Notes on Uniform Convexity and Uniform Smoothness

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1 Introduction

The unit ball of any normed vector space V is convex, though it need not be strictly convex, which would mean that for all unit vectors u and v in V,

$$\|u - v\| > 0 \to \|(u+v)/2\| < 1\ . \tag{1.1}$$

Indeed, strict convexity fails for $L^1(M, \mathcal{M}, \mu)$ and $L^\infty(M, \mathcal{M}\mu)$, even for a two-point measure space.

To see this in $L^1(M, \mathcal{M}, \mu)$, take any two *non-negative* unit vectors $u(x)$ and $v(x)$. Then of course

$$\left\|\frac{u+v}{2}\right\| = \frac{1}{2}\int_M (u(x) + v(x))\mathrm{d}\mu = \frac{1}{2}\int_M u(x)\mathrm{d}\mu + \frac{1}{2}\int_M v(x)\mathrm{d}\mu = 1\ .$$

To see this in $L^\infty(M, \mathcal{M}, \mu)$, take any two $u(x)$ and $v(x)$ to be the indicator functions of two measurable sets with different, non-zero measure. Then $u(x)$ and $v(x)$ are both unit vectors in $L^\infty(M, \mathcal{M}, \mu)$, as is their average, $(u+v)/2$.

In some normed spaces however, a *uniform* version of strict convexity holds, and this has significant consequences.

1.1 DEFINITION (Uniform convexity)**.** Let V be a vector space with norm $\|\cdot\|$. The *modulus of convexity* of V is the function δ_V defined by

$$\delta_V(\epsilon) = \inf\left\{1 - \left\|\frac{v+w}{2}\right\|\ :\ \|v - w\| < 2\epsilon\ \right\} \tag{1.2}$$

for $0 \le \epsilon \le 1$. We say that V is *uniformly convex* in case $\delta_V(\epsilon) > 0$ for all $0 < \epsilon < 1$.

If there is no ambiguity as to which space V is under consideration, we just write $\delta(\epsilon)$ in place of $\delta_V(\epsilon)$. Then, by definition, in case V is unofrmly convex, for all $0 < \epsilon < 1$ there is a $\delta(\epsilon) > 0$ so that for all *unit vectors* v,w,

$$\|v - w\| > 2\epsilon \Rightarrow \left\|\frac{v+w}{2}\right\| < 1 - \delta(\epsilon)\ , \tag{1.3}$$

which is indeed a uniform version of (1.1). The logically equivalent implication

$$\left\|\frac{v+w}{2}\right\| \geq 1-\delta(\epsilon) \Rightarrow \|v-w\| \leq 2\epsilon \tag{1.4}$$

will be used frequently in what follows.

By what we have seen just above, neither $L^1(M,\mathcal{M},\mu)$ nor $L^\infty(M,\mathcal{M},\mu)$ is uniformly convex. It turns out, however, that for $1 < p < \infty$, $L^p(M,\mathcal{M},\mu)$ *is* uniformly convex. This is easy to show for $L^2(M,\mathcal{M},\mu)$, and we begin with that:

For any f and g in we have the *parallelogram identity*

$$\|f-g\|_2^2 + \|f+g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2 \ .$$

Now take f and g to be unit vectors, so that $\|f\|_2 = \|g\|_2 = 1$. Divide through by 4, and we get

$$\left\|\frac{f+g}{2}\right\|_2^2 + \left\|\frac{f-g}{2}\right\|_2^2 = \frac{\|f\|_2^2 + \|g\|_2^2}{2} = 1 \ .$$

Therefore,

$$\left\|\frac{f+g}{2}\right\|_2 = \sqrt{1 - \left\|\frac{f-g}{2}\right\|_2^2} \ .$$

For any number a with $0 < a < 1$, $\sqrt{1-a} < 1 - a/2$, and hence,

$$\left\|\frac{f+g}{2}\right\|_2 \leq 1 - \frac{1}{2}\left\|\frac{f-g}{2}\right\|_2^2 \ .$$

Since for very small values of a, $\sqrt{1-a} <\approx 1 - a/2$, and so this computation gives us the exact modulus of convexity for $L^2(M,\mathcal{M},\mu)$, namely

$$\delta_{L^2}(\epsilon) = \frac{1}{2}\epsilon^2 \ . \tag{1.5}$$

1.1 Applications of uniform convexity to convergence questions

1.2 THEOREM (Convergence of norms plus weak convergence yields strong convergence)**.** *Let V be a uniformly convex normed space. Let $\{f_n\}$ be a weakly convergent sequence in V with $1 < p < \infty$, and let f be its limit. Then $\{f_n\}$ is strongly convergent if and only if $\|f\| = \lim_{n\to\infty}\|f_n\|$.*

Proof: If $\{f_n\}$ is strongly convergent, it must converge strongly to f, and we have already observed that it must be the case that $\|f\| = \lim_{n\to\infty}\|f_n\|$.

The converse is more subtle, and it is here that uniform convexity comes in. If $f = 0$, the strong covergence is obvious, so we may suppose that this is not the case. Then, dividing through by $\|f\|$, we may assume that $\|f\| = 1$. Since $\lim_{n\to\infty}\|f_n\| = \|f\| = 1$, we may delete a finite number of terms from the sequence to arrange that $\|f_n\| \neq 0$ so any n.

Consider the sequence $\{g_n\}$ where

$$g_n = \frac{f_n/\|f_n\| + f}{2} \ .$$

Clearly, since $\lim_{n\to\infty} \|f_n\| = \|f\| = 1$, $\{g_n\}$ also converges weakly to f.

By the weak lower semicontinuity of the norms,

$$\liminf_{n\to\infty} \|g_n\| \geq \|f\| = 1 \ . \tag{1.6}$$

But by Minkowskii's inequality,

$$1 = \frac{\|f_n\|/\|f_n\| + \|f\|}{2} \geq \|g_n\| \ , \tag{1.7}$$

and then combining (1.6) and (1.7),

$$\lim_{n\to\infty} \|g_n\| = 1 \ . \tag{1.8}$$

Now use the uniform convexity, and in particular (1.4):

$$\|g_n\| = \left\| \frac{\|f_n\|/\|f_n\| + \|f\|}{2} \right\| \geq 1 - \delta_V(\epsilon) \Rightarrow \|f_n/\|f_n\| - f\| \leq 2\epsilon \ .$$

This together with (1.8) shows that

$$\lim_{n\to\infty} \|f_n/\|f_n\| - f\| = 0 \ .$$

But

$$\|f_n - f\| = \|(f_n - f_n/\|f_n\|) + (f_n/\|f_n\| - f)\| \leq \frac{|\|f_n\| - 1|}{\|f_n\|} + \|f_n/\|f_n\| - f\| \ .$$

Hence it follows that $\lim_{n\to\infty} \|f_n - f\| = 0$. ■

Our next application is very important: It is the generalization of the Projection Lemma to general uniformly convex spaces.

1.3 THEOREM (Projection Lemma for uniformly convex spaces)**.** *Let V be a uniformly convex Banach space, and let K be a closed convex set in V. Then there exists a unique element of minimal norm in K. That is, there exists an element $v \in K$ with*

$$\|v\| < \|w\|$$

for all $w \in K$ with $w \neq v$.

Proof: Let $D = \inf\{\|w\| \mid w \in K\}$. For each positve integer n, choose $v_n \in K$ with

$$\|v_n\| \leq D + \frac{1}{n} \ .$$

Then

$$\lim_{n\to\infty} \|v_n\| = D \ ,$$

and so if $D = 0$, $\lim_{n\to\infty} v_n = 0$. Since K is closed, this means $0 \in K$, and this is our unique element of minimal norm. Therefore, assume that $D > 0$.

Normalize the v_n to obtain unit vectors, as needed for the application of uniform convexity. Let $u_n = v_n/\|v_n\|$. For large n, we have that $u_n \approx v_n/D$ since $\lim_{k\to\infty} \|v_k\| = D$. Indeed, adding and subtracting,

$$u_n = \frac{1}{D}v_n + \frac{D - \|v_n\|}{D\|v_n\|}v_n \ .$$

Therefore, for any m and n,

$$\begin{aligned}
\frac{1}{D}\left\|\frac{v_n + v_m}{2}\right\| &= \left\|\frac{u_n + u_m}{2} - \frac{D - \|v_n\|}{2D\|v_n\|}v_n - \frac{D - \|v_m\|}{2D\|v_m\|}v_m\right\| \\
&\leq \left\|\frac{u_n + u_m}{2}\right\| + \frac{(\|v_n\| - D) + (\|v_m\| - D)}{2} \ .
\end{aligned}$$

Now by the convexity of K, $(v_n + v_m)/2 \in K$ and hence $\|(v_n + v_m)/2\| \geq D$. Therefore,

$$\left\|\frac{u_n + u_m}{2}\right\| \geq 1 - \frac{(\|v_n\| - D) + (\|v_m\| - D)}{2} \geq \left(1 - \frac{1}{2n} - \frac{1}{2m}\right) \ .$$

Then by (1.4), for all $\epsilon > 0$,

$$\left(1 - \frac{1}{2n} - \frac{1}{2m}\right) \geq 1 - \delta(\epsilon) \Rightarrow \|u_n - u_m\| \leq 2\epsilon \ .$$

Thus, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since V is complete, $\{u_n\}_{n\in\mathbb{N}}$ converges in norm to $u \in V$, and this implies that $\{v_n\}_{n\in\mathbb{N}}$ converges in norm to $v := Du$. Since K is closed, $Du \in K$, and since $\|u\| = 1$, $\|v\| = \|Du\| = D$. This proves the existence of an element v of K with minimal norm.

To prove the uniqueness, let $\tilde{v}$ also be in K with $\|\tilde{v}\| = D$. Then if $\|v - \tilde{v}\| > 2D\epsilon$ for some $\epsilon > 0$, (1.3) yields

$$\left\|\frac{v + \tilde{v}}{2}\right\| = D\left\|\frac{v/D + \tilde{v}/D}{2}\right\| \leq D\left(1 - \delta(\epsilon)\right) < D \ .$$

Since $(v + \tilde{v})/2 \in K$, this would contradict the definition of D. Hence $\tilde{v} = v$, and the uniqueness is proved. ■

1.2 Applications of uniform convexity to unit normal vectors

Recall that for any normed space V, and any $v \in V$, there exists an $f \in V^*$ with $\|f\|_* = 1$ and $f(v) = \|v\|$. This is a consequence of the Hahn-Banach Theorem. However, given $f \in V^*$, there may or may not be any unit vector u in V such that $f(u) = \|f\|_*$, as we have seen in the case of $V = \mathcal{C}([0,1])$ with the uniform norm. If V is uniformly convex, things are much better.

1.4 THEOREM (Uniform Convexity and Unit Normal Vectors)**.** *Let V be a uniformly convex Banach space, and let f be any non–zero linear functional in V^*. Then there is a unique unit vector $v_f \in V$ so that*

$$f(v_f) = \|f\|_* \ .$$

Moreover, the function $f \mapsto v_f$ from V^* to V is continuous in the norm topologies at all $f \neq 0$.

The vector whose existence is asserted by the theorem is called the *unit normal vector at f* for reasons that will soon be explained.

Proof: Let K be given by

$$K = \{\ v \in V\ :\ f(v) = \|f\|_*\ \}\ .$$

K is closed, convex and non–empty. By the projection lemma, K contains a unique element v of minimal norm. Note that

$$\|f\|_* = f(v) \le \|f\|_* \|v\|$$

so $\|v\| \ge 1$.

Now we prove an upper bound on $\|v\|$. For any ϵ with $0 < \epsilon < \|f\|_*$, there is a unit vector w with $|f(w)| \ge \|f\|_* - \epsilon$. Multiplying w by a complex number of unit magnitude, we can assume that $f(w) = |f(w)|$. Now let

$$v = \frac{\|f\|_*}{f(w)} w\ .$$

Then $v \in K$, and since w is a unit vector,

$$\|v\| = \frac{\|f\|_*}{f(w)} \le \frac{\|f\|_*}{\|f\|_* - \epsilon}\ .$$

Since v is the element of K with minimal norm, we have

$$1 \le \|v\| \le \frac{\|f\|_*}{\|f\|_* - \epsilon}$$

for all ϵ with $0 < \epsilon < \|f\|_*$. This means that $\|v\| = 1$, and $v_f = v$ is the vector we seek.

Since v_f is uniquely determined, the function $v \mapsto v_f$ is well defined. We now show that it is continuous.

Let f and g in V^* be given, and let v_f and v_g be the corresponding unit vectors in V. Then

$$\begin{aligned}
\|f+g\|_* \|v_f + v_g\| &\ge \mathcal{R}\left((f+g)(v_f+v_g)\right) \\
&= \mathcal{R}\left(f(v_f) + f(v_g) + g(v_f) + g(v_g)\right) \\
&= 2(\|f\|_* + \|g\|_*) + \mathcal{R}\left(f(v_g) + g(v_f) - f(v_f) - g(v_g)\right) \\
&= 2(\|f\|_* + \|g\|_*) - \mathcal{R}\left((f-g)(v_f - v_g)\right) \\
&\ge 2\|f+g\|_* - \|f-g\|_* \|v_f - v_g\|\ .
\end{aligned} \tag{1.9}$$

Dividing through by $2\|f+g\|_*$ we get

$$\left(\frac{\|f-g\|_*}{\|f+g\|_*}\right) \left\|\frac{v_f - v_g}{2}\right\| \ge 1 - \left\|\frac{v_f + v_g}{2}\right\|\ .$$

If $\|v_f - v_g\| > 2\epsilon$, $1 - \|(v_f+v_g)/2\| \ge \delta(\epsilon)$, and since $\|(v_f - v_g)/2\| \le 1$ in any case,

$$\left(\frac{\|f-g\|_*}{2\|f\|_* - \|f-g\|_*}\right) \ge \delta(\epsilon)\ ,$$

or

$$\|f-g\|_* \ge \frac{\delta(\epsilon)}{1+\delta(\epsilon)} 2\|f\|_*\ .$$

Hence

$$\|f-g\|_* < \frac{\delta(\epsilon)}{1+\delta(\epsilon)} 2\|f\|_* \Rightarrow \|v_f - v_g\| < \epsilon\ .$$

This proves the continuity of $f \mapsto v_f$ at all $f \neq 0$. ■

We now turn to a concept that is closely related to convexity – namely uniform smoothness.

2 Uniform smoothness

2.1 Differentiability of functions on Banach spaces

Let V be a normed space with norm $\| \cdot \|$. The derivative gives the "best linear approximation" to a function.We say that a functional F on V is *Frechét differentiable* at $u \in V$ in case there is a linear functional $\ell_{F,u} \in V^*$ so that

$$F(u+v) - F(u) = \ell_{F,u}(v) + o(\|v\|)$$

or, in other words, if

$$\lim_{v \to 0} \frac{|F(u+v) - F(u) - \ell_{F,u}(v)|}{\|v\|} = 0 \ , \tag{2.1}$$

where the limit is taken in the norm sense.

There is another notion of differentiability, corresponding to the usual directional derivative. A functional F is said to be *Gateaux differentiable* at $u \in V$ in case for there is a linear functional $\ell_{F,u} \in V^*$ so that for each $v \in V$,

$$F(u+tv) - F(u) = t\ell_{F,u}(v) + o(t)$$

or, in other words, if

$$\lim_{v \to 0} \frac{|F(u+tv) - F(u) - t\ell_{F,u}(v)|}{t} = 0 \ . \tag{2.2}$$

Clearly, if a functional F is Frechét differentiable, then it is Gateaux differentiable, and the two derivatives coincide. However, there are functionals that are Gateaux differentiable, but not Frechét differentiable.

To check differentiablility from the definition, you have to know the derivative $\ell_{F,u}$, which is somewhat inconvenient. There is, however, a necessary condition for differentiablility that can be stated solely in terms of F itself. If F is Frechét differntiable at u, then for any v,

$$F(u+v) - F(u) = \ell_{F,u}(v) + o(\|v\|)$$

and

$$F(u-v) - F(u) = -\ell_{F,u}(v) + o(\|v\|) \ .$$

Summing, the terms involving $\ell_{F,u}$ cancel, and we have

$$F(u+v) + F(u-v) - 2F(u) = o(\|v\|) \ .$$

In particular, a necessary condition for Frechét differentiability the norm functional on a Banach space is that

$$\left\| \frac{u+v}{2} \right\| + \left\| \frac{u-v}{2} \right\| - \|u\| = o(\|v\|) \ .$$

This brings us to the following definition:

2.1 DEFINITION (Uniform Smoothness)**.** Let V be a Banach space with norm $\|\cdot\|$. The *modulus of smoothness* of V is the function $\rho_V(\tau)$ defined by

$$\rho_V(\tau) = \sup\left\{\left\|\frac{u+\tau v}{2}\right\| + \left\|\frac{u-\tau v}{2}\right\| - 1 \ :\ \|u\| = \|v\| = 1\right\} \tag{2.3}$$

for each $\tau \geq 0$. Then V is said to be *uniformly smooth* in case $\rho_V(\tau) = o(\tau)$, i.e., if

$$\lim_{\tau\to 0}\frac{\rho_V(\tau)}{\tau} = 0\ . \tag{2.4}$$

When there is no abiguity as to which space V is intended, we write $\rho(\tau)$ in place of $\rho_V(\tau)$.

It is easy to see that uniform smoothness fails for $L^1(M,\mathcal{M},\mu)$ and $L^\infty(M,\mathcal{M}\mu)$, even for a two-point measure space, while $L^2(M,\mathcal{M},\mu)$ is uniformly smooth. This is left as an exercise. In fact, it is a good exercsie to compute the moduli of smoothness for these spaces. The results are:

- *When* $V = L^1(M,\mathcal{M},\mu)$,

$$\delta_V(\epsilon) = 0 \qquad \text{and} \qquad \rho_V(\tau) = \tau\ .$$

- *When* $V = L^2(M,\mathcal{M},\mu)$,

$$\delta_V(\epsilon) = 1 - \sqrt{1-\epsilon^2} \qquad \text{and} \qquad \rho_V(\tau) = \sqrt{1+\tau^2} - 1\ .$$

- *When* $V = L^\infty(M,\mathcal{M},\mu)$,

$$\delta_V(\epsilon) = 0 \qquad \text{and} \qquad \rho_V(\tau) = \tau\ .$$

The next theorem gives the relation between uniform convexity and uniform smoothness. Before stating it, we introduce the notion of a *dual pair* of Banach spaces.

2.2 DEFINITION (Dual Pairs)**.** A *dual pair* of Banach spaces is a pair of Banach spaces V and W with norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively, and a bilinear form $\langle\cdot,\cdot\rangle$ on $V\times W$ so that for all $v\in V$,

$$\|v\|_V = \sup\{\ |\langle v,w\rangle|\ :\ \|w\|_W \leq 1\ \} \tag{2.5}$$

and

$$\|w\|_W = \sup\{\ |\langle v,w\rangle|\ :\ \|v\|_V \leq 1\ \}\ . \tag{2.6}$$

The primary example is that in which $W = V^*$, and

$$\langle v,w\rangle = w(v)\ .$$

Then (2.6) holds by the definition of the norm on V^*, while (2.5) holds by the Hahn-Banach Theorem, which asserts the existence of a $w\in V^*$ with $\|w\| = 1$ and $w(v) = \|v\|$.

When V and W are a dual pair, there is a map from V into W^* which assigns to v the linear functional

$$f_v(\cdot) = \langle v, \cdot \rangle \ .$$

By (2.6), and the definition of the dual norm $\|\cdot\|_*$,

$$\|f_w\|_* = \|w\|_W \ .$$

Hence the map $w \mapsto \langle \cdot, w \rangle$, which is clearly linear, is also an isometry.

However, it need not be the case that its image is all of V^*. In summary,

• *When V and W are a dual pair, W may be identified with a subset of V^* through the isometric linear transformation*

$$w \mapsto \langle \cdot, v \rangle \ .$$

Howver, it is not necessarily the case that every $f \in V^*$ is in the range of this transformation.

We now prove that when V and W are a dual pair the moduli of smoothness and convexity of the one space can be detemrined from those of the other.

2.3 THEOREM (Lindenstrauss–Day Theorem)**.** *Let V and W be a dual pair of Banach spaces. Then W is uniformly smmoth if and only if V is uniformly convex. Moreover,*

$$\rho_W(\tau) = \sup_{0 \le \epsilon \le 1} \{ \ \epsilon\tau - \delta_V(\epsilon) \ \} \ . \tag{2.7}$$

Before giving the proof of Theorem 2.3, we give three simple but important applications.

2.4 THEOREM (Uniqueness and continuity of unit tangent functionals)**.** *If V is a uniformly smooth Banach space, then for each non-zero $v \in V$, there exists a unique unit vector $f_v \in V^*$ such that $f_v(v) = \|v\|$. Moreover, the map $v \mapsto f_v$ is continuous in the norm topologies.*

Proof: The Hahn-Banach Theorem tell us that the linear functional f_v exists; the points to be shown are the uniqueness and the continuity. By the Lindenstrauss-Day Theorem, V^* is uniformly convex. Now an argument in the spirit of the one used to prove the second half of Theorem 1.4 can be applied here. This is left as an exercise. ■

2.5 THEOREM (Differentiablity of the Norm)**.** *Let V be a uniformly smooth Banach space. Then the norm on V is continuously Frechet differentiable at all $v \neq 0$ in V, and the derivaitve is given by $\mathcal{R}(f_v)$, where f_v is the unique unit vector in V^* with $f_v(v) = \|v\|$*

Proof: Since V^* is uniformly convex, for each $u \in V$, there exists a unique unit vector $f_u \in V^*$ so that $f_u(u) = 1$. Hence,

$$\begin{aligned} \|v + w\| &= f_{v+w}(v + w) \\ &\le \mathcal{R}(f_{v+w}(v)) + \mathcal{R}(f_{v+w}(w)) \\ &\le \|v\| + \mathcal{R}(f_{v+w}(w)) \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} \|v+w\| &\geq \mathcal{R}\left(f_v(v+w)\right) \\ &= \mathcal{R}\left(f_v(v)\right) + \mathcal{R}\left(f_v(w)\right) \\ &= \|v\| + \mathcal{R}\left(f_v(w)\right) \ . \end{aligned}$$

Altogether,

$$\begin{aligned} 0 \leq \|v+w\| - \|v\| - \mathcal{R}\left(f_v(w)\right) &\leq \mathcal{R}\left(f_{v+w}(w)\right) - \mathcal{R}\left(f_v(w)\right) \\ &\leq \|f_{v+w} - f_v\|_* \|w\| \ . \end{aligned}$$

Hence

$$\left| \frac{\|v+w\| - \|v\| - \mathcal{R}\left(f_v(w)\right)}{\|w\|} \right| \leq \|f_{v+w} - f_v\|_* \ ,$$

and by Theorem 2.4 we know that $\lim_{\|w\|_p \to 0} \|f_{v+w} - f_v\|_* = 0$. ■

Now let V be a Banach space, and let V^* be the dual space of linear functionals on V, and let V^{**} be the dual space of continuous linear functionals on V^*. We have seen that in case $V = L^2(M, \mathcal{S}, \mu)$, V^* can be identified with V, and so V^{**} can as well.

This is a rather special circumstance. However, it is frequently the case that $V^{**} = V$. Let V be a Banach space, and consider the isometric mapping $v \mapsto L_v \in V^{**}$, where

$$L_v(f) = f(v)$$

for all $f \in V^*$. Recall that V is called *reflexive* in case the image of this mapping is all of V^{**}, which we express by writing $V = V^{**}$.

2.6 THEOREM (Millman)**.** *A uniformly convex Banach space is reflexive.*

Proof: By the Lindenstrauss–Day Theorem, V^{**} is uniformly convex. Given any unit vector $L \in V^{**}$, and any $\epsilon > 0$, pick a unit vector $f \in V^*$ so that

$$L(f) > 1 - \epsilon \ ,$$

which is possible by the definition of the $\| \cdot \|_{**}$ norm. Now since V is uniformly convex, there is a unit vector $v \in V$ with $f(v) = 1$. Let L_v be the corresponding element of V^{**}, given by $L_v(g) = g(v)$ for al $g \in V^*$. We then have

$$\begin{aligned} \left\| \frac{L + L_v}{2} \right\|_{**} &\geq \frac{L(f) + L_v(f)}{2} \\ &= \left(\frac{L + L_v}{2} \right)(f) \\ &\geq 1 - \frac{\epsilon}{2} \ . \end{aligned}$$

It follows that

$$\left\| \frac{L - L_v}{2} \right\|_{**} \leq \eta$$

where $\eta = \sup_{0 \leq s \leq 1} \{ s \ : \ \delta_{V^{**}}(s) < \epsilon/2 \ \}$. ■

Proof of Theorem 2.3: We will use f and g to denote elements of W, and u and v to denote elements of V. We will leave subscripts off the norms as this convention makes it clear which norm is inteneded.

The first step of the proof is to show that

$$\rho_W(\tau) + \delta_V(\epsilon) \geq \tau\epsilon \tag{2.8}$$

for all $\tau \geq 0$ and all $0 \leq \epsilon \leq 1$. To see this, fix any such τ and ϵ. Take any u and v in V with $\|u\| = \|v\| = 1$ and $\|v - w\| \geq 2\epsilon$.

Since V and W are a dual pair, for any $\eta > 0$, there are unit vectors f and g in W with

$$\langle f, (u+v)/2\rangle \geq \left\|\frac{u+v}{2}\right\| - \eta \qquad \text{and} \qquad \langle f, (u-v)/2\rangle \geq \left\|\frac{u-v}{2}\right\| - \eta \ .$$

Then

$$\begin{aligned}
\rho_W(\tau) &\geq \left\|\frac{f+\tau g}{2}\right\| + \left\|\frac{f-\tau g}{2}\right\| - 1 \\
&\geq \langle (f+\tau g)/2, v\rangle + \langle (f-\tau g)/2, v\rangle - 1 \\
&= \langle f, (u+v)/2\rangle + \tau\langle g, (u-v)/2\rangle - 1 \\
&\geq \left\|\frac{u+v}{2}\right\| + \tau\left\|\frac{u-v}{2}\right\| - 1 - 2\eta \\
&\geq \left\|\frac{u+v}{2}\right\| + \tau\epsilon - 1 - 2\eta \\
&\geq -\left(1 - \left\|\frac{u+v}{2}\right\|\right) + \tau\epsilon - 2\eta
\end{aligned}$$

Hence

$$\rho_W(\tau) + \left(1 - \left\|\frac{u+v}{2}\right\|\right) \geq \tau\epsilon - 2\eta \ .$$

By the definition of δ_V, and the fact that $\eta > 0$ is arbitrary, this proves (2.8).

We now observe that if W is uniformly smooth, then V is uniformly convex. Indeed, (2.8) says that

$$\delta_V(\epsilon) \geq \sup_{\tau\geq 0}\{\epsilon\tau - \rho_W(\tau)\ \} \ .$$

But if $\lim_{\tau\to 0}\rho_W(\tau)/\tau = 0$, there is a $\tau_\epsilon > 0$ so that

$$\frac{\rho_W(\tau_\epsilon)}{\tau_\epsilon} \leq \frac{\epsilon}{2}$$

and then we have

$$\delta_V(\epsilon) \geq \frac{\epsilon\tau_\epsilon}{2} > 0 \ .$$

It is also true that when V is uniformly convex, then W is uniformly smooth. to show this, we need and upper bound on $\rho_W(\tau)$. Hence, the second step is to show that for any $\tau > 0$, any $\eta > 0$ and any unit vectors f and g in W, there is an ϵ_τ with $0 \leq \epsilon_\tau \leq 1$ so that

$$\left(\left\|\frac{f+\tau g}{2}\right\| + \left\|\frac{f-\tau g}{2}\right\| - 1\right) \leq \tau\epsilon_\tau - \delta_V(\epsilon_\tau) + \eta \ . \tag{2.9}$$

Together, (2.8) and (2.9) prove (2.8). Indeed, fixing τ, and varinging f and g, we have

$$\begin{aligned} \rho_W(\tau) &\leq \tau\epsilon_\tau - \delta_V(\epsilon_\tau) + \eta \\ &\leq \sup_{0\leq\epsilon\leq 1} \{ \ \epsilon\tau - \delta_V(\epsilon) \ \} + \eta \ . \end{aligned}$$

Since $\eta > 0$ is arbitrary, and since we have (2.8), we have (2.8).

To prove (2.9), fix any $\tau > 0$ and any unit vectors f and g in W. Fix any $\eta > 0$, and choose unit vectors u_τ and v_τ in V with

$$\langle (f + \tau g), u_\tau \rangle \geq \|f + \tau g\| - \eta \qquad \text{and} \qquad \langle (f - \tau g), v_\tau \rangle \geq \|f - \tau g\| - \eta \ . \tag{2.10}$$

Then

$$\begin{aligned} \left\| \frac{f + \tau g}{2} \right\| + \left\| \frac{f - \tau g}{2} \right\| &\leq \frac{\langle (f + \tau g), u_\tau \rangle}{2} + \frac{\langle (f - \tau g), v_\tau \rangle}{2} + \eta \\ &= \frac{\langle f, u_\tau + v_\tau \rangle}{2} + \tau \frac{\langle g, u_\tau - v_\tau \rangle}{2} + \eta \\ &\leq \left\| \frac{u_\tau + v_\tau}{2} \right\| + \tau \left\| \frac{u_\tau - v_\tau}{2} \right\| + \eta \end{aligned}$$

Now let

$$\epsilon_\tau = \left\| \frac{u_\tau - v_\tau}{2} \right\|$$

so that $0 \leq \epsilon_\tau \leq 1$ and

$$\left\| \frac{u_\tau + v_\tau}{2} \right\| \leq 1 - \delta_V(\epsilon_\tau) \ ,$$

and so (2.9) holds.

Now that (2.7) is established, we can show that when V is uniformly convex, then W is uniformly smooth. Indeed, from (2.7) it follows that for any $\tau > 0$

$$\frac{\rho_W(\tau)}{\tau} \leq \sup_{0\leq\epsilon\leq 1} \left\{ \epsilon - \frac{\delta_V(\epsilon)}{\tau} \ \right\} \ .$$

For any $\epsilon_0 > 0$, if $\tau < 1/\delta_V(\epsilon_0)$ then the right hand side is no greater than ϵ_0. This shows that if $\delta_V(\epsilon_0) > 0$ for all $\epsilon_0 > 0$, then $\lim_{\tau\to 0} \rho_W(\tau)/\tau = 0$, and so W is uniformly smooth. ■

3 Uniform convexity and smoothness in L^p spaces

We now prove an inequality due to Hanner that interpolates between Minkowski's inequality for L^1, $\|f + g\|_1 \leq \|f\|_1 + \|g\|_2$, and the parallelogram identity $\|f + g\|_2^2 + \|f - g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2)$ in L^2.

For $0 \leq r \leq 1$, define two functions $\alpha(r)$ and $\beta(r)$ by

$$\alpha(r) = (1 + r)^{p-1} + (1 - r)^{p-1} \qquad \text{and} \qquad \beta(r) = (r^{-1} + 1)^{p-1} - (r^{-1} - 1)^{p-1} \ ,$$

where $p > 1$ and $\beta(0) := \lim_{r\to 0} \beta(r) = 0$. We then have the following lemma:

3.1 LEMMA. *For all real numbers x and y and all $0 < r < 1$, when $1 < p < 2$, then*

$$|x+y|^p + |x-y|^p \geq \alpha(r)|x|^p + \beta(r)|y|^p \ , \tag{3.1}$$

and the reverse inequality holds when $2 < p < \infty$.

For $x \neq 0$, and $|y|/|x| < 1$,

$$|x+y|^p + |x-y|^p = \alpha(|y|/|x|)|x|^p + \beta(|y|/|x|)|y|^p \ . \tag{3.2}$$

Proof: Let us begin with the last part. By the definitions

$$\begin{aligned}\alpha(|y|/|x|)|x|^p + \beta(|y|/|x|)|y|^p &= |x|[(|x|+|y|)^{p-1} + (|x|-|y|)^{p-1}] \\ &+ |y|[(|x|+|y|)^{p-1} - (|x|-|y|)^{p-1}] \\ &= (|x|+|y|)^p + (|x|-|y|)^p \\ &= |x+y|^p + |x-y|^p.\end{aligned}$$

As for the first part, suppose first that $1 < p < 2$. The case $2 < p < \infty$ is similar, except that all inequalities encountered in the treatment of $1 < p < 2$ will reverse.

To deal with the case $1 < p < 2$, suppose first that $|y| \leq |x|$, and define $f(r) := \alpha(r)|x|^p + \beta(r)|y|^p$. Computing the derivative $f'(r)$, we find

$$\begin{aligned} f'(r) &= (p-1)\left(|x|^p[(1+r)^{p-2} - (1-r)^{p-2}] + |y|^p r^{-2}[-(r^{-1}+1)^{p-2} + (r^{-1}-1)^{p-2}]\right) \\ &= (p-1)|x|^p[(1+r)^{p-2} - (1-r)^{p-2}]\left(1 - \left(\frac{|y|}{r|x|}\right)^p\right) \ . \end{aligned}$$

Now for $1 < p < 2$, $p-2 < 0$, and so $[(1+r)^{p-2} - (1-r)^{p-2}] < 0$ for all $0 < r < 1$. Thus, $f'(r)$ has the same sign as $|y|/|x| - r$. That is, $f'(r) > 0$ for $r < |y|/|x|$, and $f'(r) < 0$ for $r > |y|/|x|$. Hence, f is maximized at $r = |y|/|x|$.

Combining this with the computation above, we have that, for all $0 < r < 1$,

$$|y| \leq |x| \quad \Rightarrow \quad |x+y|^p + |x-y|^p \geq \alpha(r)|x|^p + \beta(r)|y|^p \ , \tag{3.3}$$

which prove (3.1) when $|x| > |y|$.

Next, we must deal with the case $|y| > |x|$. We claim that for all $0 < r < 1$, $\alpha(r) \geq \beta(r)$. Assuming this for now, we then observe that

$$|y| > |x| \quad \Rightarrow \quad \alpha(r)|y|^p + \beta(r)|x|^p \geq \alpha(r)|x|^p + \beta(r)|y|^p \ .$$

Combining this with (3.3) for x and y interchanged, we see that (3.1) is true for all x and y.

Thus, the proof of (3.1) $1 < p < 2$ is reduced to proving that for $0 < r < 1$ an $1 < p < 2$, $\alpha(r) \geq \beta(r)$. A simple calculation of a derivative, like the one made above, now shows that $\alpha(r) - \beta(r)$ is monotone decreasing, and then the claim follows since $\alpha(1) = \beta(1) = 2^{p-1}$.

Going back over the proof, one sees that all of the inequalities reverse in case $2 < p < \infty$. ■

3.2 LEMMA. *For all real complex z and w , and all $1 < p < 2$,*

$$|z+w|^p + |z-w|^p \geq ||z| + |w||^p + ||z| - |w||^p \ ,$$

while for $2 < p < \infty$, the reverse inequality holds.

Proof: Without loss of generality, we may suppose $z = |z|$ and $w = |w|e^{i\theta}$. Then $|z+w|^p = (|z|^2+|w|^2+2|z||w|\cos(\theta))^{p/2}$ and $|z-w|^p = (|z|^2+|w|^2-2|z||w|\cos(\theta))^{p/2}$. Thus,

$$|z+w|^p + |z-w|^p = (|z|^2+|w|^2+2|z||w|\cos(\theta))^{p/2} + (|z|^2+|w|^2-2|z||w|\cos(\theta))^{p/2} \ .$$

For $1 < p < 2$,

$$t \mapsto \varphi(t) = (|z|^2+|w|^2+t)^{p/2} + (|z|^2+|w|^2-t)^{p/2}$$

is a symmetric, concave functions of t defined for $t \in [-|z|^2+|w|^2, |z|^2+|w|^2]$. It is therefore monotone decreasing in t, and hence

$$\begin{aligned}(|z|^2+|w|^2+2|z||w|\cos(\theta))^{p/2} + (|z|^2+|w|^2-2|z||w|\cos(\theta))^{p/2} \geq \\ (|z|^2+|w|^2+2|z||w|)^{p/2} + (|z|^2+|w|^2-2|z||w|)^{p/2} = ||z|+|w||^p + ||z|-|w||^p \ .\end{aligned}$$

For $2 < p < 2$, an analogous convexity argument yields the opposite inequality. ■

3.3 THEOREM (Hanner's Inequality)**.** *For any measure space $(M, \mathcal{M}, \mu)$ and any $1 < p < 2$, and any $f, g \in L^p(M, \mathcal{M}, \mu)$,*

$$\|f+g\|_p^p + \|f-g\|_p^p \geq |\|f\|_p + \|g\|_p|^p + |\|f\|_p - \|g\|_p|^p \ . \tag{3.4}$$

and

$$\|f\|_p^p + \|g\|_p^p \geq \left|\frac{\|f+g\|_p + \|f-g\|_p}{2}\right|^p + \left|\frac{\|f+g\|_p - \|f-g\|_p}{2}\right|^p \tag{3.5}$$

whenever $1 < p < 2$, while for $2 < p < \infty$, the reverse inequalities hold.

Proof: Let us first deal with $1 < p < 2$. We may assume without loss of generality that $\|g\|_p \leq \|f\|_p \neq 0$. Then choosing $r = \|g\|_p/\|f\|_p$, the easy part of Lemma 3.1 tells us that

$$|\|f\|_p + \|g\|_p|^p + |\|f\|_p - \|g\|_p|^p = \alpha(r)\|f\|_p^p + \beta(r)\|g\|_p^p = \int_M [\alpha(r)|f(x)|^p + \beta(r)|g(x)^p] \, d\mu \ . \tag{3.6}$$

Then by Lemma 3.1, this time using the less obvious part

$$\int_M [\alpha(r)|f(x)|^p + \beta(r)|g(x)^p] \, d\mu \leq \int_M (||f(x)| + |g(x)||^p + ||f(x)| - |g(x)||^p) \, d\mu \ . \tag{3.7}$$

Finally, by Lemma 3.2,

$$\int_M (||f(x)| + |g(x)||^p + ||f(x)| - |g(x)||^p) \, d\mu \leq \int_M (|f(x)+g(x)|^p + |f(x)-g(x)|^p) \, d\mu \ . \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) yields (3.4). Now (3.5) follows from (3.4) if one replaces f and g by $f+g$ and $f-g$, respectively, in it. This takes care of $1 < p < 2$. By the last two lemmas, all of the inequalities reverse in case $2 < p < \infty$. ■

It is an immediate consequence of Hanner's inequality that L^p is uniformly convex for $2 < p < \infty$. Indeed, let u and v be two unit vectors in L^p, $1 < p < 2$. Then (3.4), with the reverse sign corresponding to $2 < p < \infty$, specializes to

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \leq 1 \ .$$

Thus,

$$\left\|\frac{u+v}{2}\right\|_p \le \left(1-\left\|\frac{u-v}{2}\right\|_p^p\right)^{1/p} .$$

Then by the elementary inequality $(1-a)^{1/p} \le 1-a/p$, which is a consequence of the concavity of $a \mapsto (1-a)^{1/p}$, we have

$$\left\|\frac{u+v}{2}\right\|_p \le 1-\frac{1}{p}\left\|\frac{u-v}{2}\right\|_p^p . \tag{3.9}$$

This shows that for $2 < p < \infty$,

$$\delta_{L^p}(\epsilon) \ge \frac{1}{p}\epsilon^p . \tag{3.10}$$

It is slightly more work to extract a simple bound on $\delta_{L^p}(\epsilon)$ for $1 < p < 2$.

Let u and v be two unit vectors in L^p, $1 < p < 2$. Then (3.5) specializes to

$$1 \ge \left|\frac{1}{2}\left(\left\|\frac{u+v}{2}\right\|_p + \left\|\frac{u-v}{2}\right\|_p\right)\right|^p + \left|\frac{1}{2}\left(\left\|\frac{u+v}{2}\right\|_p - \left\|\frac{u-v}{2}\right\|_p\right)\right|^p . \tag{3.11}$$

To put this in a more convenient form, we use the following:

3.4 LEMMA (Gross's Two-Point Inequality)**.** *For all $1 \le p \le 2$, and all real number x and y,*

$$\left(\frac{|x+y|^p + |x-y|^p}{2}\right)^{1/p} \ge (x^2+(p-1)y^2)^{1/2} . \tag{3.12}$$

Proof: Note that if $y^2 > x^2$, $x^2+(p-1)y^2 < y^2+(p-1)x^2$. Hence it suffices to prove (3.12) for $y^2 \le x^2$. Also, notice the neither side of (3.12) changes if we changes the signs of either x or y. Hence, we may suppose without loss of generality that $0 \le y \le x$. Let $r = y/x$. It suffices to show that for all $0 \le r \le 1$,

$$\varphi(r) := \left(\frac{(1+r)^p + (1-r)^p}{2}\right)^{1/p} \ge \psi(r) := (1+(p-1)r^2)^{1/2} .$$

In fact, we shall show that for all $0 < r < 1$,

$$\varphi(r)^2 \ge \psi(r)^2 = 1+(p-1)r^2 . \tag{3.13}$$

We compute

$$(\varphi^p)'(r) = \frac{p}{2}\left((1+r)^{p-1} - (1-r)^{p-1}\right)$$

and hence

$$(\varphi^p)''(r) = p(p-1)\left(\frac{(1+r)^{p-2} + (1-r)^{p-2}}{2}\right)$$

Since $-1 < p-2 < 0$, $x \mapsto |x|^{p-2}$ is convex, and hence

$$\frac{(1+r)^{p-2} + (1-r)^{p-2}}{2} \ge \left(\frac{(1+r)+(1-r)}{2}\right)^{p-2} = 1 .$$

It follows that $(\varphi^p)''(r) \ge p(p-1)$ for all $0 \le r \le 1$. Then since $\varphi^p(0) = 1$ and $(\varphi^p)'(0) = 0$,

$$\varphi^p(r) \ge 1 + \frac{p(p-1)}{2}r^2 . \tag{3.14}$$

Using the elementary inequality $(1+a)^{2/p} \geq 1 + \dfrac{a}{2p}$, which is a consequence of the convexity of $a \mapsto (1+a)^{2/p}$ (since $p < 2$), we have (3.13) from (3.14). ■

Now, applying Lemma 3.4 to (3.11), we conclude

$$1 \geq \left(\left\| \frac{u+v}{2} \right\|_p^2 + (p-1) \left\| \frac{u-v}{2} \right\|_p^2 \right)^{1/2} ,$$

which in turn yields

$$\left\| \frac{u+v}{2} \right\|_p \leq \left(1 - (p-1) \left\| \frac{u-v}{2} \right\|_p^2 \right)^{1/2} .$$

Now using the elementary inequality $(1-a)^{1/2} \leq 1 - a/2$, which is a consequence of the concavity of $a \mapsto (1-a)^{1/2}$, we have

$$\left\| \frac{u+v}{2} \right\|_p \leq 1 - \frac{p-1}{2} \left\| \frac{u-v}{2} \right\|_p^2 . \tag{3.15}$$

This displays the uniform convexity of L^p, $1 < p < 2$. For such p,

$$\delta_{L^p}(\epsilon) \geq \frac{p-1}{2}\epsilon^2 . \tag{3.16}$$

We now have the following result:

3.5 THEOREM (Uniform convexity of L^p, $1 < p < \infty$)**.** *For any measure space $(M, \mathcal{M}, \mu)$ and any $L^p(M, \mathcal{M}, \mu)$ is uniformly convex. For $1 < p < 2$, one has the bound*

$$\delta_{L^p}(\epsilon) \ \geq \frac{p-1}{2}\epsilon^2 \tag{3.17}$$

while for $2 < p < \infty$, one has the bound

$$\delta_{L^p}(\epsilon) \ \geq \frac{1}{p}\epsilon^p \tag{3.18}$$

Proof: In the discussion just above (3.16) and (3.10), we have proved the uniform convexity, and the left halves of (3.17) and (3.18). ■

We are now give two proofs of the Riesz Representation Theorem for L^p, $1 < p < \infty$.

3.6 THEOREM (Riesz Representation Theorem for L^p, $1 < p < \infty$)**.** *Let $(M, \mathcal{M}, \mu)$ be any measure space. Let $1 \leq p \leq \infty$, and let $q = p/(1-p)$. Then the map from L^q into $(L^p)^*$ given by $g \mapsto \varphi_g$ where*

$$\varphi_g(f) = \int_M fg \mathrm{d}\mu , \qquad f \in L^p ,$$

is an isometry from L^q into $(L^p)^*$.

Before giveing our two proofs, let us take stock of what has been dealt with, and what remains to be dealt with. We have already seen, as a consequence of Hölder's inequality, that for every $g \in L^q$, $\|\varphi_g\|_{(L^p)^*} = \|g\|_q$, and hence $g \mapsto \varphi_g$ is an isometric map *into* $(L^p)^*$. It remains to be shown that this map is *onto* $(L^p)^*$. We now give two proofs of this.

First Proof of Theorem 3.6: Let $1 < p < \infty$. Let V be the range of the mapping $g \mapsto \varphi_g$ in $(L^p)^*$. Since the map is an isometry, and since L^q is complete. V is a closed subspace of $(L^p)^*$, If V is a proper subspace of $(L^p)^*$, there exists a non-zero $\varphi \in (L^p)^* \backslash V$ and then by the Hahn-Banach Theorem, there is an $L \in (L^p)^{**}$ such that

$$L(\varphi) = \|\varphi\|_{(L^p)^*} \neq 0 \ , \tag{3.19}$$

and $L\varphi_g) = 0$ for all $g \in L^q$.

However, by Millman's Theorem, since L^p is uniformly convex, it is reflexive, and so there exists an $f \in L^p$ so that

$$L(\psi) = \psi(f) \qquad \text{for all} \quad \psi \in (L^p)^* \ .$$

Therefore, for all $g \in L^q$, since $\varphi_g \in V$,

$$0 = L(\varphi_g) = \varphi_g(f) = \int_M gf \mathrm{d}\mu \ .$$

But since

$$\|f\|_p = \sup \left\{ \int_M gf \mathrm{d}\mu \ : \ \|g\|_q = 1 \right\} \ ,$$

it would follow that $\|f\|_p = 0$, and hence $L = 0$. This is contradicts (3.19), and hence V is not a proper subspace of $(L^p)^*$. ∎

Second Proof of Theorem 3.6: Let $1 < p < \infty$. Since L^p is uniformly convex, for each $\varphi \in (L^p)^*$, there exists a unique $f_\varphi \in L^p$ with $f_\varphi \in L^p$ and $\varphi(f_\varphi) = \|\varphi\|_{(L^p)^*}$. Then, for any $g \in L^p$, the function

$$t \mapsto \varphi\left(\frac{f_\varphi + tg}{\|f_\varphi + tg\|_p} \right)$$

has a maximum at $t = 0$.

If we assume for the moment that $t \mapsto \|f_\varphi + tg\|_p$ is differentiable at $t = 0$, then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \varphi\left(\frac{f_\varphi + tg}{\|f_\varphi + tg\|_p} \right) \bigg|_{t=0} = \mathcal{R}\varphi(g) - \|\varphi\|_{(L^p)^*} \frac{\mathrm{d}}{\mathrm{d}t} \|f_\varphi + tg\|_p \bigg|_{t=0} \ .$$

By the convexity of $x \mapsto |x|^p$, for all $0 < t < 1$ and all $x \in M$,

$$|f_\varphi(x)|^p - |f_\varphi(x) - g(x)|^p \leq \frac{|f_\varphi(x) + tg(x)|^p - |f_\varphi(x)|^p}{t} \leq |f_\varphi(x) + g(x)|^p - |f_\varphi(x)|^p \ .$$

Then, since $|f_\varphi - g|^p$, $|f_\varphi - g|^p$ and $|f_\varphi|^p$ are all integrable, The Dominated Convergence Theorem yields us

$$\lim_{t \to 0} \int_M \frac{|f_\varphi(x) + tg(x)|^p - |f_\varphi(x)|^p}{t} \mathrm{d}\mu = \int_M \lim_{t \to 0} \frac{|f_\varphi(x) + tg(x)|^p - |f_\varphi(x)|^p}{t} \mathrm{d}\mu \ .$$

Now one easily computes that for all x,

$$\lim_{t \to 0} \frac{|f_\varphi(x) + tg(x)|^p - |f_\varphi(x)|^p}{t} = \mathcal{R} |f_\varphi|^{p-2} \overline{f_\varphi}(x) g(x) \ .$$

Thus, for all $g \in L^q$, we have

$$\mathcal{R}\varphi(g) = \|\varphi\|_{(L^p)^*} \int_M \mathcal{R} |f_\varphi|^{p-2} \overline{f_\varphi}(x) g(x) \mathrm{d}\mu \ .$$

Substituting g by ig, we obtain the same result for the imaginary part, and hence

$$\varphi(g) = \|\varphi\|_{(L^p)^*} \int_M |f_\varphi|^{p-2}\overline{f_\varphi}(x)g(x)\mathrm{d}\mu \ .$$

It is now easily checked that $|f_\varphi|^{p-2}\overline{f_\varphi}$ is a unit vector in L^q, and hence φ is in the range of our isometry into $(L^p)^*$. But since φ is an arbitrary element of $(L^p)^*$, we see that our isomemtry is onto $(L^p)^*$. ■

Next, as a consequence of Theorem 3.5 the Lindenstrauss-Day Theorem ,we obtain the following result:

3.7 THEOREM (Uniform smoothness of L^p, $1 < p < \infty$)**.** *For any measure space* $(M, \mathcal{M}, \mu)$ *and any* $L^p(M, \mathcal{M}, \mu)$ *is uniformly smooth. For* $1 < p < 2$*,one has the bound*

$$\rho_{L^p}(\tau) \le \frac{1}{2(p-1)}\tau^2 \ , \tag{3.20}$$

while for $2 < p < \infty$ *and* $q = p/(p-1)$*, one has the bound*

$$\rho_{L^p}(\tau) \le \frac{1}{q}\tau^2 \ . \tag{3.21}$$

Proof: The uniform smoothness follows directly from Theorems 2.3 and 3.5. To obtain (3.20) use (2.7) to deduce

$$\rho_{L^p}(\tau) \le \sup_{0\le\epsilon\le1} \{\epsilon\tau - \delta_{L^q}(\epsilon) \ \} \ .$$

Then by (3.17) and a simple calculation, one obtains and (3.20). The proof of (3.21) is similar. ■

It follows from Theorems 2.5 and 3.7 that for $1 < p < \infty$, the norm on Ł^p is Frechét differentiable at any $f \neq 0$ in L^p, and we have seen that the derivative is the real part of the linear functional

$$g \mapsto \int_M [\|f\|_p^{1-p}|f(x)|^{p-2}\overline{f(x)}]g(x)\mathrm{d}\mu \ .$$

on L^p.

It is easy to see, by direct computation that $v_f := \|f\|_p^{1-p}|f(x)|^{p-2}$ is a unit vector in L^q, and is in fact the unit tangent vector for the linearfunctional

$$h \mapsto \int_M f(x)h(x)\mathrm{d}\mu$$

on L^q, $q = p/(p-1)$.

Therefore, by Theorem 1.4, the map $f \mapsto v_f$ is continuous from $L^p\backslash\{0\}$ into L^q. In fact, we can easily estimate the modulus of continuity. By the bound obtained at the end of the proof of Theorem 1.4, for any $f, g \in L^p$, $f \neq 0$,

$$\|f - g\|_p \le \frac{\delta_{L^q}(\epsilon)}{1 + \delta_{L^q}(\epsilon)} 2\|f\|_p \quad \Rightarrow \quad \|v_f - v_g\|_q \le \epsilon.$$

Let us suppose for instance that $2 < p < \infty$ so that we have the bound $\delta_{L^q}(\epsilon) \ge (q-1)\epsilon^2/2$. The we can simplify the implication to

$$\|f - g\|_p \le \|f\|_p(q-1)\frac{\epsilon^2}{2} \quad \Rightarrow \quad \|v_f - v_g\|_q \le \epsilon.$$

Eliminating ϵ yields

$$\|v_f - v_g\|_q \leq \left(\frac{2}{(q-1)\|f\|_p}\right)^{1/2} \|f - g\|_p^{1/2} \ .$$

That is, the map $f \mapsto v_f$ is locally (away from 0) Hölder continuous with Hölder exponent $1/2$. A similar result. A similar result, with Hölder exponent $1/q$ holds for $1 < p < 2$.

This has an important application. Let $t \mapsto f(t)$ be a strongly differentiable map from $\mathbb{R}$ into L^p, $1 < p < \infty$. That is, for each t

$$g(t) = \lim_{h \to 0} \frac{1}{h}(f(t+h) - f(t))$$

exists in norm in L^p. Suppose moreover that $t \mapsto g(t)$ is continuous in the L^p norm. This situation often arises when $f(t)$ is the solutions of some partial differential equation, for example.

Then, the function $t \mapsto \|f(t)\|_p$ is continuously differentiable on any interval on which $f(t)$ does not vanish, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_p = \int_M v_{f(t)} g(t) \mathrm{d}\mu \ .$$

Exercises:

1. Do the computations of ρ_{L^p} for $p = 1$, $p = 2$ and $p = \infty$ that are left as an exercise on page 7.

2. Complete the proof of Theorem 2.4.