Notes on the the solutions of the exercises from the second set

1 The first problem entailed showing that $L^p \cap L^r$, $1 \le p < r \le \infty$ is a Banach space when equipped with the norm

$$|||f||| = \|f\|_p + \|f\|_r \ .$$

The main point here is the completeness. Suppose $\{f_n\}$ is a Cauchy sequence for the norm $|||\cdot|||$. Then since $|||f||| \ge \|f\|_p$ and $|||f||| = \|f\|_r$, it is also a Cauchy sequence for the L^p and L^r norms. Since L^p is complete, there is some $g \in L^p$ so that $\lim_{n\to\infty} f_n = g$ in the L^p norm, and likewise there is some $h \in L^r$ so that $\lim_{n\to\infty} f_n = h$ in the L^r norm. Every L^p convergent sequence has a subsequence converging almost everywhere, and likewise for L^r, so some subsequence of $\{f_n\}$ converges to g, and then a further subsequence of this converges to h. Hence $h = g$, and so the limit – call it f – is in $L^p \cap L^r$, and $\lim_{n\to\infty} |||f_n - f||| = 0$. This shown the completeness.

Regarding the fact that for $p \le q \le r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous, since this map is linear, we need only show it is bounded. But this follows directly from

$$\|f\|_q \le \|f\|_p^{\lambda} \|f\|_r^{1-\lambda} \le \lambda \|f\|_p + (1-\lambda)\|f\|_q \le |||f|||$$

where $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$.

2 The first problem entailed showing that $L^p + L^r$, $1 \le p < r \le \infty$ is a Banach space when equipped with the norm

$$|||f||| = \inf\{\|g\|_p + \|h\|_r \ : \ g + h = f\ \} \ .$$

Here there is a bit more to do to show that this is a norm. The main thing to show is that if $|||f||| = 0$, then $f(x) = 0$ almost everywhere. To see this, suppose one has two sequences $\{g_n\}$ and $\{h\}_n$ with $f = g_n + h_n$ for all n, but

$$\lim_{n\to\infty} (\|g_n\|_p + \|h_n\|_r) = 0 \ .$$

Then of course $g_n \to 0$ in L^p and $h_n \to 0$ in L^r. Since L^p (and L^r) sequences have subsequences that converge almost everywhere, passing to a subsequence, we have

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = 0$$

for almost every x. But $f = h_n + g_n$ for all n, so $f(x) = 0$ for almost every x.

Most people proved the completeness in a nice way by adapting th proof of completeness in L^p spaces. Again for the continuity of the imbedding $L^q \to L^p + L^r$ for $p \le q \le r$, since this map is linear, it suffices to show it is bounded. So take any f in the unit ball in L^q. Then

$$f = 1_{\{|f|\le 1\}} f + 1_{\{|f|>1\}} f \ ,$$

and so, since $r > q$,

$$\|1_{\{|f|\le 1\}} f\|_r^r \le \|1_{\{|f|\le 1\}} f\|_q^q \le \|f\|_q^q = 1 \ .$$

Likewise, since $q > p$,

$$\|1_{\{|f|\ge 1\}} f\|_p^p \le \|1_{\{|f|\ge 1\}} f\|_q^q \le \|f\|_q^q = 1 \ .$$

Therefore,

$$|||f||| \le \|1_{\{|f|\ge 1\}} f\|_p^p + \|1_{\{|f|\le 1\}} f\|_r^r \le 2 \ ,$$

and so the inclusion is bounded, and the bound is no greater than 2. (However, one can improve the argument. One way is to cut at height t, instead of the arbitrary height 1, and then to optimize over t. But this is not needed here.

3 This problem was to show that if if $\{f_n\}$ is a sequence in L^p, $1 \le p < \infty$, that converges to f almost everywhere, then $\lim_{n\to\infty} \|f_n - f\|_p = 0$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. The more involved part is the "if" part, and most people did this in a good way. However, there were many complicated proofs of the "only if" part. Any proof that uses dominated convergence is a bit too complicated since in fact for it any norm, one has

$$\|f\| = \|(f - f_n) + f_n\| \le \|f - f_n\| + \|f_n\|$$

and so (swapping the roles of f and f_n)

$$\big|\|f\| - \|f_n\|\big| \le \|f - f_n\| \ .$$

Hence if f_n converges to f in norm, the norm of f_n converges to the norm of f, and this is true in any normed vector space, whether the norm is defined in terms of integrals or not.

4 This exercise involved showing that for $1 < p < \infty$, if $\{f_n\}$ is a bounded sequence in L^p that converges almost everywhere to f, then $\{f_n\}$ converges weakly to f. (Note that necessarily $f \in L^p$ by Fatou's Lemma.)

We have to show that for any $g \in L^q$, $1/q + 1/p = 1$,

$$\lim_{n\to\infty} \int_X f_n g \mathrm{d}\mu = \int_X f g \mathrm{d}\mu \ .$$

The simplest way to proceed, ignoring the hints somewhat, is to use the fact that simple functions are dense in L^q. Pick any $\epsilon > 0$, and choose $h \in L^q$ with $\|g-h\|_q < \epsilon$. Then

$$\left|\int_X (f_n - f)g\mathrm{d}\mu\right| \leq \left|\int_X (f_n - f)h\mathrm{d}\mu\right| + \left|\int_X (f_n - f)|g-h|\mathrm{d}\mu\right| \ .$$

Then by Hölder's inequality,

$$\left|\int_X (f_n - f)|g-h|\mathrm{d}\mu\right| \leq \|f_n - f\|_p\|g-h\|_q \leq 2\left(\sup_{n\geq 0}\|f_n\|_p\right)\epsilon \ .$$

Since $epsiion > 0$ is arbitrary, it suffices to show that

$$\lim_{n\to\infty}\int_X f_n h\mathrm{d}\mu = 0 \ . \qquad (*)$$

But since h is simple, $h = 0$ off a set A of finite measure, and also h is bounded. Now it is easy to use Egorov's Theorem to prove $(*)$.

6 Let $f_n(x) = \cos(2\pi nx)$ on $L^2([0,1])$. To see that $\{f_n\}$ converges weakly to zero, one can use the fact that continuously differentiable functions that vanish at $x = 0$ and $x = 1$ are dense in $L^2([0,1])$. Let g be any such function. Then for $n > 0$, integrating by parts,

$$\int_0^1 f_n g(x)\mathrm{d}x = \frac{1}{2\pi n}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\sin(2\pi nx)g(x)\mathrm{d}x = -\frac{1}{2\pi n}\int_0^1 \sin(2\pi nx)g'(x)\mathrm{d}x \ .$$

Thus, by the Schwarz inequality,

$$\left|\int_0^1 f_n g(x)\mathrm{d}x\right| \leq \frac{1}{2\pi n}2^{-1/2}\|g'\|_2 \ ,$$

and this clearly goes to zero as $n \to \infty$. Since such functions g are dense in $L^2([0,1])$, $\{f_n\}$ converges weakly to zero.

7 Recall that for any measurable function and any $1 \leq p \leq infty$, with $1/q + 1/p = 1$,

$$\|f\|_p = \sup_{\|g\|_g \leq 1}\int_X fg\mathrm{d}\mu \ .$$

This is the *duality formula.*

Hence once you have shown that

$$\int\left[\int K(xy)f(y)\mathrm{d}y\right]g(x)\mathrm{d}y \leq \phi(1/2)\|f\|_2\|g\|_2 \ ,$$

which is a special case of the result of part (a), you have that

$$\|Kf\|_2 \leq \phi(1/2)\|f\|_2 \ , \qquad \text{and hence} \qquad \|K\| \leq \phi(1/2) \ .$$

There is no need to redo the computation of part (a) again for the norm computation. A similar remark applies in the final problem.