$. Moreover, by the $*$-isomorphism established above, $1_E(A) = (1_E(A))^*$ and $(1_E(A))^2 = 1_E(A)$. Hence $1_E(A)$ is an orthogonal projection. Moreover, if $f$ is any Baire class 1 function with support in $E$, for all $\eta \in \mathscr{H}$,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = f(A)\eta \ ,$$

while if the support of $F$ lies in $E^c$,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = 0 \ .$$

**3.10 DEFINITION.** Let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint. The set of operators of the form $1_E(A)$, where $E \subset \mathbb{R}$ is such that $1_E$ is a Baire class 1 function, is the set of *spectral projections of A*

In particular, let $E = (-\infty, 0) \cup (0, \infty)$. Then $1_E(A)$ is a spectral projection, and by what we have noted above, $1_E(A)\eta = 0$ if and only if $A\eta = 0$. That is, $1_E(A)$ is the orthogonal projection onto the orthogonal complement of $\ker(A)$, or what is the same, the orthogonal projection onto the closure of the range of $A$. Combining the result obtained in this section, we have:

**3.11 LEMMA.** *$A \in \mathscr{B}(\mathscr{H})$ be self adjoint. Then the orthogonal projection onto the closure of the range of $A$ belongs to the strong operator topology closure of $\mathscr{C}(\{1, A\})$.*

Let $P_1$ and $P_2$ be two orthogonal projections in $\mathscr{B}(\mathscr{H})$ with ranges $\mathscr{K}_1$ and $\mathscr{K}_2$ respectively. Then $P_1 \vee P_2$ denotes the orthogonal projection onto $\mathscr{K}_1 + \mathscr{K}_2$, and $P_1 \wedge P_2$ denotes the orthogonal projection onto $\mathscr{K}_1 \cap \mathscr{K}_2$.

**3.12 THEOREM.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the strong operator topology. Then $\mathscr{A}$ is the norm closure of the span of the orthogonal projections contained in $\mathscr{A}$. Moreover, if $P_1$ and $P_2$ are two orthogonal projections in $\mathscr{A}$, then $P_1 \vee P_2$ and $P_1 \wedge P_2$ both belong to $\mathscr{A}$.*

*Proof.* Let $A \in \mathscr{A}$ be self adjoint. Fix $n \in \mathbb{N}$, and define $E_j = [j/h, (j+1)/n)$. Let $k \in \mathbb{N}$, $k \geq \|A\|$. Define $f_n(\lambda) = \sum_{j=-k}^{k} \frac{j}{n} 1_{E_j}(\lambda)$. Define $f(\lambda) = \lambda$. Then $\sup_{\lambda \in \sigma(A)}\{|f_n(\lambda) - f(\lambda)| \leq 1/n$, and $A = f(A)$. Hence $\|A - f_n(A)\| \leq 1/n$, and $A$ is a finite linear combination of the orthogonal projections $E_j(A)$, which belong to $\mathscr{A}$ on account of it closure in the strong operator topology. Hence every self adjoint element of $\mathscr{A}$, is the norm closure of the span of the orthogonal projections contained in $\mathscr{A}$, and since $\mathscr{A}$ is a $*$ algebra, every element in it is a sum of two self adjoint elements. This proves the first part.

For the second part. note that the spectrum of $P_1 \vee P_2$ lies in $\{0, 1, 2\}$. Let $f$ be any continuous real valued function with $f(0) = 0$, $f(1) = f(2) = 1$. Then $P_1 \vee P_2 = f(P_1 + P_2)$, and $f(P_1 + P_2) \in cA$. The proof for $P_1 \wedge P_2$ is similar. □

## 3.3 The polar decomposition

**3.13 DEFINITION** (Operator absolute value)**.** Let $\mathscr{H}$ be a Hilbert space and let $A \in \mathscr{B}(\mathscr{H})$. Then the operator absolute value of $A$ is the operator $|A|$ defined by

$$|A| = \sqrt{A^* A} \ , \tag{3.14}$$

where the square root is taken using the Abstract Spectral Theorem.

**3.14 REMARK.** *One should not be misled by the notation: It is not in general true that $|AB| = |A||B|$, or that $|A^*| = |A|$ or even that $|A + B| \leq |A| + |B|$.*

**3.15 LEMMA.** *For all $A \in \mathscr{B}(\mathscr{H})$, there is a unique partial isometry $U$ in $\mathscr{B}(\mathscr{H})$ such that $A = U|A|$, and $\mathrm{ran}(U) = \mathrm{ran}(A)$ and $\mathrm{ran}(U^*) = \ker(A)^\perp$. Moreover, $U$ belongs to the strong closure of $\mathscr{C}(\{1, A, A^*\})$.*

*Proof.* For each $t > 0$, define the operator $U_t := A(t1 + |A|)^{-1}$, noting that $1t + ||A|$ is invertible. For $s, t > 0$, the Resolvent Identity yields

$$U_t - U_s = (s - t)A[(t1 - |A|)^{-1}(s1 - |A|)^{-1}] \ .$$

Hence for any $\xi \in \mathscr{H}$, and $0 < s < t$,

$$\begin{aligned}\|(U_t - U_s)\xi\|^2 &= (s-t)^2\langle \xi, |A|^2(t1 - |A|)^{-2}(s1-|A|)^{-2}\xi\rangle_{\mathscr{H}} \\ &= (t-s)^2\int_{\sigma(|A|)} \frac{\lambda^2}{(t+\lambda)^2(s+\lambda)^2}\mathrm{d}\mu_\xi \\ &\leq \int_{\sigma(|A|)\backslash\{0\}} \frac{t^2}{(t+\lambda)^2}\mathrm{d}\mu_\xi\end{aligned}$$

Since $0 \leq t^2/(t+\lambda)^2 \leq 1$ for all $\lambda > 0$, and since $\lim_{t\to 0} t^2/(t+\lambda)^2 = 0$ for all $\lambda > 0$, the Lebesgue Dominated Convergence Theorem yields $\lim_{t\to 0}\left(\sup_{s<t}\{\|(U_t - U_s)\xi\|^2\}\right) = 0$. Thus, the strong limit $U = \lim_{t\to 0} U_t$ exists. Note that $U|A| = \lim_{t\to 0} U_t|A| = \lim_{t\to 0} f_t(|A|)$ where $f_t(\lambda) = \lambda/(t+\lambda)$. since $\lim_{t\to 0} f_t(\lambda) = 1_{(0,\infty)}\lambda$ for all $\lambda \geq 0$, it follows from Theorem 3.9 that $\lim_{t\to 0} f_t(|A|) = 1_{(0,\infty)}(|A|) = 1 - 1_{\{0\}}(|A|)$. Since $1_{\{0\}}(|A|)$ is the projector onto the null space of $|A|$, which is the null space of $A$, $A1_{\{0\}}(|A|) = 0$, and hence

$$U|A| = A\ . \tag{3.15}$$

Next note that $U^*U = \lim_{t\to\infty} f_t^2(|A|)$ with $f_t(\lambda) = \lambda/(t+\lambda)$ once more. It follows that

$$U^*U = 1_{(0,\infty)}(|A|) \tag{3.16}$$

which is the orthogonal projection onto $\ker(A)^\perp$. It follows from (3.15) that $\mathrm{ran}(U) = \mathrm{ran}(A)$, and hence $U$ is a partial isometry from $\ker(A)^\perp$ onto $\mathrm{ran}(A)$. □

Taking the adjoint of (3.15), we obtain $A^* = |A|U^*$ and hence $AA^* = UA^*AU^*$. Squaring both sides and observing that $AU^*U = A$ follows from (3.16), we obtain $(AA^*)^2 = U(A^*A)^2U^*$. An induction now yields $(AA^*)^n = U(A^*A)^nU^*$ for all $n$, and then taking a polynomial approximation to the square root, we conclude that

$$U|A|U^* = |A^*|\ , \tag{3.17}$$

and then since $A^* = |A|U^* = U^*U|A|U^*$,

$$A^* = U^*(U|A|U^*) \tag{3.18}$$

is the polar decomposition of $A^*$.

**3.16 THEOREM.** *Let $\mathscr{A}$ be a unital $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that contained that is closed in the strong operator topology. Then for all $A \in \mathscr{A}$, the components $|A|$ and $U$ of the polar decomposition $A = U|A|$ both belong to $\mathscr{A}$. Moreover, for all $A \in \mathscr{A}$, the orthogonal projections onto* $\ker(A)^\perp$ *and* $\overline{\mathrm{ran}(A)}$ *both belong to $\mathscr{A}$.*

*Proof.* Since the strong operator topology is weaker than the norm topology, $\mathscr{A}$ is a $C^*$ algebra, and hence $|A| = \sqrt{A^*A} \in \mathscr{A}$. By Lemma 3.15, $U \in \mathscr{A}$. Since $\mathscr{A}$ is a $*$-algebra, $U^*U$ and $UU^*$ both belong to $\mathscr{A}$, by Lemma 3.15, these are, respectively, the orthogonal projections onto $\ker(A)^\perp$ and $\overline{\mathrm{ran}(A)}$. □

### 3.4 Three theorems on strong closure

The self-adjoint operators in $\mathscr{B}(\mathscr{H})$ are partially ordered by the relation $A \geq B$ in case $A - B$ is positive. Using this partial order, we introduce the notion of a directed set.

**3.17 DEFINITION.** A set $\mathcal{S}$ os set adjoint operators in $\mathscr{B}(\mathscr{H})$ is an *upward directed set* in case for all $A, B \in \mathcal{S}$, there exists $C \in \mathcal{S}$ such that $C \geq A$ and $C \geq B$. Likewise, it is a *downward directed set* in case for all $A, B \in \mathcal{S}$, there exists $C \in \mathcal{S}$ such that $C \geq A$ and $C \leq B$.

The next theorem gives a condition for set of self adjoint operators in $\mathscr{B}(\mathscr{H})$ to have a maximal element in its strong operator topology closure. Note that $\mathcal{S}$ is downward directed if and only if $\mathcal{S}$ is upward directed; there is a trivial restatement in terms of downward directed sets.

**3.18 THEOREM** (Vigier's Theorem)**.** *let $\mathcal{S}$ be an upward directed set of self adjoint operators in $\mathscr{B}(\mathscr{H})$ Suppose also that $\mathcal{S}$ is norm bounded in $\mathscr{B}(\mathscr{H})$. Then there exists a unique $A_{\max}$ in the strong operator topology closure of $\mathcal{S}$ such that $A_{\max} \geq A$ for all $A \in \mathcal{S}$. Any self adjoint operator $C \in \mathscr{B}(\mathscr{H})$ such that $C \geq A$ for all $A \in \mathcal{S}$ satisfies $C \geq A$.*

*Proof.* Pick any $A_0 \in \mathcal{S}$, and define $\widetilde{\mathcal{S}} = \{A - A_0 \ : A \in \mathcal{S}\ , A \geq A_0\}$. Then $\widetilde{\mathcal{S}}$ has is a bounded upward directed set, and it consists of positive operators $B$ such that for some $K$ and all $B \in \widetilde{\mathcal{S}}$, $\|B\| \leq K$. It suffices to prove the theorem for $\widetilde{\mathcal{S}}$.

Define a function $q$ on $\mathscr{H}$ by $q(\xi) = \sup\{\langle \xi, B\xi \rangle \ : B \in \widetilde{\mathcal{S}}\}$, and then define a function $b$ on $\mathscr{H} \times \mathscr{H}$ by polarization:

$$b(\eta, \xi) = \frac{1}{4}[q(\eta + \xi) - q(\eta - \xi)] - \frac{i}{4}[q(\eta + i\xi) - q(\eta - i\xi)] \ .$$

It is evident that for any finite set of vectors in $\mathscr{H}$, and any $\epsilon > 0$, we can find $A \in \widetilde{\mathcal{S}}$ such that $q(\zeta) - \epsilon \leq \langle \zeta, A\zeta \rangle \leq q(\zeta)$ for all $\zeta$ in the set, and hence on any finite set, $b(\eta, \xi)$ can be uniformly approximated by a function of the form $\langle \eta, A\xi \rangle$, $A \in \widetilde{\mathcal{S}}$. It follows that $b$ is a bounded sesquilinear form on $\mathscr{H}$, and hence there is an operator $B \in \mathscr{B}(\mathscr{H})$, $\|B\| \leq K$, such that $b(\eta, \xi) = \langle \eta, B\xi \rangle$ for all $\eta, \xi \in \mathscr{H}$. For all $\xi \in \mathscr{H}$, $\langle \xi, B\xi \rangle = q(\xi) \geq \langle \xi, A\xi \rangle$ for all $A \in \widetilde{\mathcal{S}}$, and any $C \in \mathscr{B}(\mathscr{H})$ such that $C \geq A$ for all $A \in \widetilde{\mathcal{S}}$ satisfies $\langle \xi, C\xi \rangle \geq q(\xi)$, and hence any such operatror satisfies $C \geq B$.

Finally, for all $\xi \in \mathscr{H}$, we may take a sequence $\{A_n\}$ in $\widetilde{\mathcal{S}}$ such that

$$\lim_{n \to \infty} \langle \xi, A_n\xi \rangle = q(\xi) = \langle \xi, B\xi \rangle \ .$$

and hence $\lim_{n \to \infty} \langle \xi, B - A_n\xi \rangle = 0$. Then for some unit vector $\eta$,

$$\|(B - A_n)\xi\| = \langle \eta, (B - A_n)\xi \rangle = \langle (B - A_n)^{1/2}\eta, (B - A_n)^{1/2}\xi \rangle = \|B - A_n\| \langle \xi, (B - A_n)\xi \rangle \ .$$

Then since $\|B - A_n\| \leq 2K$, $\lim_{n \to \infty} \|(B - A_n)\xi\| = 0$. Hence $B$ is in the closure of $\widetilde{\mathcal{S}}$ in the strong operator topology, and $B$ is the desired operator $A_{\max}$. □

**3.19 DEFINITION.** If $\mathcal{S}$ is a bounded upward directed set of self adjoint operators in $\mathscr{B}(\mathscr{H})$, the unique operator $B$ in the strong closure of $\mathcal{S}$ such that $B \geq A$ for all $A \in \mathcal{S}$, is called the *least upper bound* of $\mathcal{S}$, and denote l.u.b.$(\mathcal{S})$. Likewise if $\mathcal{S}$ is a bounded downward directed set of self adjoint operators in $\mathscr{B}(\mathscr{H})$, the unique operator $B$ in the strong closure of $\mathcal{S}$ such that $B \leq A$ for all $A \in \mathcal{S}$ is called the *greatest bound* of $\mathcal{S}$, and denote g.l.b.$(\mathcal{S})$.

Vigier's theorem has a number of consequences; here is the first of these.

**3.20 COROLLARY.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the strong operator topology. Then $\mathscr{A}$ is unital.*

*Proof.* Let $\mathcal{S}$ denote the set of all orthogonal projections contained in $\mathscr{A}$. Evidently $\|P\| \leq 1$ for all $P \in \mathcal{S}$, and by Theorem 3.12, for all $P, Q \in \mathcal{S}$, $P \vee Q \in \mathcal{S}$. Then by Theorem 3.18, there exists $P_{\max} = \text{l.u.b.}(\mathcal{S})$ in the closure of $\mathcal{S}$ in the strong operator topology, and hence $\in \mathscr{A}$, such that $P_{\max} \geq P$ for all $P \in \mathcal{S}$. $P_{\max}$ is evidently an orthogonal projection whose range includes the range of any other orthogonal projection $P$ in $\mathscr{A}$. Then by Theorem 3.12 once more, $P_{\max}A = AP_{\max} = A$ for all $A \in \mathscr{A}$. □

Note that the multiplicative identity $P_{\max}$ provided by Corollary 3.20, need not by the identity in $\mathscr{B}(\mathscr{H})$. However, every operator $A$ in $\mathscr{A}$ annihilates the range of $P_{\max}^{\perp}$, and hence nothing is lost if we restrict $\mathscr{A}$ to the range of $P_{\max}$, and on this subspace of $\mathscr{H}$, it is the identity operator.

**3.21 DEFINITION.** Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the subset of $\mathscr{B}(\mathscr{H})$ given by

$$\mathcal{S}' = \{\ A \in \mathscr{B}(\mathscr{H})\ :\ AB - BA = 0 \quad \text{for all} \quad b \in \mathcal{S}\ \}\ .$$

We write $[A, B] = AB - BA$ to denote the em commutator of $A$ and $B$.

**3.22 LEMMA.** *Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$. The commutant $\mathcal{S}'$ of $\mathcal{S}$ has the following properties:*

*(1) $\mathcal{S}'$ is a closed in the weak operator topology on $\mathscr{B}(\mathscr{H})$, and contains the identity* 1.

*(2) $\mathcal{S}'$ is a subalgebra of $\mathscr{B}(\mathscr{H})$.*

*(3) If $\mathcal{S}$ is closed under the involution, then so is $\mathcal{S}'$, so that $\mathcal{S}'$ is a weakly closed $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity.*

*Proof.* It is evident that $1 \in \mathcal{S}'$. Moreover, for any $\zeta, \xi \in \mathscr{H}$ and $B \in \mathcal{S}$, define the linear functional $\varphi_{\zeta,\xi,B}$ on $\mathscr{B}(\mathscr{H})$ by

$$\varphi_{\zeta,\xi,B}(A) = \langle \zeta, ([A,B])\xi \rangle_{\mathscr{H}} = \langle \zeta, A(B\xi) \rangle_{\mathscr{H}} - \langle (B^*\zeta), A\xi \rangle_{\mathscr{H}}\ .$$

Since $\varphi_{\zeta,\xi,b}$ is weakly continuous, $\varphi_{\zeta,\xi,b}^{-1}(\{0\})$ is weakly closed. Then since

$$\mathcal{S}' = \bigcap \{\ \varphi_{\zeta,\xi,B}^{-1}(\{0\})\ :\ \zeta, \xi \in \mathscr{H}\ ,\ B \in \mathcal{S}\ \}\ ,$$

*(1)* is proved. *(2)* is evident, and the *(3)* follows from the fact that $([A,B])^* = ([B^*, A^*])$ together with *(1)* and *(2)*. □

**3.23 THEOREM** (von Neumann Double Commutant Theorem)**.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity. Then $\mathscr{A}''$ is the weak operator topology closure of $\mathscr{A}$.*

*Proof.* Since $\mathscr{A}$ is convex, the weak and strong operator topology closures of $\mathscr{A}$ coincide. Hence is suffices to show that for all $A \in \mathscr{A}''$, every strong neighborhood of $a$ contains some $B \in \mathscr{A}$. That is, it suffices to show that for all $n \in \mathbb{N}$ and all $\{\eta_1, \ldots, \eta_n\} \subset \mathscr{H}$, and all $\epsilon > 0$, there is some $B \in \mathscr{A}$ such that $\|(B - A)\eta_j\| < \epsilon$ for all $j = 1, \ldots, n$.

Let $\widehat{\mathscr{H}} = \mathscr{H} \oplus \cdots \oplus \mathscr{H}$, the direct sum of $n$ copies of $\mathscr{H}$. The elements of $\mathscr{B}(\widehat{\mathscr{H}})$ are $n \times n$ matrices $[B_{i,j}]$ with entries in $\mathscr{B}(\mathscr{H})$.

Let $\widehat{\mathscr{A}}$ be the algebra of all operators on $\widehat{\mathscr{H}}$ the form $[A\delta_{i,j}]$ with $A \in \mathscr{A}$. Evidently, its commutator $\widehat{\mathscr{A}'}$ consists of all $[B_{i,j}]$ with each $B_{i,j} \in \mathscr{A}'$. Thus, for all $A \in \mathscr{A}''$, $[A\delta_{i,j}] \in \widehat{\mathscr{A}''}$.

Let $\eta = \eta_1 \oplus \cdots \oplus \eta_n$, and define $\mathscr{K} = \overline{\widehat{\mathscr{A}}\eta}$, which is a closed subspace of $\widehat{\mathscr{H}}$ that is invariant under $\widehat{\mathscr{A}}$. Let $P$ be the orthogonal projection onto $\mathscr{K}$. Since $\mathscr{K}$ is invariant under $\widehat{\mathscr{A}}$, it is evident that $P \in \widehat{\mathscr{A}'} \subset \widehat{\mathscr{A}'''}$. But then $PB = BP$ for all $B \in \widehat{\mathscr{A}''}$, and hence $\mathscr{K}$ is invariant under $\widehat{\mathscr{A}''}$ In particular, for all $A \in \mathscr{A}''$, $\mathscr{K}$ is invariant under $[A\delta_{i,j}]$.

Since $\mathscr{A}$ contains the identity, $\eta \in \mathscr{K}$, so that $A\eta_1 \oplus \cdots \oplus A\eta_n \in \mathscr{K}$. Therefore, by the definition of $\mathscr{K}$ as the closure of $\widehat{\mathscr{A}}\eta$, for all $\epsilon > 0$, there exists $B \in \mathscr{A}$ such that

$$\|B\eta_1 \oplus \cdots \oplus B\eta_n - A\eta_1 \oplus \cdots \oplus A\eta_n\|^2_{\widehat{\mathscr{H}}} \le \epsilon^2 \ .$$

□

Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, and let $\mathscr{M}$ be its closure in the strong operator topology, which is also a $*$-subalgebra. Then $B_{\mathscr{A}}$, the unit ball in $\mathscr{A}$, is contained in $B_{\mathscr{M}}$, the unit ball in $\mathscr{M}$, and $B_{\mathscr{M}}$ is strongly closed. Hence the strong closure of $B_{\mathscr{A}}$, $\overline{B_{\mathscr{A}}}$, is contained in $B_{\mathscr{M}}$. Since the norm function is lower semicontinuous in the strong topology, it is not immediately clear that $B_{\mathscr{A}}$ is strongly dense in $B_{\mathscr{M}}$. However, this, and more, is true.

**3.24 THEOREM** (Kaplansky's Density Theorem)**.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, and let $\mathscr{M}$ be its closure in the strong operator topology, Then $B_{\mathscr{A}}$ is strongly dense in $B_{\mathscr{M}}$, and the self-adjoint part of $B_{\mathscr{A}}$ is strongly dense in the self adjoint part of $B_{\mathscr{M}}$*

*Proof.* Let $\mathscr{A}_{\text{s.a}}$ denote the space of self adjoint elements in $\mathscr{A}$, and let $\mathscr{M}_{\text{s.a}}$ denote the space of self adjoint elements in $\mathscr{M}$. Let $\mathscr{N}$ denote the space self adjoint elements of $\mathscr{B}(\mathscr{H})$, and note that $\mathscr{N}$ is the nullspace of the real linear map $X \mapsto X - X^*$. Since the involution is weakly continuous, $\mathscr{N}$ is weakly, and hence strongly closed. Since $\mathscr{M}_{\text{s.a.}} = \mathscr{M} \cap \mathscr{N}$, $\mathscr{M}_{\text{s.a.}}$ is strongly closed. Therefore, the strong closure of $\mathscr{A}_{\text{s.a}}$, $\overline{\mathscr{A}_{\text{s.a}}}$, satisfies $\overline{\mathscr{A}_{\text{s.a}}} \subset \mathscr{M}_{\text{s.a.}}$. In fact, $\overline{\mathscr{A}_{\text{s.a}}} = \mathscr{M}_{\text{s.a.}}$. To see this it suffices to show that $\mathscr{A}_{\text{s.a}}$ is weakly dense in $\mathscr{M}_{\text{s.a}}$ since, being convex, its weak closure is the same as its strong closure.

Therefore, let $Y \in \mathscr{M}_{\text{s.a.}}$. Then if $V_{\epsilon,\xi_1,\dots,xi_n}$ is any weak basic neighborhood of 0, as in (3.2) $Y + V_{\epsilon,\xi_1,\dots,\xi_n}$ contains some $X \in \mathscr{A}$, i.e., there is an $X \in \mathscr{A}$ such that $|\langle \xi_j, (Y - X)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$. Then since $Y$ is self adjoint, $|\langle \xi_j, (Y - X^*)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$, and then by convexity, with $Z = \frac{1}{2}(X + X^*)$, $|\langle \xi_j, (Y - Z)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$, and $Z \in \mathscr{A}_{\text{s.a.}}$. Hence $Y$ is in the weak closure of $\mathscr{A}_{\text{s.a.}}$. This completes the proof that $\overline{\mathscr{A}_{\text{s.a}}} = \mathscr{M}_{\text{s.a.}}$.

We now prove the second part of the theorem. By Corollary 3.20, $\mathscr{M}$ is unital even if $\mathscr{A}$ is not. Let 1 denote the multiplicative identity in $\mathscr{M}$. Without loss of generality, we may suppose that $\mathscr{A}$ is norm closed; i.e., that $\mathscr{A}$ is a $C^*$ algebra. Consider the function $f : [-1,1] \to [-1,1]$ given by

$$f(\lambda) = \frac{2\lambda}{1 + \lambda^2} \ .$$

The is is easy to see that $f$ a homeomorphism from $[-1,1]$ onto $[-1,1]$. The inverse function is given by

$$g(t) := \begin{cases} 0 & t = 0 \\ \frac{1}{t} - \sqrt{\frac{1-t^2}{t^2}} & t > 0 \\ \frac{1}{t} + \sqrt{\frac{1-t^2}{t^2}} & t < 0 \end{cases} .$$

for $t \in [-1, 1]$, but notice that the formula for $g$ makes sense for all $t \in \mathbb{R}$. Likewise, the formula for $f$ makes sense for all $\lambda \in \mathbb{R}$. Since both $f(0) = 0$ and $g(0) = 0$, we may apply the Abstract Spectral Theorem to define $f(A)$ and $g(A)$ for all $A \in \mathscr{A}_{\text{s.a.}}$, even when $\mathscr{A}$ is not unital. Of course, $\mathscr{M}$ is a unital $C^*$ algebra, and we may define $f(B)$ and $g(B)$ for all $B \in \mathscr{M}_{\text{s.a.}}$. Now let $B \in B_{\mathscr{M}} \cap \mathscr{M}_{\text{s.a.}}$, and define $\widehat{B} = g(B)$. We then choose $\widehat{A} \in \mathscr{A}_{\text{s.a.}}$ to belong to an appropriate neighborhood of $\widehat{B}$, and then define $A = f(\widehat{A})$. Evidently $A \in \mathscr{A}_{\text{s.a.}} \cap B_{\mathscr{A}}$. By what we have noted above, $f(\widehat{B}) = f(g(B)) = fg(B) = B$.

Now compute, using $\widehat{B}^2 - \widehat{A}^2 = \widehat{A}(\widehat{B} - \widehat{A}) + (\widehat{B} - \widehat{A})\widehat{B}$ in the last step,

$$\begin{aligned} A - B &= f(\widehat{A}) - f(\widehat{B}) \\ &= 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A}(1+\widehat{B}^2) - (1+\widehat{A}^2)\widehat{B}\right)\frac{1}{1+\widehat{B}^2} \\ &= 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A} - \widehat{B}\right)\frac{1}{1+\widehat{B}^2} + 2\frac{\widehat{A}}{1+\widehat{A}^2}\left(\widehat{B}^2 - \widehat{A}^2\right)\frac{\widehat{B}}{1+\widehat{B}^2} \\ &= 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A} - \widehat{B}\right)\frac{1}{1+\widehat{B}^2} - 2\frac{\widehat{A}^2}{1+\widehat{A}^2}\left(\widehat{A} - \widehat{B}\right)\frac{1}{1+\widehat{B}^2} - 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A} - \widehat{B}\right)\frac{\widehat{B}^2}{1+\widehat{B}^2} . \end{aligned}$$

Note that $\left\|\frac{1}{1+\widehat{A}^2}\right\|, \left\|\frac{\widehat{A}^2}{1+\widehat{A}^2}\right\| < 1$, and therefore if $\widehat{A}$ satisfies

$$\left\|(\widehat{A} - \widehat{B})\frac{1}{1+\widehat{B}^2}\xi\right\| < \frac{\epsilon}{6} \quad \text{and} \quad \left\|(\widehat{A} - \widehat{B})\frac{\widehat{B}^2}{1+\widehat{B}^2}\xi\right\| < \frac{\epsilon}{6}\},$$

then $\|(A - B)\xi\| < \epsilon$. This proves that for every strong neighborhood $U$ of $B$, there is a strong neighborhood $V$ of $\widehat{B}$ so that if $\widehat{A} \in \mathscr{A}_{\text{s.a.}}$ is chosen in $V$, then $A = f(\widehat{A}) \in U$. Since $A \in \mathscr{A}_{\text{s.a.}} \cap B_{\mathscr{A}}$, this shows that $\mathscr{A}_{\text{s.a.}} \cap B_{\mathscr{A}}$ is strongly dense in $\mathscr{M}_{\text{s.a.}} \cap B_{\mathscr{M}}$.

To pass to the general case, consider $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$, consisting of $2 \times 2$ block matrices $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2.1} & B_{2,2} \end{bmatrix}$ with entries in $\mathscr{B}(\mathscr{H})$; we identify $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ with $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. A neighborhood basis at 0 for the strong topology in $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ is given by the sets $\widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$ consisting of operators of the form $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2.1} & B_{2,2} \end{bmatrix}$ in $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ such that $B_{i,j} \in U_{\epsilon,\xi_1,\ldots,\xi_n}$ with $U_{\epsilon,\xi_1,\ldots,\xi_n}$ specified in (3.1). It is then easy to see that $\mathscr{A} \otimes M_2(\mathbb{C})$ is dense in $\mathscr{M} \otimes M_2(\mathbb{C})$. For $B \in \mathscr{M}$, define

$$\widetilde{B} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} . \tag{3.19}$$

Then $\|\widetilde{B}\| = \|B\|$, so that for $B \in B_{\mathscr{M}}$, by what we have just proved, for any neighborhood $\widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$, there is $\widetilde{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathscr{A} \otimes M_2(\mathbb{C})$ with $A_{2,1} = A_{1,2}^*$ and $A_{1,1}, A_{2,2} \in \mathscr{A}_{\text{s.a.}}$ and $\|\widetilde{A}\| \leq 1$, such that $\widetilde{A} - \widetilde{B} \in \widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$. Then $A_{1,2} - B \in U_{\epsilon,\xi_1,\cdots\xi_n}$, and since $\|\widetilde{A}\| \leq 1$, $\|A_{1,2}\| \leq 1$. □

We now come to an important class of operator algebras singled out by von Neumann.

**3.25 DEFINITION.** A von Neumann algebra is $C^*$ subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity on $\mathscr{B}(\mathscr{H})$ and is closed in the weak operator topology.

Replacing the strong operator topology by the weal operator topology would not make the definition any more inclusive since the same convex sets (and hence subspaces) of $\mathscr{B}(\mathscr{H})$ are closed for the two topologies. Corollary 3.20 shows that any weakly closed $C^*$ subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is unital, and the unit is an orthogonal projection in $\mathscr{B}(\mathscr{H})$. Restricting the operators in $\mathscr{A}$ to the range of the m to the range of this projection, nothing essential is lost, and then the identity in the algebra is the identity operator on the smaller Hilbert space.

## 3.5 Trace class operator and the $\sigma$-weak topology

**3.26 DEFINITION** (Trace class)**.** Let $\mathscr{H}$ be a s Hilbert space. an operator $A \in \mathscr{B}(\mathscr{H})$ is *trace class* in case

$$\|A\|_1 := \sup\left\{ \sum_{j=1}^n |\langle \eta_j, A\xi_j\rangle| \ , \{\eta_1,\dots,\eta_n\} \ , \ \{\xi_1,\dots,\xi_n\} \text{ orthonormal} \ , \ n \in \mathbb{N} \right\} < \infty \ . \tag{3.20}$$

The set of trace class operator is denoted by $\mathscr{T}(\mathscr{H})$. the function $A \mapsto \|A\|_1$ on $\mathscr{T}(\mathscr{H})$ defined by (3.20) is called the *trace norm* on $\mathscr{T}(\mathscr{H})$; this terminology will be justified below.

**3.27 THEOREM.** *For any Hilbert space $\mathscr{H}$, $\mathscr{T}(\mathscr{H})$ is a (non-closed) $*$-ideal in $\mathscr{B}(\mathscr{H})$, and the function $A \mapsto \|A\|_1$ is a norm on $A$ under which the involution $*$ is isometric.*

*Moreover, for all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$,*

$$\|AB\|_1 \leq \|B\|\|A\|_1 \quad \text{and} \quad \|BA\|_1 \leq \|B\|\|A\|_1 \ . \tag{3.21}$$

*and there is equality in both inequalities in* (3.21) *whenever $B$ is unitary.*

*Proof.* Since $|\langle \eta, (A+B)\xi\rangle| \leq |\langle \eta, A\xi\rangle| + |\langle \eta, B\xi\rangle|$, it is evident that $\mathscr{T}(\mathscr{H})$ is closed under addition, and moreover that $\|A+B\|_1 \leq \|A\|_1 + \|B\|_1$. It is also evident that $\mathscr{T}(\mathscr{H})$ is closed under scalar multiplication. It is evident that for all $A \in \mathscr{B}(\mathscr{H})$, $\|A\|_1 \geq \|A\| = \sup\{|\langle \eta, A\xi\rangle| \ : \|\eta\| = \|xi\| = 1\}$. Hence $\|A\|_1 = 0$ if and only if $A = 0$. This shows that the trace norm is indeed a norm. Since $|\langle \xi, A^*\eta\rangle| = |\langle \eta, A\xi\rangle|$, $\|A^*\|_1 = \|A\|_1$ and hence $\mathscr{T}(\mathscr{H})$ is closed under the involution, and the involution is isometric in the trace norm,

Finally, let $A \in \mathscr{T}(\mathscr{H})$, and let $U$ be unitary in $\mathscr{B}(\mathscr{H})$. Then Since $\{U\eta_1,\dots,U\eta_n\}$ is orthonormal if and only if $\{\eta_1,\dots,\eta_n\}$ is orthonormal, it follows that $AU$ and $UA$ are trace class, and that $\|AU\|_1 = \|UA\|_1 = \|A\|_1$.

Now $\epsilon > 0$, and let $B \in \mathscr{B}(\mathscr{H})$ satisfy $\|B\| < 1-\epsilon$. Then for $m \in \mathbb{N}$. $B$ has the form $B = \frac{1}{m}\sum_{j=1}^m U_j$, where each $j$ is unitary. Then $\|BA\|_1 \leq \dfrac{1}{m}\displaystyle\sum_{j=1}^m \|U_jA\|_1 = \|A\|_1$. Since $\epsilon > 0$ is arbitrary, it follows that for all $\|B\|$ in the unit ball of $\mathscr{B}(\mathscr{H})$, $\|BA\|_1 \leq \|A\|_1$, and then that for all $B \in \mathscr{B}(\mathscr{H})$, the first inequality in (3.21) is valid. The same argument (or use of the isometry property of the involution) shows proves the second inequality. Finally, (3.21) shows that $\mathscr{T}(\mathscr{H})$ is an ideal in $\mathscr{B}(\mathscr{H})$. □

**3.28 LEMMA.** *For all $A \in \mathscr{T}(\mathscr{H})$, and all $\epsilon > 0$, there is a finite rank operator $A_\epsilon$ such that $\|A - A_\epsilon\|_1 < \epsilon$. Moreover, $\ker(A)^\perp$ and $\overline{\mathrm{ran}(A)}$ are separable subspaces of $\mathscr{H}$, and hence there exists a separable subspace $\mathscr{K}$ of $\mathscr{H}$ such that $A|_{\mathscr{K}^\perp} = 0$.*

*Proof.* Let $A \in \mathscr{T}(\mathscr{H})$, and let $A = U|A|$ be its polar decomposition. Then $|A| = U^*A \in \mathscr{T}(\mathscr{H})$, and $\||A|\|_1 = \|A\|_1$. By the Cauchy-Schwarz inequality

$$\||A|\|_1 := \sup\left\{\sum_{j=1}^n |\langle \xi, |A|\xi_j\rangle| \ , \{\xi_1, \dots, \xi_n\} \text{ orthonormal} \ , \ n \in \mathbb{N}\right\} < \infty \ . \tag{3.22}$$

Fix $\epsilon > 0$, and let $P_\epsilon$, be the spectral projection of $|A|$ for the interval $[\epsilon, \|A\|]$. If $\{\eta_1, \dots, \eta_m\}$ is an orthonormal set in the range of $P_\epsilon$, then $\sum_{j=1}^m \langle |\eta_j, |A|\eta_j\rangle| \le \epsilon m$, and hence $m \le \|A\|_1/\epsilon$.

Let $P = \lim_{\epsilon\to 0} P_\epsilon$ which exists in the strong operator topology, and is the orthogonal projection onto $\ker(|A|)^\perp = \ker(A)^\perp$. Since the range of each $P_\epsilon$ is finite dimensional, the range of $P$ is a separable subspace of $\mathscr{H}$. Replacing $A$ with $A^*$, we see that the closed range of $A$ is also a separable subspace of $A$. Let $\mathscr{H}$ be the closed span of $\ker(A)^\perp \cup \overline{\mathrm{ran}(A)}$. Then $\mathscr{K}$ is separable and invariant under $A$, and $A|_{\mathscr{K}^\perp} = 0$.

Moreover, for all $\xi$, $\langle \xi, |A|\xi\rangle - \langle P_\epsilon\xi, |A|P_\epsilon\xi\rangle = \langle(\xi - P_\epsilon\xi), |A|\xi\rangle + \langle P_\epsilon\xi, |A|(\xi - P_\epsilon\xi)\rangle$, and then by the strong convergence of $P_\epsilon$ to $\epsilon$, for all $\{\xi_1, \dots, \xi_m\}$ orthonormal,

$$\lim_{\epsilon\to 0}\sum_{j=1}^m \langle \xi_j, P_\epsilon|A|P_\epsilon\xi_j\rangle = \sum_{j=1}^m \langle \xi_j, |A|\xi_j\rangle \ .$$

Thus for all $\delta > 0$, there exists $\epsilon > 0$ such that $\|P_\epsilon|A|P_\epsilon\| \ge \||A|\|_1 - \delta$. Fix any such $\epsilon$, and suppose that $\||A| - P_\epsilon|A|P_\epsilon\|_1 \ge \delta$. We shall show this leads to contradiction. Choose an orthonormal set $\{xi_1, \dots, \xi_r\}$ such that $\sum_{j=1}^r \langle \xi_j, P_\epsilon|A|P_\epsilon\xi_j\rangle \ge \|P_\epsilon|A|P_\epsilon\|_1 - \delta/2$, and choose an orthonormal set $\{\eta_1, \dots, \eta_s\}$ such that $\sum_{j=1}^s \langle \eta_j, (||A| - P_\epsilon|A|P_\epsilon)\eta_j\rangle > \delta$. Since $|A| - P_\epsilon|A|P_\epsilon \ge 0$, adding vectors to the orthonormal set only increases the sum. Adding finitely many vectors to $\{\eta_1, \dots, \eta_s\}$ we may suppose that the span of this set includes $\{xi_1, \dots, \xi_r\}$. Then we may extend this to an orthonormal basis $\{\xi_1, \dots, \xi_s\}$ of the span of $\{\eta_1, \dots, \eta_s\}$. Then

$$\begin{aligned}
\sum_{j=1}^s \langle \eta_j |A| \eta_j\rangle &= \sum_{j=1}^s \langle \eta_j P_\epsilon|A|P_\epsilon\eta_j\rangle + \sum_{j=1}^s \langle \eta_j(|A| - P_\epsilon|A|P_\epsilon)\eta_j\rangle \\
&\ge \sum_{j=1}^s \langle \eta_j P_\epsilon|A|P_\epsilon\eta_j\rangle + \delta = \sum_{j,k,\ell=1}^s \langle \eta_j, \xi_k\rangle\langle \xi_k P_\epsilon|A|P_\epsilon\xi_\ell\rangle\langle \xi_\ell, \eta_j\rangle + \delta \\
&= \sum_{k,\ell=1}^s \left(\left(\sum_{j=1}^s \langle \xi_\ell, \eta_j\rangle\langle \eta_j, \xi_k\rangle\right)\langle \xi_k P_\epsilon|A|P_\epsilon\xi_\ell\rangle\right) + \delta \\
&= \sum_{k=1}^s \langle \xi_k P_\epsilon|A|P_\epsilon\xi_k\rangle + \delta \ge \|A\|_1 + \delta/2 \ .
\end{aligned}$$

This contradiction proves that $\||A| - P_\epsilon|A|P_\epsilon\|_1 < \delta$. □

**3.29 LEMMA.** *Let $A \in \mathscr{T}(\mathscr{H})$ be positive, and let $\mathscr{K}$ be any separable subspace of $\mathscr{H}$ that is invariant under $A$ and is such that $A|_{\mathscr{K}^\perp} = 0$. Then for any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$ of $\mathscr{K}$,*

$$\|A\|_1 = \sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle \ . \tag{3.23}$$

*Proof.* Let $A \in \mathscr{T}(\mathscr{H})$ be a positive operator on a separable Hilbert space $\mathscr{H}$, and let $\{\eta_j\}_{j\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ be two orthonormal bases for $\mathscr{H}$. Then

$$\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=1}^{\infty} \|A^{1/2}\eta_j\|_{\mathscr{H}} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left|\langle \xi_k, A^{1/2}\eta_j\rangle\right|^2 \ .$$

Since infinite series of non-negative terms may be summed in any order, the right hand side is actually symmetric in $\{\eta_j\}_{j\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$. Therefore, by symmetry in the two orthonormal bases,

$$\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{k=1}^{\infty} \langle \xi_k, A\xi_k\rangle \ , \tag{3.24}$$

showing that $\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle$ depends only on $A$, and not the particular orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$.

Now suppose that $\mathscr{H}$ is not separable. Let $A \in cT(\mathscr{H})$ be positive, and let $\mathscr{K}$ be a separable subspace of $\mathscr{H}$ that "carries" $A$, as in Lemma 3.28. Let $\epsilon > 0$, $\{\xi_1, \dots, \xi_m\}$ be orthonormal in $\mathscr{H}$ be such that $\sum_{j=1}^{m} \langle \xi_j, A\xi_j\rangle \geq \|A\|_1 - \epsilon$. Let $\widetilde{\mathscr{K}}$ be the span of $\mathscr{K} \cap \{\xi_1, \dots, \xi_m\}$. Let $\{\xi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\widetilde{\mathscr{K}}$ extending $\{\xi_1, \dots, \xi_m\}$.

Let $\{\eta_j\}_{j\in\mathbb{N}}$ be any orthonormal basis of $\mathscr{K}$. If necessary, add finitely many vectors with indices less than 1 such that $\{\eta_j\}_{j>p}$ is an orthonormal basis for $\mathscr{K}$. all vectors with indices $p \leq j < 1$ are in $\ker(A)$. Then by the invariance proved above

$$\|A\|_1 \geq \sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=p}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=1}^{\infty} \langle \xi_j, A\xi_j\rangle \geq \sum_{j=1}^{m} \langle \xi_j, A\xi_j\rangle \geq \|A\|_1 - \epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, this proves (3.23). □

For any $A \in \mathscr{T}(\mathscr{H})$, write $A = X + iY$, with $X, Y$ self adjoint. Since $X = \frac{1}{2}(A + A^*)$, $\|X\|_1 \leq \|A\|_1$ and likewise $\|Y\|_1\| \leq \|A\|_1$. Then $XP_+ = \frac{1}{2}(|X| + X)$ and $X_- = \frac{1}{2}(|X| - X)$, so that $\|X_+\|_1, \|X_-\|_1 \leq \|A\|_1$, and likewise $\|Y_+\|_1, \|Y_-\|_1 \leq \|A\|_1$. By what we have just proved, for any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$,

$$\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=1}^{\infty} \langle \eta_j, X_+\eta_j\rangle - \sum_{j=1}^{\infty} \langle \eta_j, X_-\eta_j\rangle + i\sum_{j=1}^{\infty} \langle \eta_j, Y_=\eta_j\rangle - i\sum_{j=1}^{\infty} \langle \eta_j, Y_-\eta_j\rangle$$

converges absolutely and is independent of the orthonormal basis.

**3.30 DEFINITION** (Trace)**.** For all $A \in \mathscr{T}(\mathscr{H})$, the *trace* of $A$, $\mathrm{Tr}[A]$, is defined by

$$\mathrm{Tr}[A] = \sum_{j=1}^{\infty} \langle \eta_j A\eta_j\rangle$$

where $\{\eta_j\}$ is any orthonormal basis of of any separable subspace $\mathscr{K}$ invariant under $A$ with $A|_{\mathscr{K}^\perp} = 0$.

**3.31 THEOREM** (Properties of the trace)**.** *The functional* $A \mapsto \mathrm{Tr}[A]$ *is linear on* $\mathscr{T}(\mathscr{H})$ *and* $\mathrm{Tr}[A^*] = \mathrm{Tr}[A]^*$ *for all* $A \in \mathscr{T}(\mathscr{H})$*. Moreover:*
*(i) For all* $A \in \mathscr{T}(\mathscr{H})$ *and all* $B \in \mathscr{B}(\mathscr{H})$

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA] \tag{3.25}$$

*(ii) For all* $A \in \mathscr{T}(\mathscr{H})$ *and all* $B \in \mathscr{B}(\mathscr{H})$

$$|\mathrm{Tr}[AB]| \leq \|A\|_1 \|B\| \tag{3.26}$$

*(iii) For all* $\zeta, \xi \in \mathscr{H}$*, and all* $B \in \mathscr{B}(\mathscr{H})$*,* $\mathrm{Tr}\big[|\zeta\rangle\langle\xi|A\big] = \langle \xi, A\zeta\rangle_{\mathscr{H}}$*.*

*Proof.* The linearity is evident, and since $\langle \eta, A\eta\rangle_{\mathscr{H}} = \langle \eta, A^*\eta\rangle_{\mathscr{H}}^*$, it follows that $\mathrm{Tr}[A^*] = \mathrm{Tr}[A]^*$. In view of the linearity and the fact that every $B \in \mathscr{B}(\mathscr{H})$ is a linear combination of 4 unitaries, it suffices to prove (3.25) when $B$ is unitary.. But then $AB = B^*(BA)B$ and since $\{B\eta_j\}_{j\in\mathbb{N}}$ is an orthonormal basis when $\{\eta_j\}_{j\in\mathbb{N}}$ is,

$$\mathrm{Tr}[AB] = \sum_{j=1}^{\infty} \langle \eta_j, AB\eta_j\rangle = \sum_{j=1}^{\infty} \langle B\eta_j, BAB\eta_j\rangle = \mathrm{Tr}[BA]\ .$$

For part *(ii)*, note that evidently for all $A \in \mathscr{T}(\mathscr{H})$, $|\mathrm{Tr}[A]| \leq \|A\|_1$, and hence for all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$, $|\mathrm{Tr}[AB]| \leq \|AB\|_1 \leq \|A\|_1\|B\|$, using (3.21) in the last step.

Part *(iii)*) is a simple computation using any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$ in which $\eta_1 = \|\eta\|^{-1}\eta$ when $\eta \neq 0$, and is trivial otherwise. □

**3.32 DEFINITION.** The set of all finite rank operators on $\mathscr{H}$ is evidently a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, and its norm closure in $\mathscr{B}(\mathscr{H})$ is a $C^*$-subalgebra of $\mathscr{B}(\mathscr{H})$ denoted by $\mathscr{C}(\mathscr{H})$.

It is easy to see that $\mathscr{B}(\mathscr{H})$ consists of all compact operators on $\mathscr{H}$; that is all operators that take weakly convergent sequences to strongly convergent sequences, or equivalently, all operators that take the unit ball on $\mathscr{B}(\mathscr{H})$ into a relatively compact subset of $\mathscr{B}(\mathscr{H})$. We shall not need these facts, and do not prove them. Our interest in $\mathscr{C}(\mathscr{B})$ is that introducing it paves the way for a simple proof that $\mathscr{T}(\mathscr{H})$ is complete in the trace norm.

**3.33 THEOREM.** $\mathscr{T}(\mathscr{H})$ *is a Banach space in the metric given by the trace norm Moreover:*
*(i) For every* $A \in \mathscr{T}(\mathscr{H})$ *define a linear functional* $\phi_A$ *on* $\mathscr{C}(\mathscr{H})$ *by*

$$\phi_A(X) = \mathrm{Tr}[AX] \quad \text{for all} \quad X \in \mathscr{C}(\mathscr{H})\ . \tag{3.27}$$

*The mapping* $A \mapsto \phi_A$ *is an isometric isomorphism of* $\mathscr{T}(\mathscr{H})$ *onto* $\mathscr{C}(\mathscr{H})^*$*.*
*(i) For every* $B \in \mathscr{B}(\mathscr{H})$ *defines a linear functional* $\psi_B$ *on* $\mathscr{T}(\mathscr{H})$ *by*

$$\psi_B(X) = \mathrm{Tr}[BX] \quad \text{for all} \quad X \in \mathscr{T}(\mathscr{H})\ . \tag{3.28}$$

*The mapping* $B \mapsto \psi_B$ *is an isometric isomorphism of* $\mathscr{B}(\mathscr{H})$ *onto* $\mathscr{T}(\mathscr{H})^*$*.*

*Proof.* Once we have proved *(i)*, the completeness of $\mathscr{T}(\mathscr{H})$ follows since the dual of a Banach space is always complete. The proofs of *(i)* and *(ii)* are almost the same:

For any $\zeta, \zeta' \in \mathscr{H}$, consider the rank-one operator $|\zeta\rangle\langle\zeta'|$. Then

$$\||\zeta\rangle\langle\zeta'|\| = \||\zeta\rangle\langle\zeta'|\|_1 = \|\zeta\|_{\mathscr{H}}\|\zeta'\|_{\mathscr{H}} \ . \tag{3.29}$$

For $\phi \in \mathscr{C}(\mathscr{H})^*$, define the sesquilinear form $b_\phi(\zeta,\zeta') := \phi(|\zeta\rangle\langle\zeta'|)$. Then

$$b_\phi(\zeta,\zeta') = |\phi(|\zeta\rangle\langle\zeta'|)| \leq \|\phi\|\||\zeta\rangle\langle\zeta'|\| = \|\phi\|\|\zeta\|_{\mathscr{H}}\|\zeta'\|_{\mathscr{H}} \ .$$

By the Riesz Lemma, there is an $A_\phi \in \mathscr{B}(\mathscr{H})$ with $\|A_\phi\| \leq \|\phi\|$ such that for all $\zeta, \zeta' \in \mathscr{H}$, $q_\phi(\zeta,\zeta') = \langle\zeta, A\zeta'\rangle_{\mathscr{H}}$. Then for $\zeta,\zeta' \in \mathscr{H}$, $\phi(|\zeta\rangle\langle\zeta'|) = \mathrm{Tr}\big[A_\phi|\zeta\rangle\langle\zeta'|\big]$, and then by linearity, $\phi(X) = \mathrm{Tr}[A_\phi X]$ for all finite rank $X$. Since finite rank operators are dense in $\mathscr{C}(\mathscr{H})$, this is valid for all $X \in \mathscr{C}(\mathscr{H})$.

We now show that $A_\phi \in \mathscr{T}(\mathscr{H})$. Let $A = U|A_\phi|$ be the polar decomposition of $A_\phi$. Let $\{\eta_1,\dots,\eta_n\}$ be any set of $n$ orthonormal vectors in $\mathscr{H}$, and define the finite rank partial isometry $V = \sum_{j=1}^n |\eta_j\rangle\langle U\eta_j|$. Then $\sum_{j=1}^n \langle\eta_j, |A_\phi|\eta_j\rangle = \mathrm{Tr}[AV] = \phi(V) \leq \|\phi\|\|V\| = \|\phi\|$. Since $n$ is arbitrary, $A_\phi \in \mathscr{T}(\mathscr{H})$, and $\|A_\phi\|_1 \leq \|\phi\|$. Evidently $A = 0$ if and only if $\phi = 0$, and hence $\phi \mapsto A_\phi$ is a one-to-one linear isomorphism from $\mathscr{C}(\mathscr{H})^*$ into $\mathscr{T}(\mathscr{H})$, and it is onto since for all $A \in \mathscr{T}(\mathscr{H})$, $\phi_A \in \mathscr{C}(\mathscr{H})^*$, and evidently the image of $\phi_A$ under this isomorphism is $A$. We have seen above that $\|A\|_1 \leq \|\phi_A\|$. Moreover, for all finite rank $X$, $\phi_A(X) = \mathrm{Tr}[AX] \leq \|A\|_1\|X\|$, so that $\|\phi_A\| \leq \|A\|_1$. Together with $\|A\|_1 \leq \|\phi_A\|$, this proves that $\|A\|_1 = \|\phi_A\|$, and hence the isomorphism is isometric. This completes the proof of *(i)*.

Let $\psi \in \mathscr{T}(\mathscr{H})^*$. Define a sesquilinear form $q_\psi$ on $\mathscr{H} \times \mathscr{H}$ by $q_\phi(\zeta,\zeta') = \phi(|\zeta\rangle\langle\zeta'|)$ for all $\zeta,\zeta' \in \mathscr{H}$. By (3.29),

$$|\psi(|\zeta\rangle\langle\zeta'|)| \leq \|\psi\|\||\zeta\rangle\langle\zeta'|\|_1 = \|\psi\|\|\zeta|_{\mathscr{H}}\|\zeta'\|_{\mathscr{H}} \ .$$

By the Riesz Lemma, there exists $B \in \mathscr{B}(\mathscr{H})$ with $\|B\| \leq \|\psi\|$ such that for all $\zeta,\zeta' \in \mathscr{H}$, $q_\psi(\zeta,\zeta') = \langle\zeta, B\zeta'\rangle$. But then for $\zeta,\zeta' \in \mathscr{H}$, $\psi(|\zeta\rangle\langle\zeta'|) = \langle\zeta, B\zeta'\rangle = \mathrm{Tr}[B|\zeta\rangle\langle\zeta'|]$. and then by linearity, $\psi(X) = \mathrm{Tr}[BX]$ for all finite rank $X$. Since finite rank operators are dense in $\mathscr{T}(\mathscr{H})$ in the trace norm, this is valid for all $X \in \mathscr{T}(\mathscr{H})$. This shows that $B \mapsto \psi_B$ is a linear map of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{T}(\mathscr{H})^*$. By (3.26), $\|\psi_B\| \leq \|B\|$. Since $\|B\| = \sup\{\ \langle\zeta, B\zeta'\rangle\ :\ \|\zeta\|_{\mathscr{H}}, \|\zeta'\|_{\mathscr{H}} = 1\ \}$ and since

$$|\langle\zeta, B\zeta'\rangle| = |\mathrm{Tr}\big[B|\zeta\rangle\langle\zeta'|\big]| = |\psi_B(|\zeta\rangle\langle\zeta'|)| \leq \|\psi_B\|\|\zeta\|_{\mathscr{H}}\|\zeta'\|_{\mathscr{H}} \ ,$$

we also have $\|B\| \leq \|\psi_B\|$. Hence $\|\psi_B\| = \|B\|_1$, and the map is isometric and hence also one-to-one.. □

**3.34 DEFINITION.** The $\sigma$-weak operator topology on $\mathscr{B}(\mathscr{H})$ is the weak-$*$ topology on $\mathscr{B}(\mathscr{H})$, which by Theorem 3.33 is the weakest topology making all of the functions $B \mapsto \mathrm{Tr}[AB]$, $A \in \mathscr{T}(\mathscr{H})$, continuous.

by Theorem 3.6, the $\sigma$-weak operator topology is stronger than the weak operator topology, which is the weakest topology making all of the functions $B \mapsto \mathrm{Tr}[AB]$, $A$ finite rank, continuous. By Lemma 3.28 and (3.26), the relative topologies on the unit ball of $\mathscr{B}(\mathscr{H})$ induced by the $\sigma$-weak operator topology and the -weak operator topology coincide.

Now let $\mathscr{M}$ be a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$. Since $\mathscr{M}$ is weakly closed in $\mathscr{B}(\mathscr{H})$, it is $\sigma$-weakly closed. Define $\mathscr{M}^\perp$ to be the *anihilator* of $\mathscr{M}$ in the dual pairing between $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$. That is,

$$\mathscr{M}^\perp = \{A \in \mathscr{T}(\mathscr{H}) \ : \ \mathrm{Tr}[AB] = 0 \text{ for all } A \in \mathscr{M} \ \} \ .$$

Evidently, $\mathscr{M}^\perp$ is norm closed in $\mathscr{T}(\mathscr{H})$. We define $\mathscr{M}_*$ to be the Banach space $\mathscr{T}(\mathscr{H})/\mathscr{M}^\perp$ consisting of equivalence classes in $\mathscr{T}(\mathscr{H})$ under the equivalence relation $A \sim C$ if and only if $A - C \in \mathscr{M}^\perp$, and we equip $\mathscr{M}_*$ with the quotient norm. Then, by a standard result in the theory of Banach spaces, the dual of $\mathscr{M}_*$ is isometrically isomorphic to $\mathscr{M}$.

To see this, first note that $\mathscr{M}$ is the anihilator of $\mathscr{M}^\perp$ in the dual pairing between $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$. That is:

$$\mathscr{M} = \{B \in \mathscr{B}(\mathscr{H}) \ : \ \mathrm{Tr}[AB] = 0 \text{ for all } A \in \mathscr{M}^\perp \ \} =: \mathscr{M}^{\perp\perp} \ . \tag{3.30}$$

This is a conseqence of the Hahn-Banach Theorem: Clearly $\mathscr{M} \subset \mathscr{M}^{\perp\perp}$. If the containment were proper, there would exist $X \in \mathscr{M}^{\perp\perp}$ not belonging to the $\sigma$-weakly closed subspace $\mathscr{M}$, and then by the Hahn-Banach Theorem, there would exist a $\sigma$-weakly continuous linear functional $\phi$ on $\mathscr{B}(\mathscr{H})$ with $\phi(\mathscr{M}) = 0$ and $\psi(X) = 1$, and every $\sigma$-weakly continuous linear functional on $\mathscr{B}(\mathscr{H})$ is of the form $B \mapsto \mathrm{Tr}[AB]$ for some $A\ in \mathscr{T}(\mathscr{H})$. But this is impossible since then $\phi \in \mathscr{M}^\perp$ and $X \in \mathscr{M}^{\perp\perp}$ which is incompatible with $\phi(X) = 1$.

Let $\psi \in (\mathscr{M}_*)^*$. Let $Q : \mathscr{T}(\mathscr{H}) \to \mathscr{M}_*$ be the quotient map, which, by the definition of the quotient norm, is a contraction. Consequently, $\|\psi \circ Q\|_{(\mathscr{T}(\mathscr{H}))^*} \leq \|\psi\|_{(\mathscr{M}_*)^*}$. Thus $\psi \circ Q \in (\mathscr{T}(\mathscr{H}))^*$, and since $Q$ vanishes on $\mathscr{M}^\perp$, $\psi \circ Q \in \mathscr{M}^{\perp\perp} = \mathscr{M}$.

Conversely, suppose that $\phi \in \mathscr{M} = (\mathscr{M}^\perp)^\perp$, Define the linear functional $\widehat{\phi}$ on $\mathscr{M}_*$ by $\widehat{\phi}(\{A\}) = \phi(A)$, which is well-defined since $\phi$ vanishes on $\mathscr{M}^\perp$. Moreover. for all $C \sim A$, $|\widehat{\phi}(\{A\})| = |\phi(C)| \leq \|\phi\|\|C\|$. Taking the infimum over $C \sim A$, $|\widehat{\phi}(\{A\})| \leq \|\phi\|\|\{A\}\|$. That is, $\|\widehat{\phi}\|_{(\mathscr{M}_*)^*} \leq \|\phi\|$, By construction, for all $A \in \mathscr{T}(\mathscr{H})$, $\widehat{\phi} \circ Q(A) = \widehat{\phi}(\{A\}) = \phi(A)$; That is, $\widehat{\phi} \circ Q = \phi$.

Therefore, the map $\psi \mapsto \psi \circ Q$ is a linear map from $(\mathscr{M}_*)^*$ onto $\mathscr{M} = \mathscr{M}^{\perp\perp}$, and its inverse on $\mathscr{M}$ is the map $\phi \mapsto \widehat{\phi}$ defined in the previous paragraph. Since both maps are contractions, they are both isometries; that is $\psi \mapsto \psi \circ Q$ is a linear isometry of $(\mathscr{M}_*)^*$ onto $\mathscr{M}$. We have proved:

**3.35 THEOREM.** *Every von Neumann algebra $\mathscr{M}$ is isomorphic as a Banaoh space to the dual of a Banach space. More precisely, $\mathscr{M}$ is isomorphic to the dual of $\mathscr{M}_*$ where $\mathscr{M}_*$ is the quotient space $\mathscr{T}(\mathscr{H})/\mathscr{M}^\perp$ where $\mathscr{M}^\perp$ is the subspace of $\mathscr{T}(\mathscr{H})$ annihilated by $\mathscr{M}$ in the dual pairing of $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$.*

# 4 $C^*$ algebras as operator algebras

## 4.1 Representations of $C^*$ algebras

**4.1 DEFINITION.** A *representation* of a $C^*$-algebra $\mathscr{A}$ is a $*$-homomorphism $\pi$ from $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. For any subspace $\mathscr{K}$ of $\mathscr{H}$, we define

$$\pi(\mathscr{A})\mathscr{K} = \{ \ \pi(A)\eta \ : \ a \in \mathscr{A} \ , \ \eta \in \mathscr{K} \ \} \ .$$

A subspace $\mathscr{K}$ of $\mathscr{H}$ is *invariant under* $\pi$ in case $\pi(\mathscr{A})\mathscr{K} \subset \mathscr{K}$. The representation $\pi$ is *irreducible* in case no non-trivial subspace $\mathscr{K}$ of $\mathscr{H}$ is invariant under $\pi$. The representation $\pi$ is *non-degenerate* in case $\overline{\pi(\mathscr{A})\mathscr{H}} = \mathscr{H}$. Let $\pi_1$ and $\pi_2$ be two representations of $\mathscr{A}$ on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then $\pi_1$ and $\pi_2$ are *equivalent representations* of $\mathscr{A}$ in case there exists a unitary transformation from $U$ from $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that for all $A \in \mathscr{A}$,

$$\pi_2(A)U = U\pi_1(A) \ .$$

**4.2 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\pi$ be a non-zero representation of it as an algebra of operators on some Hilbert space $\mathscr{H}$. Then a closed subspace $\mathscr{K}$ of $\mathscr{H}$ is invariant under $\pi(\mathscr{A})$ if and only if the orthogonal projection of $\mathscr{H}$ onto $\mathscr{K}$ belongs to $(\pi(\mathscr{A}))'$.*

*Proof.* First, $\mathscr{K}$ is invariant under $\pi(\mathscr{A})$ if and only if $\mathscr{K}^\perp$ is invariant under $\pi(\mathscr{A})$. To see this, let $\zeta \in \mathscr{K}^\perp$ and $\xi \in \mathscr{K}$, and $A \in \mathscr{A}$. If $\mathscr{K}$ is in variant, $\pi(A^*)\xi \in \mathscr{K}$, and hence

$$\langle \pi(A)\zeta, \xi \rangle_{\mathscr{H}} = \langle \zeta, \pi(A^*)\xi \rangle_{\mathscr{H}} = 0 \ .$$

Thus the invariance of $\mathscr{K}$ implies the invariance of $\mathscr{K}^\perp$, and then by symmetry, the reverse implication is valid as well.

Now let $P$ be the orthogonal projection onto $\mathscr{K}$. Then when $\mathscr{K}$ is invariant, for all $A \in \mathscr{A}$,

$$0 = P\pi(A)(1-P) = P\pi(A) - P\pi(A)P = P\pi(A) - \pi(A)P$$

where the last equality is true since the range of $\pi(A)P$ lies in $\mathscr{K}$. Therefore, $P \in (\pi(\mathscr{A}))'$. Conversely, if $P \in (\pi(\mathscr{A}))'$ and $\xi \in \mathscr{K}$, then for all $A \in \mathscr{A}$,

$$\pi(A)\xi = \pi(A)P\xi = P\pi(A)\xi \in \mathscr{K} \ ,$$

which shows the invariance of $\mathscr{K}$. □

Lemma 4.2 permits us to make the following definition:

**4.3 DEFINITION.** For a representation $\pi$ of a $\mathbb{C}^*$ algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and a non-zero projector $P \in (\pi(\mathscr{A}))'$, $\pi_P$ is the subrepresntation obtained by restricting $\pi$ to $\mathrm{ran}(P)$.

**4.4 THEOREM.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\pi$ be a non-zero representation of it as an algebra of operators on some Hilbert space $\mathscr{H}$ . Then $\pi$ is irreducible if and only if $(\pi(\mathscr{A}))'$ consists of scalar multiples of the identity.*

*Proof.* If $(\pi(\mathscr{A}))'$ consists of scalar multiples of the identity, then $(\pi(\mathscr{A}))'$ contains no non-trivial orthogonal projections, and hence by Lemma 4.2, $\pi$ is irreducible. On the other hand, if $(\pi(\mathscr{A}))'$ contains some operator that is not a multiple of the identity, then it contains a self adjoint operator $A$ that is not a multiple of the identity. Any such $A \in (\pi(\mathscr{A}))'$ has a non-trivial spectral projection that is also in $(\pi(\mathscr{A}))'$ since $(\pi(\mathscr{A}))'$ is a strongly closed $*$-algebra containing $A$. □

## 4.2 States on a $C^*$ algebra

**4.5 DEFINITION.** Let $\mathscr{A}$ be a $C^*$ algebra. A linear functional $\varphi$ on $\mathscr{A}$ regarded as a Banach space, is *positive* in case $\varphi(A) \geq 0$ for all $A \in \mathscr{A}^+$. A positiver linear functional $\varphi$ is *faithful* in case

$$\varphi(A^*A) = 0 \quad \Rightarrow \quad A = 0 \ . \tag{4.1}$$

Every self-adjoint $A \in \mathscr{A}$ is the difference of two elements of $\mathscr{A}^+$: For $f(t) := \max\{0, t\}$, define $A_1 = f(A)$ and $A_2 = A - f(A)$. Then $A = A_1 - A_2$, $A_1, A_2 \in \mathscr{A}^+$. Moreover, with this choice of $A_1$ and $A_2$, $\|A_1\|, \|A_2\| \leq \|A\|$ and $A_1 A_2 = A_2 A_1 = 0$. In this case we say that $A_1$ is the *positive part* of $A$, written $A_+$, and $A_2$ is the *negative part of* $A$, written $A_-$. Consequently, for a positive linear functional $\varphi$, $\varphi(A) = \varphi(A_+) - \varphi(A_-)$, and hence $\varphi$ is real on the self-adjoint elements of $\mathscr{A}$.

Every $A \in \mathscr{A}$ has a canonical decomposition

$$A = X_1 - X_2 + i(Y_1 - Y_2) \tag{4.2}$$

where $X_1, X - 2, Y_1, Y_2 \in \mathscr{A}^+$, and $\|X_1\|, \|X_2\|, \|Y_1\|, \|Y_2\| \leq \|A\|$. To see this, define

$$X := \frac{A + A^*}{2} \quad \text{and} \quad Y = \frac{A - A^*}{2i} \ ,$$

and then decompose $X$ and $Y$ into their positive and negative parts.

For all positive linear functionals $\varphi$ on a $\mathbb{C}^*$ algebra $A$, the map

$$(A, B) \mapsto \varphi(A^* B) =: \langle A, B \rangle_\varphi$$

defines a (possibly degenerate) inner product on $\mathscr{A}$; this inner product is non-degenerate if and only if $\varphi$ is faithful. In any case, the fact that $\langle A, A \rangle_\varphi \geq 0$ for all $A \in \mathscr{A}$ yields the Cauchy-Schwarz inequality:

$$|\langle A, B \rangle_\varphi| \leq \langle A, A \rangle_\varphi^{1/2} \langle B, B \rangle_\varphi^{1/2} \ . \tag{4.3}$$

**4.6 LEMMA** (Positivity and continuity)**.** *Let $\mathscr{A}$ be a $C^*$ algebra. Every positive linear functional is bounded, and*

$$\|\varphi\| = \sup_{A \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(A)|\} \ . \tag{4.4}$$

*If furthermore $\mathscr{A}$ is unital,*

*(1) For all $\varphi \in \mathscr{A}^*_+$, $\|\varphi\| = \varphi(1)$.*

*(2) Every bounded linear functional $\varphi$ such that $\|\varphi\| = \varphi(1)$ is positive.*

*Proof.* As noted above, each $A \in B_{\mathscr{A}}$ has decomposition $A = X_1 - X_2 + i(Y_1 - Y_2)$ with $X_1$, $X_2$, $Y_1$, $Y_2$ all in $\mathscr{A}^+ \cap B_{\mathscr{A}}$. Therefore, for any positive linear functional $\varphi$,

$$\sup_{\|A\| \leq 1} \{|\varphi(A)|\} \leq 4 \sup_{A \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(A)|\} \ .$$

Suppose $\varphi$ is a positive linear functional, and that $\varphi \in \mathscr{A}^*$ is not bounded, Then by the remarks just above, there is a sequence $\{X_j\}_{j \in \mathbb{N}}$ in $\mathscr{A}^+$ with $\|X_j\| = 1$ and $|\varphi(X_j)| \geq 4^j$ for all $j$. Define

$$X := \sum_{j=1}^{\infty} 2^{-j} X_j \ .$$

Since $\mathscr{A}^+$ is closed, $X \in \mathscr{A}^+$, Moreover for all $n > m$, $\sum_{j=m+1}^{n} 2^{-j} X_j \in \mathscr{A}^+$, and taking $n \to \infty$, and using the closure of $\mathscr{A}^+$ once more, for all $m \in \mathbb{N}$, $X - \displaystyle\sum_{j=1}^{m} 2^{-j} X_j \in \mathscr{A}^+$. Therefore,

$$\varphi(X) \geq \sum_{j=1}^{m} 2^{-j} \varphi(X_j) \geq \sum_{j=1}^{m} 2^j \ ,$$

and this evidently yields a contradiction for $m$ large enough. Next, let $\epsilon > 0$, and pick $A \in B_{\mathscr{A}}$ such that $|\varphi(A)| + \epsilon \geq \|\varphi\|$. By Theorem 2.31, there exists $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\|AE - A\| \leq \epsilon$. Then $|\varphi(AE)| \geq |\varphi(A)| - \|\varphi\|\epsilon$, and by the Cauchy-Schwarz inequality,

$$|\varphi(AE)| \leq \left(\varphi(A^*A)(\varphi(E^2)\right)^{1/2} \leq \sup_{X \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(X)|\}$$

since both $A^*A$ and $E^2$ belong to $\mathscr{A}^+ \cap B_{\mathscr{A}}$. Since $\epsilon > 0$ is arbitrary, this proves (4.4).

For the second part, suppose that $\mathscr{A}$ is unital. Since $\|\varphi\| \geq \varphi(1) \geq \varphi(A)$ for all $A \in \mathscr{A}^+ \cap B_{\mathscr{A}}$, *(1)* is an immediate consequence of (4.4).

To prove *(2)*, suppose that $\varphi \in \mathscr{A}^*$ and $\varphi(1) = \|\varphi\|$. If $\varphi = 0$, it is positive. If $\varphi \neq 0$, we may divide by $\|\varphi\|$ and thus may suppose that $\|\varphi\| = \varphi(1) = 1$.

We claim that for all $\varphi \in \mathscr{A}^*$ such that $\varphi(1) = \|\varphi\|$, $\varphi(A)$ belongs to the convex hull of $\sigma_{\mathscr{A}}(A)$ for all $A \geq 0$ in $\mathscr{A}$. To see this suppose that the closed disc of radius $r$ centered on $\lambda$ contains $\sigma_{\mathscr{A}}(A)$. Then $\lambda 1 - A$ is normal, and its spectrum is continued in $\{\lambda - t \ : \ t \in \sigma_{\mathscr{A}}(A)\}$, and hence the spectral radius of $\lambda 1 - A$ is at most $r$. Since $\lambda 1 - A$ is normal, $\|\lambda 1 - A\| \leq r$. Therefore,

$$|\lambda - \varphi(A)| = |\varphi(\lambda 1 - A)| \leq \|\lambda 1 - A\| \leq r \ .$$

Thus for all $r > 0$ and $\lambda \in \mathbb{C}$, $\varphi(A)$ is contained in the closed disc of radius $r$ centered on $\lambda$ contains $\sigma_{\mathscr{A}}(A)$. The intersection over all such discs is the convex hull of $\sigma_{\mathscr{A}}(A)$. □

By the first part of Lemma 4.6, for every $C^*$ algebra $\mathscr{A}$, every positive linear functional on $\mathscr{A}$ belongs to $A^*$, the Banach space dual to $\mathscr{A}$. We write $A^*_+$ to denote the set of positive linear functionals.

**4.7 DEFINITION** (State and quasi-state)**.** Let $\mathscr{A}$ be a $C^*$ algebra. A *state* on $\mathscr{A}$ is an element $\varphi$ of $\mathscr{A}^*_+$ with $\|\varphi\| = 1$. A *quasi-state* on $\mathscr{A}$ is an element $\varphi$ of $\mathscr{A}^*_+$ with $\|\varphi\| \leq 1$. We write $S_{\mathscr{A}}$ to denote the set of state on $\mathscr{A}$, and $Q_{\mathscr{A}}$ to denote the set of quasi-states. We equip both $S_{\mathscr{A}}$ and $Q_{\mathscr{A}}$ with the relative weak-$*$ topology.

Notice that $Q_{\mathscr{A}} = \cap_{A \in \mathscr{A}^+} \{\varphi \in \mathscr{A}^* \ : \ \|\varphi\| \leq 1 \quad \text{and} \quad \varphi(A) \geq 0\}$, which is evidently a weak-$*$ closed subset of the unit ball in $\mathscr{A}^*$. Then by the Banach-Alaoglu Theorem, $Q_{\mathscr{A}}$ is weak-$*$ compact.

Suppose that $\mathscr{A}$ is not unital, and let $\mathscr{A}_1$ be the $C^*$ algebra obtained by adjoining a unit in the canonical manner. Let $\varphi \in Q_{\mathscr{A}}$, and define $\widetilde{\varphi}$ on $\mathscr{A}_1$ by

$$\widetilde{\varphi}((\lambda, A)) = \|\varphi\|\lambda + \varphi(A) \ . \tag{4.5}$$

Let $(\lambda, A) \in \mathscr{A}_1^+$.Then $\lambda \geq 0$ and $\sigma_{\mathscr{A}}(A) \subset [-\lambda, \infty)$. Hence

$$\widetilde{\varphi}((\lambda, A)) = \|\varphi\|\lambda + \varphi(A_+) - \varphi(A_-) \geq \|\varphi\|\lambda - \varphi(A_-) \ .$$

Since $\sigma_{\mathscr{A}}(A) \subset [-\lambda, 0]$, $\|A_-\| \leq \ \lambda$. Therefore $\varphi(A_-) \leq \|\varphi\|\|A_-\| \leq \|\varphi\|\lambda$. Thus, $\widetilde{\varphi}((\lambda, A)) \geq 0$, showing that $\widetilde{\varphi}$ is positive. By part *(1)* of Lemma 4.6, $\| \wr \varphi\| = \widetilde{\varphi}(1) = \|\varphi\|$, so that $\widetilde{\varphi} \in Q_{\mathscr{A}_1}$, and if $\varphi \in S_{\mathscr{A}}$, then $\widetilde{\varphi} \in S_{\mathscr{A}}$.

Let $\psi \in S_{\mathscr{A}_1}$. Define a positive linear function $\psi|_{\mathscr{A}}$ on $\mathscr{A}$ by

$$\psi|_{\mathscr{A}}(A) = \psi((0, A)) \ .$$

Evidently $\|\psi|_{\mathscr{A}}\| \leq \|\psi\| = 1$, so that $\psi|_{\mathscr{A}} \in Q_{\mathscr{A}}$. Let $\psi_0$ denote the state on $\mathscr{A}_1$ given by $\psi_0((\lambda, A)) = \lambda$. For $\psi \in S_{\mathscr{A}_1}$, $\psi\ neq \psi_0$, $\|\psi|_{\mathscr{A}}\| > 0$. Then with $c := \|\psi|_{\mathscr{A}}\|$, $c^{-1}\psi|_{\mathscr{A}} \in S_{\mathscr{A}}$, and

$$\psi((\lambda, A)) = \lambda + \psi((0, A)) = \lambda + \psi|_{\mathscr{A}}(A) = (1-c)\lambda + c\left(\lambda + c^{-1}\psi|_{\mathscr{A}}(A)\right) \ .$$

That is,

$$\psi = c\psi_0 + (1-c)\widetilde{c^{-1}\psi|_{\mathscr{A}}} \ ,$$

which shows that every state $\psi \in S_{\mathscr{A}_1}$ is a convex combination of $\psi_0$ and a state of the form $\widetilde{\varphi}$, $\varphi \in S_{\mathscr{A}}$.

By Lemma 4.6, when $\mathscr{A}$ is unital, when $\varphi \in \mathscr{A}^*_+$, $\|\varphi\| = \varphi(1)$, and hence in this case, reasoning as above,

$$S_{\mathscr{A}} = \cap_{A \in \mathscr{A}^+} \{\varphi \in \mathscr{A}^* \ : \ \varphi(A) \geq 0 \quad \text{and} \quad \varphi(1) = 1\}$$

which displays $S_{\mathscr{A}}$ as the intersection over a family of weak-$*$ closed subsets of the unit ball of $\mathscr{A}^*$. Hence in the unital case, the state space is compact. When $\mathscr{A}$ is not unital, we still have that

$$Q_{\mathscr{A}} = \cap_{A \in \mathscr{A}^+} \{\varphi \in \mathscr{A}^* \ : \ \varphi(A) \geq 0\} \cap B_{\mathscr{A}^*} \ ,$$

which displays $Q_{\mathscr{A}}$ as the intersection over a family of weak-$*$ closed subsets of the unit ball of $\mathscr{A}^*$. Hence $Q_{\mathscr{A}}$ is compact in the weak-$*$ topology.

**4.8 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra. For all self adjoint $A \in \mathscr{A}$, there exists a state $\varphi$ such that $|\varphi(A)| = \|A\|$.*

*Proof.* We may assume $A \neq 0$. First suppose that $\mathscr{A}$ is unital. Consider the $C^*$ algebra $C(\{1, A\})$ generated by 1 and $A$. This is a commutative $C^*$ algebra, and by Corollary 1.33, there exists a character $\varphi_0$ of $C(\{1, A\})$ such that $|\varphi_0(A)| = \nu(A)$, which equals $\|A\|$. Since $\varphi_0$ is a character $\varphi(1) = 1$. Then by Lemma 4.6, $\varphi_0 \in \mathscr{A}^*_+$, and so $\varphi$ is a state on $C(\{1, A\})$.

By the Hahn-Banach Theorem, there is a norm preserving extension $\varphi$ of $\varphi_0$ (as a linear functional) to $\mathscr{A}$. Then $\varphi(1) = \varphi_0(1) = 1$, and hence by Lemma 4.6, $\varphi$ is a state, and since $\varphi$ extends $\varphi_0$, $\varphi(A) = \|A\|$.

When $\mathscr{A}$ is not unital, consider $\mathscr{A}_1$, the $C^*$ algebra $\mathscr{A}_1$ obtained by adjoining a unit in the canonical manner. Then $(0, A)$ is self adjoint in $\mathscr{A}_1$, and by what we have just proved, there is a character $\psi$ on $\mathscr{A}_1$ such that $\psi((0, A)) = \|(0, A)\| = \|A\|$. Define $\varphi$ to be the quasi-state $\varphi = \psi|_{\mathscr{A}}$; i.e., for all $B \in \mathscr{A}$, $\varphi(B) = \psi((0, B))$. Since $\varphi(A) = \|A\|$, $\|\varphi\| = 1$, and $\varphi$ is a state on $\mathscr{A}$. □

We have seen that for a unital $C^*$ algebra $\mathscr{A}$, $S_{\mathscr{A}}$ is compact in the weak-$*$ topology, and it is evidently convex. The Krein-Milman Theorem says that every non-empty convex set in $\mathscr{A}^*$ that is compact in the weak-$*$ topology is the convex hull of its extreme points. Hence there exist extreme points in $\mathscr{A}^*_{+,1}$.

**4.9 DEFINITION** (Pure state)**.** Let $\mathscr{A}$ be a unital $C^*$ algebra. A *pure state* is an extreme point of $\mathscr{A}^*_{+,1}$.

**4.10 THEOREM.** *Let $\mathscr{A}$ be a unital $C^*$ algebra. For all self adjoint $A \in \mathscr{A}$, there exists a pure state $\varphi$ such that $|\varphi(a)| = \|a\|$.*

*Proof.* By Lemma 4.8, the set $\mathcal{S}$ of states $\varphi$ such that $\varphi(A) = \|A\|$ is non-empty, and evidently it is convex and closed in the weak-$*$ topology. By the Krein-Milman, $\mathcal{S}$ has at least one extreme point $\psi$. We now show that $\psi$ is extreme in $S_{\mathscr{A}}$ as well as in $\mathcal{S}$.

Suppose that $\psi_1, \psi_2 \in S_{\mathscr{A}}$ and that $\psi = t\psi_1 + (1-t)\psi_2$ for some $t \in (0,1)$. Evaluating both sides at $A$,

$$\|A\| = \psi(A) = t\psi_1(A) + (1-t)\psi_2(A) \leq t\|A\| + (1-t)\|A\| = \|A\| \ .$$

Hence $\psi_1, \psi_2 \in \mathcal{S}$, and so $\psi_1 = \psi_2 = \psi$. □

**4.11 DEFINITION.** Let $\pi$ be a representation of a $C^*$ algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. A vector $\eta \in \mathscr{H}$ is cyclic for $\pi$ in case $A \mapsto \pi(A)\eta$ has dense range, and is a *separating* vector for $\pi$ in case $A \mapsto \pi(A)\eta$ is injective. If a cyclic vector exists, then $\pi$ is a *cyclic representation.*

For any representation $\pi$ of $\mathscr{A}$ on $\mathscr{H}$, and any *unit vector* $\eta \in \mathscr{H}$, the functional $\varphi_\eta \in \mathscr{A}^*$ defined by

$$\varphi_\eta(A) = \langle \eta, \pi(A)\eta \rangle_{\mathscr{H}} \tag{4.6}$$

is a state. Evidently, $\varphi_\eta(A^*A) = \langle \eta, \pi(A^*A)\eta \rangle_{\mathscr{H}} = \|\pi(A)\eta\|^2_{\mathscr{H}}$, and hence $\eta$ is separating for $\pi$ is and only if $\varphi_\eta$ is faithful. The next theorem links cyclicity and irreducibility.

**4.12 THEOREM.** *Let $\pi$ be a representation of a $C^*$ algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and let $\eta$ be a cyclic unit vector for $\pi$. Then with $\varphi_\eta$ denoting the state defined in (4.6). Then $\pi$ is irreducible iff and only if $\varphi_\eta$ is pure.*

The heart of the matter is the following lemma:

**4.13 LEMMA.** *Let $\pi$ be a representation of a unital $C^*$ algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and let $\eta$ be a cyclic unit vector for $\pi$. Then with $\varphi_\eta$ denoting the state defined in (4.6), suppose that $\psi \in S_{\mathscr{A}}$, and that for some $r \in (0,\infty)$,*

$$\psi(A) \leq r\varphi_\eta(A) \quad \text{for all} \quad A \in \mathscr{A} \ .$$

*Then there is a positive operator $X \in (\pi(\mathscr{A}))'$ such that $\|X\| \leq r$ and for all $A, B \in \mathscr{A}$,*

$$\psi(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta \rangle_{\mathscr{H}} \ . \tag{4.7}$$

*Proof.* Define a sesquilinear form $q$ on $\pi(\mathscr{A})\eta$ by $q(\pi(A)\eta, \pi(B)\eta) = \psi(A^*B)$. We have

$$|q(\pi(A)\eta, \pi(B)\eta)| \leq r|\langle \pi(A)\eta, \pi(B)\eta \rangle_{\mathscr{H}}| \leq r\|\pi(A)\|_{\mathscr{H}}\|\pi(B)\|_{\mathscr{H}} \ .$$

Since $\eta$ is cyclic, $q$ is densely defined on $\mathscr{H}$ and extends to a sesquilinear form on all of $\mathscr{H}$, still denoted by $q$, that satisfies $|q(\zeta,\xi)| \leq r\|\zeta\|_{\mathscr{H}}\|\xi\|_{\mathscr{H}}$ for all $\zeta, \xi \in \mathscr{H}$. By Reisz's Lemma, there exists a self adjoint operator $X \in \mathscr{B}(\mathscr{H})$ such that $q(\zeta,\xi) = \langle \zeta, X\xi \rangle_{\mathscr{H}}$ for all $\zeta, \xi \in \mathscr{H}$, and $\|X\| \leq r$. Since $q(\zeta,\zeta) \geq 0$ for $\zeta$ in the dense set $\pi(\mathscr{A})\eta$, $X$ is positive.

Finally, note that for all $A, B, C \in \mathscr{A}$, $A^*(BC) = (B^*A)^*C$, and hence $\psi(A^*(BC)) = \psi((B^*A)^*C)$. This means that $q(\pi(A)\eta, \pi(B)\pi(C)\eta) = q(\pi(B^*)\pi(A)\eta, \pi(C)\eta)$ which is the same as

$$\langle \pi(A)\eta, X\pi(B)\pi(C)\eta \rangle_{\mathscr{H}} = \langle \pi(A)\eta, \pi(B)X\pi(C)\eta \rangle_{\mathscr{H}} \ .$$

Thus for all $\zeta, \xi$ in a dense subset of $\mathscr{H}$, $\langle \zeta, X\pi(B)\xi \rangle_{\mathscr{H}} = \langle \zeta, \pi(B)X\xi \rangle_{\mathscr{H}}$ and this shows that $X$ commutes with $\pi(B)$ for arbitrary $b \in \mathscr{A}$. □

*Proof of Theorem 4.12.* Suppose that $\pi$ is irreducible. Let $\psi_1, \psi_2$ be two states such that $\varphi_\eta = t\psi_1 + (1-t)\psi_2$ for some $t \in (0,1)$. By Lemma 4.13, applied to $\psi_1$, which satisfies $\psi_1 \leq t^{-1}\varphi_\eta$, there is a positive $X \in (\pi(\mathscr{A}))'$

$$\psi_1(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta\rangle_{\mathscr{H}} \quad \text{for all} \quad A, B \in \mathscr{A} \ . \tag{4.8}$$

Since $\pi$ is irreducible, $X$ must be a scalar multiple of the identity. Since $\psi_1$ is a state, taking $A = B = 1$ in (4.8), $1 = \psi(1) = \langle \eta, X\eta\rangle_{\mathscr{H}}$, which shows that $X = 1$. Then taking $A = 1$ in (4.8) shows that $\psi(B) = \varphi_\eta(B)$ so that $\psi_1 = \varphi_\eta$. By symmetry, $\psi_2 = \varphi_\eta$ as well, and this proves $\varphi_\eta$ is extreme.

For the converse, suppose that $\pi$ is not irreducible. Then there exists a projection $P \in (\pi(\mathscr{A})')$ such that neither $P$ nor $P^\perp$ is zero. Suppose that $P\eta = 0$. Then for all $A \in \mathscr{A}$, $P(\pi(A)\eta) = \pi(A)P\eta = 0$ and this would mean that $P$ vanishes on a sense subspace, which is not the case. Hence $\|P\eta\|_{\mathscr{H}} > 0$, and the same reasoning shows that $\|P^\perp\eta\|_{\mathscr{H}} > 0$. Define $\eta_1 = \|P\eta\|_{\mathscr{H}}^{-1}P\eta$ and $\eta_2 = \|P^\perp\eta\|_{\mathscr{H}}^{-1}P^\perp\eta$. For all $A \in \mathscr{A}$,

$$\langle \eta_1, \pi(A)\eta_2\rangle_{\mathscr{H}} = \langle P\eta_1, \pi(A)P^\perp\eta_2\rangle_{\mathscr{H}} = \langle \eta_1, PP^\perp\pi(A)\eta_2\rangle_{\mathscr{H}} = 0 \ .$$

Define $t \in (0,1)$ by $t = \|P\eta\|_{\mathscr{H}}^2$. Since $\|P\eta\|_{\mathscr{H}}^2 + \|P^\perp\eta\|_{\mathscr{H}}^2 = 1$, $\|P^\perp\eta\|_{\mathscr{H}}^2 = 1 - t$. Then by the orthogonality proved just above, for all $A \in \mathscr{A}$,

$$\begin{aligned}\varphi_\eta(A) &= \langle [\sqrt{t}\eta_1 + \sqrt{1-t}\eta_2], \pi(A)[\sqrt{t}\eta_1 + \sqrt{1-t}\eta_2]\rangle_{\mathscr{H}} \\ &= t\langle \eta_1, \pi(A)\eta_1\rangle_{\mathscr{H}} + (1-t)\langle \eta_2, \pi(A)\eta_2\rangle_{\mathscr{H}} \ ,\end{aligned}$$

and this displays $\varphi_\eta$ as a non-trivial convex combination of states. Hence $\varphi_\eta$ is not extreme. □

## 4.3 Grothendieck's Decomposition Theorem

**4.14 DEFINITION.** Let $\mathscr{A}$ be a $C^*$ algebra. A linear functional $\phi \in \mathscr{A}^*$ is *Hermitian* in case $\phi(A^*) = \overline{\phi(A)}$ for all $A \in \mathscr{A}$. The real vector space of all Hermitian elements of $\mathscr{A}^*$ is denoted $(\mathscr{A}^*)_{\text{s.a.}}$. Let $(\mathscr{A}_{\text{s.a.}})^*$ denote the real Batch space dual to the real Banach space $\mathscr{A}_{\text{s.a.}}$. Note that the elements of $(\mathscr{A}^*)_{\text{s.a.}}$ are bounded complex linear functionals on $\mathscr{A}$, while the elements of $(\mathscr{A}_{\text{s.a.}})^*$ are bounded real linear functionals on $\mathscr{A}_{\text{s.a.}}$.

**4.15 LEMMA.** *A linear functional $\phi \in \mathscr{A}^*$ is Hermitian if and only if $\phi : \mathscr{A}_{\text{s.a.}} \to \mathbb{R}$. For $\phi \in (\mathscr{A}^*)_{\text{s.a.}}$, define $\phi_R$ to be the restriction of $\phi$ to $\mathscr{A}_{\text{s.a}}$. For $\varphi \in (\mathscr{A}_{\text{s.a.}})^*$, define $\phi_C : \mathscr{A} \to \mathbb{C}$ by*

$$\varphi_C(X + iY) = \varphi(X) + i\varphi(Y) \tag{4.9}$$

*for $X, Y \in \mathscr{A}_{\text{s.a.}}$. Then $\varphi_C \in (\mathscr{A}^*)_{\text{s.a.}}$, and the map $\varphi \mapsto \varphi_C$ is an isometric isomorphism of $(\mathscr{A}_{\text{s.a.}})^*$ onto $(\mathscr{A}^*)_{\text{s.a.}}$.*

*Proof.* Evidently if $\phi$ is Hermitian and $A \in \mathscr{A}_{\text{s.a.}}$, then $\phi(A) \in \mathbb{R}$. Conversely , if $\phi \in \mathscr{A}^* : \mathscr{A}_{\text{s.a.}} \to \mathbb{R}$, then $\phi$ is Hermitian, since any $A \in \mathscr{A}$ can be written as $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$, and then $\phi(A^*) = \phi(X) - i\phi(Y) = \overline{\phi(X + iY)} = \overline{\phi(A)}$.

If $\phi \in \mathscr{A}^*$ is Hermitian, then the restriction $\phi_R$ of $\phi$ to the real Banach space $\mathscr{A}_{\text{s.a.}}$ is a bounded real linear functional on $\mathscr{A}_{\text{s.a.}}$, and since $\phi$ is determined by its action on $\mathscr{A}_{\text{s.a.}}$, $\phi \mapsto \phi_R$ is one-to-one.

For any $\varphi \in (\mathscr{A}_{\text{s.a.}})^*$, define an extension $\varphi_C$ to $\mathscr{A}$ by (4.9). Then $\varphi_C(i(X+iY)) = i\varphi(X) - i\varphi(Y) = i\varphi_C(X+iY)$, and from here one readily sees that $\varphi_C$ is complex linear functional on $\mathscr{A}$, and moreover. $\varphi_C$ is Hermitian, and the restriction of $\varphi_C$ to $\mathscr{A}_{\text{s.a.}}$ is simply $\varphi$ itself.

Furthermore, for $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$, and if $\mathscr{A} \in B_{\mathscr{A}}$, then $X, Y \in B_{\mathscr{A}_{\text{s.a.}}}$. It follows immediately that for all $A \in B_{\mathscr{A}}$, $|\varphi_C(A)| \leq 2\|\varphi\|$, so that so that $\varphi_C$ is a Hermitian element of $\mathscr{A}^*$. In fact, $\|\varphi_C\| = \|\varphi\|$. To see this, fix $\epsilon > 0$, and choose $A \in B_{\mathscr{A}}$ such that $|\varphi(A)| + \epsilon > \|\varphi_C\|$. Choose $\theta \in [0, 2\pi)$ so that $e^{i\theta}\varphi(A) > 0$. The replace $A$ by $e^{i\theta}A$, and write $A = X+iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$. Then $\varphi_C(A) = \varphi(X) + i\varphi(Y)$ is real, so that $\varphi(Y) = 0$ and

$$\|\varphi_C\| - \epsilon \leq \varphi_C(A) = \varphi(X) \leq \|\varphi\|\|X\| \leq \|\varphi\|$$

since $\|X\| \leq \|A\| \leq 1$. Hence $\|\varphi_C\| \leq \|\varphi\|$, and since $\varphi$ is the restriction of $\varphi_C$ to $\mathscr{A}_{\text{s.a.}}$, the opposite inequality is trivially true. □

**4.16 LEMMA.** *The unit ball in $(\mathscr{A}^*)_{\text{s.a.}}$ is the convex hull of $S_{\mathscr{A}}$ and $-S_{\mathscr{A}}$.*

*Proof.* Let $K$ be the convex hull of $S_{\mathscr{A}}$ and $-S_{\mathscr{A}}$. This is the image of $[0,1] \times S_{\mathscr{A}} \times S_{\mathscr{A}}$ under the map $(t, \varphi, \psi) \mapsto (1-t)\varphi - t\psi$, which is continuous with the natural product topology on the domain. As the continuous image of a compact set, $K$ is compact. Let $K_R$ be set of restrictions of elements of $K$ to $\mathscr{A}_{\text{s.a.}}$, Then $K_R$ is weak $*$-compact and convex in $(\mathscr{A}_{\text{s.a.}})^*$.

Suppose that there exists some $\psi$ in the unit ball of $(\mathscr{A}_{\text{s.a.}})^*$ that does not belong to $K_P$. Then by the Hahn-Banach Separation Theorem, there is an $A \in \mathscr{A}_{\text{s.a.}}$ and a $t \in \mathbb{R}$ such that $\psi(A) > t$, but $\varphi(A) \leq t$ for all $\varphi \in K$. Since $-\varphi \in K$ whenever $\varphi \in K$, $|\varphi(A)| \leq t$. By Theorem 4.8, $\|A\| \leq t$. But then $\psi(A) > t$ is impossible for $\psi$ in the unit ball of $(\mathscr{A}_{\text{s.a.}})^*$. Hence no such $A$ exists. By Lemma 4.9, $K$ is then all of $(\mathscr{A}^*)_{\text{s.a.}}$. □

**4.17 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra. For every two positive linear functionals $\varphi$ and $\psi$ on $\mathscr{A}$, the following are equivalent:*

*(1)* $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$

*(2) For all $\epsilon > 0$, and all $A \in \mathscr{A}^+$, there exists $X, Y \in \mathscr{A}^+$ with $X + Y \in B_{\mathscr{A}}$ and $X - Y \in B_{\mathscr{A}}$ such that*

$$\varphi(X) < \epsilon \ , \quad \psi(Y) < \epsilon \ , \quad \varphi(Y) + \epsilon > \|\varphi\| \quad \text{and} \quad \psi(X) + \epsilon > \|\psi\| \ . \tag{4.10}$$

*and*

$$\|A(X+Y) - A\| = \|(X+Y)A - A\| < \epsilon \ . \tag{4.11}$$

*Furthermore, when $\mathscr{A}$ is unital, one may take $X$ and $Y$ to satisfy $X + Y = 1$, so that* (4.11) *is trivially true for all $A$.*

*Proof.* Suppose that *(1)* is valid. Pick $B, C \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\varphi(B) + \epsilon > \|\varphi\|$ and $\psi(C) + \epsilon > \|\psi\|$. Since $\varphi - \psi \in (\mathscr{A}_{\text{s.a.}})^*$, for all $\epsilon > 0$, there exists $D$ in the unit ball of $\mathscr{A}_{\text{s.a.}}$ such that

$$\varphi(D) - \psi(D) + \epsilon \geq \|\varphi - \psi\| \ .$$

Suppose $\mathscr{A}$ is not unital. By Theorem 2.31 applied to $\{A, B, C, D\}$, there exits $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\|ZE - Z\| < \epsilon$ and $\|EZ - Z\| < \epsilon$ when $Z$ is any element of $\{A, B, C, D\}$ Then $EZE - Z = E(ZE - Z) + EZ - Z$ so that $\|EZE - Z\| < 2\epsilon$ when $Z$ is any element of $\{A, B, C, D\}$.

Let $\mathscr{A}_1$ be the $C^*$ algebra obtained by adjoining an identity to $\mathscr{A}$ in the canonical manner. Then for self adjoint $Z$ in $B_{\mathscr{A}}$, the spectrum of $1 \pm Z$ lies in $[0,2]$ and hence it follows that $E(1 \pm Z)E = E^2 \pm EZE$ have spectrum in $[0,2]$. Thus, $E^2 \geq EZE$ and $\frac{1}{2}(E^2 \pm EZE)$ both belong to $\mathscr{A}^* \cap B_{\mathscr{A}}$.

If $\mathscr{A}$ is unital, things are much simpler: We may simply take $E = 1$. In either case, with $E = 1$ when $\mathscr{A}$ is unital, by *(1)*,

$$\varphi(EAE) - \psi(EAE) + (1 + 2\|\varphi\| + 2\|\psi\|)\epsilon \geq \|\varphi\| + \|\psi\| \geq \varphi(E^2) + \psi(E^2) \ . \tag{4.12}$$

Define

$$X := \frac{1}{2}(E^2 - EAE) \quad \text{and} \quad Y := \frac{1}{2}(E^2 + EAE) \ .$$

Then as noted above, $X, Y \in \mathscr{A}^+$, and $X + Y = E^2 \in B_{\mathscr{A}}$ while $X - Y = -EAE \in B_{\mathscr{A}}$. Hence, with these definitions, (4.12) becomes

$$\psi(Y) + \varphi(X) \leq \frac{1}{2}(1 + 2\|\varphi\| + 2\|\psi\|)\epsilon \ . \tag{4.13}$$

Also, $\varphi(X + Y) = \varphi(E^2) \geq \varphi(EBE) \geq \|\varphi\| - (1 + 2\|\varphi\|)\epsilon$. Then by (4.13),

$$\varphi(Y) \geq \|\varphi\| - (2 + 3\|\varphi\| + \|\psi\|)\epsilon \ ,$$

In the same way, we obtain a similar lower bound for $\psi(X)$. Finally, since $X + Y = E^2$ and $E^2A - A = E(EA - E) + (EA - a)$, $\|E^2A - A\| \leq 2\epsilon$ Then since $\epsilon > 0$ is arbitrary, this proves that *(1)* implies *(2)*. note that when $\mathscr{A}$ is unital $E^2 = 1$, and hence $X + Y = 1$.

Suppose that *(2)* is valid for any $A \in \mathscr{A}^+$, e.g., $A = 0$. Pick $\epsilon > 0$, and suppose that $X$ and $Y$ satisfy the conditions in *(2)*. Then $X - Y \in B_{\mathscr{A}}$ and $(\varphi - \psi)(Y - X) \geq \|\varphi\| + \|\psi\| - 4\epsilon$. Hence $\|\varphi - \psi\| \geq \|\varphi\| + \|\psi\|$, and the opposite inequality is trivial. □

**4.18 DEFINITION.** Let $\mathscr{A}$ be a $C^*$ algebra. Two positive linear functionals on $\mathscr{A}$ are *mutually singular* in case $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$, in which case we write $\varphi \perp \psi$.

**4.19 THEOREM** (Grothendieck's Decomposition Theorem)**.** *Let $\mathscr{A}$ be a $C^*$ algebra and let $\phi \in (\mathscr{A}^*)_{\text{s.a.}}$. Then $\phi$ has a unique decomposition $\phi = \varphi - \psi$ where $\varphi$ and $\psi$ are mutually singular positive linear functionals on $\mathscr{A}$.*

*Proof.* We may suppose without loss of generality that $\|\phi\| = 1$. By Lemma 4.16, there are $\phi_1, \phi_2 \in Q_{\mathscr{A}}$ and $t \in [0,1]$ such that $\phi = (1-t)\phi_1 - t\phi_2$. Define $\varphi = (1-t)\phi_1$ and $\psi = t\phi_2$. Then

$$1 = \|\phi\| = \|\varphi - \psi\| \leq \|\varphi\| + \|\psi\| = (1-t)\|\phi_1\| + t\|\phi_2\| \leq 1 \ .$$

Hence equality must hold throughout, and hence $\varphi$ and $\psi$ are mutually singular. This proves the existence of such a decomposition.

Now let $\phi = \varphi - \psi$ be one such decomposition, and suppose that $\phi = \widetilde{\varphi} - \widetilde{\psi}$ is another. Pick $\epsilon > 0$ and $A \in \mathscr{A}^+$, and pick $X, Y$ satisfying the conditions of *(2)* of Lemma 4.17 for the first decomposition $\phi = \varphi - \psi$. As we have observed above, $\phi(Y - X) \geq \|\phi\| - 4\epsilon$. Hence $\widetilde{\varphi}(Y) + \widetilde{\psi}(X) - \widetilde{\varphi}(X) - \widetilde{\psi}(Y) \geq \|\widetilde{\varphi}\| + \|\widetilde{\psi}\| - 4\epsilon$. It follows that

$$\widetilde{\varphi}(X), \widetilde{\psi}(Y) \leq 4\epsilon \ .$$

Next, $\varphi(Y) = \phi(Y) + \psi(Y) = \widetilde{\varphi}(Y) - \widetilde{\psi}(Y) + \psi(Y)$ and hence $\widetilde{\varphi}(Y) \geq \|\varphi\| - 5\epsilon$. It follows that $\|\widetilde{\varphi}\| \geq \|\varphi\|$. By symmetry, $\|\widetilde{\varphi}\| = \|\varphi\|$, and hence $\widetilde{\varphi}(Y) \geq \|\widetilde{\varphi}\| - 5\epsilon$. A similar argument applies to $X$ and $\widetilde{\psi}$. Thus, replacing $\epsilon$ by $5\epsilon$, we have that (4.11) is of satisfied with the *same* pair $X, Y$ for both decompositions, and (4.11) is independent of the decomposition. Hence for each $A \in \mathscr{A}^+$, and each $\epsilon > 0$, there is a d single pair $X, Y$ satisfying the conditions in *(2)* of Lemma 4.17. Using this, we show the two decompositions coincide.

By the Cauchy-Schwarz inequality, $|\varphi(AX)| \leq \left(\varphi(X^{1/2}AX^{1/2})\varphi(X)\right)^{1/2} \leq \epsilon$, and likewise for $\widetilde{\varphi}(AX)$, $\psi(AY)$ and $\widetilde{\psi}(AY)$. Hence

$$|\varphi(AX)| \ , |\widetilde{\varphi}(AX)| \ , |\psi(AY)| \ , \ |\widetilde{\psi}(AY)| < \epsilon \ . \tag{4.14}$$

Now for $\epsilon > 0$, let $X, Y$ satisfy the conditions *(2)* of Lemma 4.17 for both decompositions. Then

$$\begin{aligned} |\varphi(AY) - \widetilde{\varphi}(AY)| &\leq |\varphi(A(X+Y)) - \widetilde{\varphi}(A(X+Y))| + |\varphi(AX) - \widetilde{\varphi}(AX)| \\ &\leq |\varphi(A(X+Y)) - \widetilde{\varphi}(A(X+Y))| + 2\epsilon \ . \end{aligned}$$

Since $\|A(X+Y)\| < \epsilon$, $|(\varphi(A(X+Y)) - \widetilde{\varphi}(A(X+Y))) - (\varphi(A) - \widetilde{\varphi}(A))| \leq 2\|\varphi\|\epsilon$ . Therefore, since $\varphi - \widetilde{\varphi} = \psi - \widetilde{\psi}$,

$$\begin{aligned} |\varphi(A) - \widetilde{\varphi}(A))| &\leq 2(1 + \||\varphi\|)\epsilon + |\varphi(AY) - \widetilde{\varphi}(AY)| \\ &= 2(1 + \||\varphi\|)\epsilon + |\widetilde{\psi}(AY) - \psi(AY)| \leq 2(2 + \||\varphi\|)\epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, and since Hermitian linear functional are determined by their values on $\mathscr{A}^+$, $\widetilde{\varphi} = \varphi$. □

For $\varphi \in \mathscr{A}^*$, define $\varphi^*$ by

$$\varphi^*(A) = \overline{\varphi(A^*)} \quad \text{for all} \quad A \ in \mathscr{A} \ . \tag{4.15}$$

Note that $\varphi$ is Hermitian if and only if $\varphi^* = \varphi$. In any case, $\dfrac{1}{2}(\varphi + \varphi^*)$ and $\dfrac{1}{2i}(\varphi - \varphi^*)$ are Hermitian, and

$$\varphi = \frac{1}{2}(\varphi + \varphi^*) + i\frac{1}{2}(\varphi + \varphi^*) \ ,$$

so that every $\varphi \in \mathscr{A}^*$, is a linear combination of two elements of $(\mathscr{A}^*)_{\text{s.a.}}$.

## 4.4 Normal functionals

**4.20 DEFINITION.** Let $\mathscr{M}$ be a von Neumann algebra, and let $\mathscr{M}_*$ be its predual, as described in Theorem 3.35. The second dual of $\mathscr{M}_*$, $(\mathscr{M}_*)^{**}$ is $\mathscr{M}^*$, and we may identify $\mathscr{M}_*$ with a subspace of $\mathscr{M}^*$ using the standard embedding of $\mathscr{M}_*$ into $(\mathscr{M}_*)^{**}$. A linear functional $\phi \in \mathscr{M}^*$ is a *normal functional* if and only if it is in the image of $\mathscr{M}_*$ under this canonical embedding. We use $\mathscr{M}_*$ to denote the set of normal functionals on $\mathscr{M}$.

By Theorem Theorem 3.35, every $\phi \in \mathscr{M}_*$ is of the form $\phi(B) = \text{Tr}[AB]$ for some $A \in \mathscr{T}(\mathscr{H})$. Therefore, a neighborhood base at 0 for the weak-$*$ topology on $\mathscr{M}$ is given by the sets

$$V_{A_1,\dots,A_m,\epsilon} = \{B \in \mathscr{M} \ : \ |\text{Tr}[A_jB]| < \epsilon \ , j = 1\dots, m \ \} \ .$$

Since $\mathscr{B}(\mathscr{H})$ is the dual of $\mathscr{T}(\mathscr{H})$, there same sets are a neighborhood base at 0 for the relative $\sigma$-weak topology on $\mathscr{M}$ considered as a subset of $\mathscr{B}(\mathscr{H})$.

For any Banach space $X$, a linear functional $\phi \in X^{**}$ is continuous with respect to the weak-$*$ topology on $X^*$ if and only if it is in the image of the canonical embedding of $X$ into $X^{**}$. Hence $\phi \in \mathscr{M}^*$ is normal if and only if it is continuous in the weak-$*$ topology on $\mathscr{M}$, and by what we have just observed, this is the case if and only if $\phi$ is continuous on $\mathscr{M}$ equipped with the relative $\sigma$-weak topology.

**4.21 THEOREM.** *Let $\mathscr{M}$ be a von Neumann algebra, and let $\phi \in \mathscr{M}_*$. Then the following are equivalent:*

*(i) $\phi$ normal.*

*(ii) $\phi$ is continuous on $\mathscr{M}$ when $\mathscr{M}$ is equipped with the $\sigma$-weak topology.*

*(iii) for all $\epsilon > 0$, there exists $A \in \mathscr{T}(\mathscr{H})$ such that $\phi(B) = \mathrm{Tr}[AB]$ for all $B \in \mathscr{M}$ and such that $\|A\|_1 \leq \|\phi\|_{\mathscr{M}_*} + \epsilon$.*

*Proof.* The discussion just before the statement of the thoerem proves the equivalence of *(i)* and *(ii)*. The equivalence of *(i)* and *(iii)* is an immediate consequence of Theorem 3.35. □

**4.22 COROLLARY.** *Let $\mathscr{M}$ be a von Neumann algebra. Then $\mathscr{M}_*$ is a norm closed subspace of $\mathscr{M}$. Furthermore, let $\phi$ be a normal functional on a von Neumann algebra $\mathscr{M}$. Then:*

*(i) $\phi^*$ is also normal, and hence the Hermitian and skew-Hermitian parts of $\phi$ are normal.*

*(ii) For all $B \in \mathscr{M}$, define $B\phi$ to be the linear functional $B\phi(X) = \phi(BX)$ for all $X \in \mathscr{M}$, and define $\phi B$ be the linear functional $\phi B(X) = \phi(XB)$ for all $X \in \mathscr{M}$. Then $B\phi$ and $\phi B$ are also normal.*

*(iii) Let $\phi$ be a Hermitian normal functional on $\mathscr{M}$, and let $\phi = \phi_+ - \phi_-$ be its Grothendieck decomposition. then $\phi_+$ and $\phi_-$ are normal.*

*Proof.* Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{M}_*$ converging in norm to $\phi \in \mathscr{M}^*$. since the canonical embedding is isometric, $\{\phi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence sequence in $\mathscr{M}_*$, and $\mathscr{M}_*$ is complete. Hence $\phi \in \mathscr{M}_*$.

Let $\phi$ be normal. By Theorem 4.21, there is $A \in \mathscr{T}(\mathscr{H})$ such that $\phi(X) = \mathrm{Tr}[AX]$ for all $X \in \mathscr{M}$, and conversely, any linear functional of this form is normal.

To prove *(i)*, note that $\phi^*(X) = \overline{\mathrm{Tr}[AX^*]} = \mathrm{Tr}[XA^*] = \mathrm{Tr}[A^*X]$. since $A^* \in \mathscr{T}(\mathscr{B})$, $\phi^* \in \mathscr{M}_*$, and then of course $\frac{1}{2}(\phi + \ \phi^*)$ and $\frac{1}{2i}(\phi - \phi^*)$ belong to $\mathscr{M}_*$.

To prove *(ii)*, note that $B\phi^*(X) = \mathrm{Tr}[BAX]$ and $BA \in \mathscr{T}(\mathscr{H})$. The same reasoning applies to $\phi B$.

To prove *(iii)*, pick $\epsilon > 0$, and then by the final part of Lemma 4.17, since $\mathscr{M}$ is unital, there exist $Y \geq 0$ with $Y \in B_{\mathscr{M}}$ such that $\phi_+(1-Y), \phi_-(Y) < \epsilon$. Then for any $X \in \mathscr{M}$,

$$
\begin{aligned}
|\phi_+(X) - Y\phi(X)| &= |\phi_+((1-Y)X) - \phi_-(YX)| \\
&\leq (\phi_+(1-Y)\phi_+(X^*(1-Y)X))^{1/2} + (\phi_-(Y)\phi_+(X^*YX))^{1/2} \\
&\leq \epsilon(\|\phi_+\| + \|\phi_-\|)\|X\| = \epsilon\|\phi\|\|X\| \ .
\end{aligned}
$$

since $Y\phi$ is normal by part *(ii)*, and since $X$ and $\epsilon > 0$ are arbitrary, $\phi_+$ is in the norm closure of $\mathscr{M}_*$, which is $\mathscr{M}_*$ itself. □

**4.23 THEOREM** (Sakai's Polar Factorization Theorem). *Let $\mathscr{M}$ be a von Neumann algebra, and let $\phi \in \mathscr{M}_*$. Then there exists a unique positive normal functional $|\phi|$ such that $\||\phi|\| = \|\phi\|$ and such that*

$$|\phi(X)|^2 \leq \|\phi\||\phi|(X^*X) \tag{4.16}$$

*for all $X \in \mathscr{M}$. Furthermore, there is a partial isometry $U \in \mathscr{M}$ such that $\phi = U|\phi|$ and $|\phi| = U^*\phi$.*

*Proof.* Let $\|\phi\| = 1$. The set $\{X \in B_{\mathscr{M}} \ \ \phi(X) = 1\}$ is a non-empty, convex, $\sigma$-weakly compact subset of $B_{\mathscr{M}}$. Hence this set contains an extreme point $U^*$, which is also evidently an extreme point of $B_{\mathscr{M}}$. By the Russo-Dye Theorem, $U^*$ is unitary, and hence $U$ is unitary.

Define $|\phi| := U^*\phi$, which is normal by Corollary 4.22, then

$$\||\phi|\| \geq |\phi(1)| = \phi(U^*) = 1 \geq \||\phi\| \ ,$$

and therefore $|\phi| \geq 0$.

By the definition of $|\phi|$, $\phi = U|\phi|$, and hence for all $X$, $\phi(X) = |\phi(UX)$. now (4.16) follows from the Cauchy-Schwarz inequality.

It remains to prove the uniqueness. Suppose $\psi$ is another positive normal functional with $\|\psi\| = 1$ and $|\psi(X)|^2 \leq \psi(X^*X)$ for all $X$. Then for $X$ self adjoint,

$$(|\phi|(X)|)^2 = |\phi(U^*X)|^2 \leq \psi(XUU^*X) = \psi(X^2) \ .$$

Replacing $X$ by $1+\epsilon X$, this yields $(|\phi|(1+\epsilon X))^2 \leq \psi((1+\epsilon X)^2)$, and since $|\phi|(1) = \psi(1) = 1$, and since $\epsilon > 0$ is arbitrary, this yields. $|\phi|(X) \leq \psi(X)$ for all self adjoint $X$. Replacing $X$ with $-X$, we get that $|\psi| = \psi$. □

## 4.5 The GNS construction, Sherman's Theorem, and some applications

A construction due to Gelfand, Neumark and Segal, known as the GNS construction, associates to every state $\varphi$ on an $C^*$ algebra $\mathscr{A}$ a representation $\pi$ of $\mathscr{A}$ on a Hilbert space built out of $\mathscr{A}$ itself and the state $\varphi$.

**4.24 THEOREM** (The GNS construction). *Let $\mathscr{A}$ be a unital $C^*$ algebra with identity 1, and let $\varphi$ be a state on $\mathscr{A}$. Then there exists a Hilbert space $\mathscr{H}$ and a cyclic representation $\pi$ of $\mathscr{A}$ on $\mathscr{H}$ with a distinguished cyclic unit vector $\eta$ such that for all $A \in \mathscr{A}$,*

$$\varphi(A) = \langle \eta, \pi(A)\eta\rangle_{\mathscr{H}} \ . \tag{4.17}$$

*The representation $\pi$ is irreducible if and only if $\varphi$ is a pure state.*

*Proof.* Let $\langle A, B\rangle_\varphi$ be the possibly degenerate inner product on $\mathscr{A}$ defined by $\langle A, B\rangle_\varphi = \varphi(A^*B)$. Define

$$\mathscr{N} := \{ \ a \in \mathscr{A} \ : \ \langle A, A\rangle_\varphi = 0 \ \} \ .$$

Since $\varphi$ is continuous, $\mathscr{N}$ is closed. In fact, $\mathscr{N}$ is a closed left ideal. To see this, consider $B \in \mathscr{A}$ and $A \in \mathscr{N}$. Then

$$\langle BA, BA\rangle_\varphi = \varphi(A^*B^*BA) = \langle A, B^*BA\rangle_\varphi \leq \langle A, A\rangle_\varphi^{1/2}\langle B^*BA, B^*BA\rangle_\varphi^{1/2} = 0 \ .$$

A similar but simpler argument shows that $\mathscr{N}$ is a subspace.

Now consider the vector space $\mathscr{A}/\mathscr{N}$. With $\sim$ denoting equivalence mod $\mathscr{N}$, we have

$$A \sim A' \quad \text{and} \quad B \sim B' \quad \Rightarrow \quad \langle A, B\rangle_\varphi = \langle A', A'\rangle_\varphi \ ,$$

and hence we may define a *non-degenerate* inner product on $\mathscr{A}/\mathscr{N}$ by $\langle \{A\}, \{B\}\rangle = \langle A, B\rangle_\varphi$. Let $\mathscr{H}$ be the completion of $\mathscr{A}/\mathscr{N}$ in the corresponding Hilbertian norm, and let $\langle \cdot, \cdot\rangle_{\mathscr{H}}$ denote the resulting inner product on $\mathscr{H}$.

For $A \in \mathscr{A}$, let $\pi(A)$ denote the linear operator on $\mathscr{A}/\mathscr{N}$ defined by $\pi(A)\{B\} = \{AB\}$ which is well-defined since $\mathscr{N}$ is a left ideal. Next note that since $B^*A^*AB = \|A\|^2 B^*B + B^*(\|A^*A\|1 - A^*A)B$, and $B^*(\|A^*A\|1 - A^*A)B$ is positive,

$$\|\pi(A)\{B\}\|^2_{\mathscr{H}} = \varphi(B^*A^*AB) \leq \|A\|^2\varphi(B^*B) = \|A\|^2\|\{B\}\|^2_{\mathscr{H}} \ .$$

Since $\mathscr{A}/\mathscr{N}$ is dense in $\mathscr{H}$, $\pi(A)$ extends to a bounded operator on $\mathscr{H}$ with $\|\pi(A)\| \leq \|A\|$. It is evident that $\pi$ is a homomorphism of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$, and note that for all $X, Y \in \mathscr{A}$,

$$\langle \{X\}, \pi(A)\{Y\}\rangle_{\mathscr{H}} = \varphi(X^*AY) = \varphi((A^*X)^*Y = \langle \pi(A)\{X\}, \{Y\}\rangle_{\mathscr{H}} \ ,$$

showing that $\pi(A^*) = \pi(A)^*$, and thus $\pi$ is a $*$-homomorphism.

The representation $\pi$ is cyclic since for all $A \in \mathscr{A}$, $\{A\} = \{A1\} = \pi(A)\{1\}$, showing that $\eta := \{1\}$ is a cyclic vector for $\pi$. Finally, note that $\langle \eta, \pi(A)\eta\rangle_{\mathscr{H}} = \varphi(1^*A1) = \varphi(A)$, and this proves (4.17). The final statement now follows from Theorem 4.12. □

**4.25 COROLLARY.** *Let $\mathscr{A}$ be a unital $C^*$ algebra. For every non-zero $A \in \mathscr{A}$, there is a representation $\pi$ of $\mathscr{A}$ such that $\|\pi(A)\| = \|A\|$.*

*Proof.* By Lemma 4.8, there exists $\varphi \in S_{\mathscr{A}}$ such that $|\varphi(A^*A)| = \|A\|^2$. Let $\pi$ be the GNS representation of $\mathscr{A}$ associated to $\varphi$, and $\eta$ the associated distinguished cyclic unit vector. Then

$$\|\pi(a)\eta\|^2_{\mathscr{H}} = \langle \eta\pi(A^*A)\eta\rangle_{\mathscr{H}} = \varphi(A^*A) = \|A\|^2 \ ,$$

showing that $\|\pi(A)\| \geq \|A\|$, and since it is automatic that $\|\pi(A)\| \leq \|A\|$, $\|\pi(A)\| = \|A\|$. □

We now arrive at the Non-Commutative Gelfand-Neumark Theorem:

**4.26 THEOREM** (Non-Commutative Gelfand-Neumark Theorem)**.** *Every $C^*$ algebra $\mathscr{A}$ with an identity is isometrically $*$-isomorphic to a $C^*$ algebra of operators.*

*Proof.* For each $\varphi \in S_{\mathscr{A}}$, let $\pi_\varphi$ be the representation of $A$ on $\mathscr{B}(\mathscr{H}_\varphi)$ provided by the GNS construction. Define $\mathscr{H} = \bigoplus_{\varphi\in S_{\mathscr{A}}} \mathscr{H}_\varphi$, and define $\pi = \bigoplus_{\varphi\in S_{\mathscr{A}}} \pi_\varphi$. For each $A \in \mathscr{A}$, there exists $\varphi \in S_{\mathscr{A}}$ such that $\|\pi_\varphi(A)\| = \|A\|$, and hence $\pi(A)\| = \|A\|$. □

Theorem 4.26 allows us to estimate norms in a $C^*$ algebra using vectors in a Hilbert space. The next lemma illustrates this.

**4.27 LEMMA.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $A, B \in \mathscr{A}_{\text{s.a.}}$, Then $\|A + iB\| \geq \|A\|$*

*Proof.* By Theorem 4.26, we may assume that $\mathscr{A}$ is a $C^*$ subalgebra of $\mathscr{B}(\mathscr{H})$. Note that Either $\|A\|$ or $-\|A\|$ belongs to $\sigma(A)$; replacing $A$ by $-A$, we may assume $\|A\| \in \sigma(A)$. Then $(\|A\|1 - A)^2$ is a positive operator on $\mathscr{H}$ with zero in its spectrum. Hence for all $\epsilon > 0$, there exist a unit vector $\xi \in \mathscr{H}$ such that $\|(A - \|A\|1)\xi\|^2 = \langle \xi, (\|A\|1 - A)^2\xi\rangle < \epsilon^2$ since otherwise $(\|A\|1 - A)^2$ would be invertible. Let $\xi$ be such a unit vector. Then since $0 \leq \langle \xi(A - \langle \xi, A\xi\rangle)^2\xi\rangle = \langle \xi, A^2\xi\rangle - \langle \xi, A\xi\rangle^2$, for all $\epsilon < \|A\|$,

$$\langle \xi A^2\xi\rangle \geq \langle \xi, A\xi\rangle^2 = (\|A\| + \langle \xi, (A - 1\|A\|\xi)\rangle)^2 \geq (\|A\| - \epsilon)^2 \ . \tag{4.18}$$

Note that $|\langle A\xi, B\xi\rangle - \|A\|\langle \xi, B\xi\rangle| \leq |\langle (A - \|A\|1)\xi, B\xi\rangle \leq \epsilon\|B\|$, and hence $|\langle B\xi, A\xi\rangle - \|A\|\langle \xi, B\xi\rangle| \leq \epsilon\|B\|$. Using this with (4.18),

$$\begin{aligned} \|(A + iB)\|^2 \leq \|(A + iB)\xi\|^2 &= \langle \xi(A^2 + B^2)\xi\rangle - i(\langle A\xi, B\xi\rangle - \langle B\xi, A\xi\rangle) \\ &\geq (\|A\| - \epsilon)^2 - 2\epsilon\|B\| \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. □

As an application of Lemma 4.27 we first prove a Lemma of Kadison [11]. In the rest of this section, it is more the proof of Kadison's Lemma than the statement that will be of use to us. The proof is a variant of the "Ahren's Trick"; see [13, p. 24] for more discussion.

We first define some classes of maps between $C^*$ algebras with which we shall be concerned in what follows, one of which figures in Kadison's Lemma.

**4.28 DEFINITION.** Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$-algebras. A linear transformation $\Phi : \mathscr{A} \to \mathscr{B}$ is *positive* in case $\Phi(A) \geq 0$ whenever $A \geq 0$. When $\mathscr{A}$ and $\mathscr{B}$ are unital then $\Phi$ is unital in case $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$. $\Phi$ is Hermitian in case $\Phi(A)^* = \Phi(A^*)$ for all $A \in \mathscr{A}$. Finally, suppose that $\mathscr{B}$ is a $C^*$ subalgebra of $\mathscr{A}$. Then $\Phi$ is a *projection* in case $\Phi(B) = B$ for all $B \in \mathscr{B}$.

**4.29 LEMMA** (Kadison's Lemma)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be $C^*$ algebras. Let $\Psi : \mathscr{A} \to \mathscr{B}$ be a unital linear transformation such that $\|\Psi(X)\| \leq \|X\|$ when $X$ is normal. Then $\Psi$ is Hermitian.*

*Proof.* Let $A \in \mathscr{A}$, $A = A^*$, $\|A\| = 1$. Write $\Psi(A) = B + iC$, $B, C \in \mathscr{B}_{\text{s.a}}$. If $C \neq 0$, there is some $\lambda \in \mathbb{R}\backslash\{0\}$ in $\sigma(C)$. Replacing $A$ by $-a$ if needed, we may suppose that $\lambda > 0$. For all $t > 0$,

$$\|A + it1\| \leq (1 + t^2)^{1/2} = t(1 + t^{-2})^{1/2} \leq t + (2t)^{-1} \ ,$$

and hence $\|A + it1\| \leq t + \lambda$ for $t \in (0, (2\lambda)^{-1})$. Since $\lambda \in \sigma(C)$, $t + \lambda \leq \|C + t\| \leq \|B + iC + it\|$, where we used Lemma 4.27 in the lest step. But $\|B + iC + it\| = \Psi(A + it1)\| = \|A + it\|$ since $A + it1$ is normal. Altogether we have $\|A + it1\| < \|A + it1\|$, and this contradiction proves the claim. □

With $\pi = \bigoplus_{\varphi \in S_{\mathscr{A}}} \pi_\varphi$, by the von Neumann Double Commutant Theorem, $(\pi(\mathscr{A}))''$ is a von Neumann algebra in which $\pi(\mathscr{A})$ is strongly dense. This von Neumann algebra, known as the *enveloping von Neumann algebra*, turns out to be the bidual of $\mathscr{A}$ considered as a Banach space. This fact is extremely useful for studying linear maps $\Phi$ from one $C^*$ algebra $\mathscr{A}$ to another, $\mathscr{B}$. The bidual $\Phi^{**} : \mathscr{A}^{**} \to \mathscr{B}^{**}$ has the same norm as $\Phi$, Moreover, we can identify $\mathscr{A}$ with a subspace of $\mathscr{A}^{**}$ using the canonical embedding, which identifies $A \in \mathscr{A}$ with $L_A$, the evaluation functional $L_A(\psi) = \psi(A)$ on $\mathscr{A}^*$. Then for all $\psi \in \mathscr{B}*$,

$$\Phi^{**}(L_A)(\psi) = L_A(\Phi^*\psi) = (\Phi^*\psi)(A) = \psi(\Phi(A)) = L_{\Phi(A)}(\psi)\ .$$

That is, identifying $\mathscr{A}$ with its image under the canonical embedding, $\Phi^{**}|_{\mathscr{A}} = \Phi$. Once we know that the bidual of a $C^*$ algebra is isometrically isomorphic to to its enveloping von Neumann algebra, we have a norm preserving extension of $\Phi : \mathscr{A} \to \mathscr{B}$ to a map, that we still denote by $\Phi^{**}$, from $(\pi(\mathscr{A}))''$ to $(\pi(cB))''$. In the investigation of $\Phi^{**}$, and hence $\Phi$, we may then use the rich structure that come along with von Neumann algebras; e.g., the measurable functional calculus and the consequent fact that von Neumann algebras are generated by the projections that they contain. Hence the following theorem is fundamentally important.

**4.30 THEOREM** (Sherman's Theorem)**.** *Let $\mathscr{A}$ be a $C^*$ algebra, and let $\mathscr{H} = \bigoplus_{\varphi \in S_{\mathscr{A}}} \mathscr{H}_\varphi$, and $\pi = \bigoplus_{\varphi \in S_{\mathscr{A}}} \pi_\varphi$ be given as in Theorem 4.26. Let $\mathscr{M} := (\pi(A))''$. Let $A^{**}$ be the bidual of $\mathscr{A}$ considered as a Banach space. Then $A^{**}$ is isometrically isomorphic to $\mathscr{M}$.*

*Proof.* We have seen that every element of $\mathscr{A}^*$ is a linear combination of at most 4 states on $\mathscr{A}$. For $\varphi \in S_{\mathscr{A}}$, let $\xi_\varphi$ be the unit vector in $\mathscr{H}_\varphi$ such that for all $A \in \mathscr{A}$, $\varphi(A) = \langle \xi_\varphi, \pi(A)\xi_\varphi\rangle$; we identify $\xi_\varphi$ with a vector in $\mathscr{H}$ in the obvious manner. Note that $\|\phi\| = \sup_{A\in B_{\mathscr{A}}}\{|\phi(A)\} = \sup_{A\in B_{\mathscr{A}}}\{|\widehat{\phi}(\pi(A))\}$, and by Kaplansky's Density Theorem, $\pi(B_{\mathscr{A}}) = B_{\pi(\mathscr{A})}$ is dense in $B_{\mathscr{M}}$ in the strong, and hence weak, topology on $B_{\mathscr{M}}$, which is the same as the $\sigma$-weak topology on $B_{\mathscr{M}}$. Hence

$$\|\phi\| = \sup_{X\in B_{\mathscr{M}}}\{|\widehat{\phi}(\pi(A))\} = \widehat{\phi}\ .$$

Therefore, $\phi \mapsto \widehat{\phi}$ is an isometric isomorphism from $\mathscr{A}^*$ into $\mathscr{M}_*$, and it is surjective since for any $\varphi \in \mathscr{M}_*$, $\varphi|_{\mathscr{A}} \in \mathscr{A}^*$, and the extension of $\varphi|_{\mathscr{A}} \in \mathscr{A}^*$ to $\mathscr{M}$ is $\varphi$, again using the weak density of $\pi(\mathscr{A})$ in $\mathscr{M}$, and the weak continuity of $\varphi|_{\mathscr{A}}$. Then $\mathscr{A}^{**}$ is isometrically isomorphic to $(\mathscr{M}_*)^* = \mathscr{M}$. □

Going forward, it will be useful to identify $\mathscr{A}$ with $\pi(\mathscr{A})$, and to write $\mathscr{A}''$ to denote the universal enveloping von Neumann algebra of $\mathscr{A}$.

**4.31 COROLLARY.** *Let $\mathscr{A}$ and $\mathscr{B}$ be two $C^*$ algebras, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a bounded linear transformation. Then there is a norm preserving extension of $\Phi$, denoted by $\Phi^{**}$, that maps $\mathscr{A}''$ to $\mathscr{B}''$.*

*Proof.* The proof has already been given in the remarks preceding the statement of Sherman's Theorem. □

We now come to an important application of this corollary.

**4.32 THEOREM** (Tomiyama's Theorem)**.** *Ler $\mathscr{A}$ be a unital $C^*$ algebra, and let $\mathscr{B}$ be a unital $C^*$ subalgebra of $\mathscr{A}$. Let $\Phi$ be a norm 1 projection from $\mathscr{A}$ to $\mathscr{B}$. Then $\Phi$ is unital and positive, and*

$$\Phi(B_1AB_2) = B_1\Phi(A)B_2 \quad \text{for all } B_1, B_2 \in \mathscr{B}\ ,\ A\ in \mathscr{A} \tag{4.19}$$

*and for all $A \in \mathscr{A}$,*

$$\Phi(A)^*\Phi(A) \le \Phi(A^*A) \tag{4.20}$$

*Proof of Tomiyama's Theorem.* By Sherman's Theorem and its Corollary, $\Phi^{**}$ may be regarded as a norm one linear map from $\mathscr{A}''$ to $\mathscr{B}''$, whose restriction fo $\mathscr{B}$ is the identity, since $\Phi^{**}|_{\mathscr{A}} = \Phi$, and $\Phi$ is a projection onto $\mathscr{B}$. More is true: $\Phi^{**}(B) = B$ for all $B \in \mathscr{B}''$. To see this, note that if $\Lambda \in \mathscr{B}^{**} \subset \mathscr{A}^{**}$, and $\psi \in B^*$, then $(\Phi^{**}\Lambda)(\psi) = \Lambda(\Phi^*(\psi)) = \Lambda(\psi \circ \Phi) = \Lambda(\psi)$. Hence $\Phi^{**}$ is the identity on $B^{**}$, and thus $\Phi^{**}$ is a norm one projection of $\mathscr{A}''$ onto $\mathscr{B}''$. Therefore, it suffices to prove the theorem when $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras; we now assume this is the case.

First, since $\Phi$ maps $\mathscr{A}$ onto $\mathscr{B}$ and $1 \in \mathscr{B}$, there is some $A \in \mathscr{A}$ with $1 = \Phi(A)$. Hence $\Phi(1) = \Phi^2(A) = \Phi(A) = 1$. Hence $\Phi$ is unital.

We next show that $\Phi$ is Hermitian. Arguing as in Kadison's Lemma, let $A \in \mathscr{A}_{\text{s.a.}}$ and write $\Phi(A) = B + iC$ where $A$ and $B$ are self adjoint, and suppose that some $\lambda > 0$ belongs to the spectrum of $C$. Then for $t > 0$, since $\Phi(1) = 1$,

$$t + \lambda \le \|C + t1\| \le \|B + iC + it1\| = \|\Phi(A + it1)\| \le \|A + it1\| \le (t^2 + \|A\|^2)^{1/2} \le t + \frac{\|A\|^2}{2t} \ .$$

This is impossible for large $t$. Hence $C$ has no positive spectrum. Repeating the argument with $-A$ in place of $A$, we conclude that $C$ has no negative spectrum, and hence $C = 0$.

The next step is to show that $\Phi$ is positive; here Tomiyama cites the argument argument of Kadison [11]. Suppose that $A \in \mathscr{A}^+$, but that $\Phi(A)$ has spectrum in $(-\infty, 0)$. We may assume without loss of generality that $0 \le A \le 1$. Then $\|1 - A\| \le 1$, but $\Phi(1 - A) = 1 - \Phi(A)$ is self adjoint and has spectrum in $(1, \infty)$, so that $\|\Phi(1 - A)\| > 1$. This contradiction proves that $\Phi$ is positive.

Now let $0 \le A \le 1$, $A \in \mathscr{A}$, and let $P$ be a projection in $\mathscr{B}$, Then $P \ge PAP$ and hence $P \ge \Phi(PAP)$. It follows that $P\Phi(PAP)P = \Phi(PAP)$. Therefore, for all $X \in \mathscr{A}$,

$$P\Phi(PXP)P = \Phi(PXP) \ . \tag{4.21}$$

Now pick $X \in \mathscr{A}$, $\|X\| \le 1$, and define $Y := PX(1 - P)$. Our immediate goal is to show that

$$P\Phi(Y)P = 0 \ . \tag{4.22}$$

First note that for all $t \in \mathbb{R}$,

$$\|Y + tP\| = \|(Y^* + tP)(Y + tP)\|^{1/2} = \|(1 - P)X^*PX(1 - P) + t^2P\|^{1/2} \le (1 + t^2)^{1/2} \ .$$

Therefore, $\|\Phi Y + tP\| \le (1 + t^2)^{1/2}$ for all $t \in \mathbb{R}$.

However,

$$\|\Phi Y + tP\| \ge \|P(\Phi Y)P + tP\| \ge \left\| \frac{P(\Phi Y)P + P(\Phi Y)^*P}{2} + tP \right\| \ .$$

Let $\lambda$ belong to the spectrum of $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P)$. If $\lambda > 0$, then for $t > 0$, $\|\Phi Y + tP\| \ge \lambda + t$. If $\lambda < 0$, and $t < 0$, $\|\Phi Y + tP\| \ge -\lambda - t$. Either way, there is a contradiction for suitable values of $t$. Hence $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P) = 0$. The same reasoning shows that $\frac{1}{2i}(P(\Phi Y)P - P(\Phi Y)^*P) = 0$, and thus (4.22) is proved.

A similar argument applied to $\|Y + t(1 - P)\|$ proves that

$$(1 - P)(\Phi Y)(1 - P) = 0 \ . \tag{4.23}$$

At this point we have $\Phi Y = (1-P)\Phi Y P + P\Phi Y(1-P)$. Therefore, for $t > 0$,

$$\Phi Y + t(1-P)\Phi Y P = P\Phi Y(1-P) + (t+1)(1-P)\Phi Y P \ .$$

Since the norm of block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is given by $\max\{\|A\|, \|B\|\}$, for all sufficiently large $t$,

$$\|\Phi Y + t(1-P)\Phi Y P\| = (t+1)\|(1-P)\Phi Y P\| \ .$$

However, since $\Phi$ is a contractive projection,

$$\begin{aligned} \|\Phi Y + t(1-P)\Phi Y P\| &\leq \|Y + t(1-P)\Phi Y P\| \\ &= \|PX(1-P) + t(1-P)\Phi Y P\| \\ &= \max\{\|PX(1-P)\|, t\|(1-P)\Phi Y P\|\} \ . \end{aligned}$$

For large $t$, $(t+1)\|(1-P)\Phi Y P\| \leq t\|(1-P)\Phi Y P\|$, and hence $\|(1-P)\Phi Y P\| = 0$.

At this point we have $(1-P)\Phi Y P = 0$, $(1-P)(\Phi Y)(1-P) = 0$ and $P(\Phi Y)P = 0$. It follows that

$$\Phi Y = P\Phi Y(1-P) \ . \tag{4.24}$$

Now define $Z := (1-P)XP$, which is just like the definition of $Y$, except with $P$ and $1-P$ interchanged. By Symmetry, we have

$$\Phi Z = (1-P)\Phi Z P \ . \tag{4.25}$$

Now for any $X \in \mathscr{A}$, we have, using (4.21) once with $P$, and once with $(1-P)$ in place of $P$,

$$\begin{aligned} \Phi X &= \Phi(PXP + PX(1-P) + (1-P)XP + (1-P)X(1-P)) \\ &= P\Phi(PXP)P + \Phi Y + \Phi Z + (1-P)\Phi((1-P)X(1-P))(1-P) \end{aligned}$$

Therefore, by (4.24) and (4.25) $(1-P)\Phi X P = (1-P)\Phi Z P = \Phi Z = \Phi((1-P)XP)$ and then, also using (4.21), $P\Phi X P = P\Phi(PXP)P = \Phi(PXP)$. Summing, $(\Phi X)P = \Phi(XP)$. Since a von Neumann algebra is generated by the projection it contains, for all $A \in \mathscr{B}$, $\Phi(X)A = \Phi(XA)$, and taking adjoints, since $\Phi$ preserves self-adjointness, $A\Phi X = \Phi(AX)$. This proves (4.19).

Next, since $(X - \Phi X)^*(X - \Phi X) \geq 0$ and since $\Phi$ preserves positivity, using (4.19) twice,

$$0 \leq \Phi(X^*X - X^*\Phi X - \Phi X^* X + (\Phi X)^*\Phi X) = \Phi(X^*X) - (\Phi X)^*\Phi X \ ,$$

and this proves (4.20) □

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