Notes on Banach Spaces of Measures

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1 intorduction

Given a topological space $(X, \mathcal{U})$, there are three natural normed vector spaces of continuous functions:

1.1 DEFINITION (Spaces of continuous functions)**.** Let $(X, \mathcal{U})$ be a topological space. We define:

(i) $\mathcal{C}_c(X)$ is the normed vector space of continuous, compactly supported functions f on X with values in $\mathbb{C}$ on which the norm, $\|\cdot\|_\infty$, is given by $\|f\|_\infty = \sup_{\{} \ |f(x)| \ : \ x \in X \ \}$.

(ii) $\mathcal{C}_0(X)$ is the normed vector space of continuous functions f on X with values in $\mathbb{C}$ such that for eaxh $\epsilon > 0$, there exists a compact set $K_{\epsilon,f}$ such that $|f(x)| < \epsilon$ for all x outside of $K_{\epsilon,f}$. The norm is once again $\|\cdot\|_\infty$, given by $\|f\|_\infty = \sup_{\{} \ |f(x)| \ : \ x \in X \ \}$.

(iii) $\mathcal{C}_b(X)$ is the normed vector space of continuous functions f on X with values in $\mathbb{C}$ such that $\|f\|_\infty = \sup_{\{} \ |f(x)| \ : \ x \in X \ \} < \infty$. The norm is once again $\|\cdot\|_\infty$.

When X is not compact, $\mathcal{C}_c(X)$ is not complete. Take for instance $X = R^n$. Let ϕ_n be a continuous radial decreasing function function with $\phi_n(x) = 1$ for $|x| \le n$, and $\phi_n(x) = 0$ for $|x| \ge 2n$. Now let f be given by $f(x) = e^{-x^2}$, and define $f_n = \phi_n f$. Then it is clear that for $n > m$,

$$\|f_n - f_m\|_\infty \le e^{-m^2}$$

and so $\{f_m\}$ is a Cauchy sequence in $\mathcal{C}_c(X)$. But clearly there is no function $g \in \mathcal{C}_c(X)$ such that $\lim_{n\to\infty} \|f_n - g\|_\infty = 0$.

However, it is easy to see that $\mathcal{C}_0(X)$ is the norm closure of $\mathcal{C}_c(X)$, and both it and $\mathcal{C}_b(X)$ are Banach spaces.

1.2 THEOREM. *$\mathcal{C}_0(X)$ equipped with the sup norm is a Banach space. If X is a locally compact Hausdorff space, then the subsapce $\mathcal{C}_c(X)$ is dense.*

Proof: If $\{f_n\}$ is a Cauchy seuqence in $\mathcal{C}_0(X)$, then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. Hence the limit exists for each x, and defines a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \ .$$

Given the uniform convergence, it is easy to check that $f \in \mathcal{C}_0(X)$ so that $\mathcal{C}_0(X)$ is complete.

To see that $\mathcal{C}_c(X)$ is dense, pick $f \in \mathcal{C}_0(X)$ and $\epsilon > 0$. Let K_ϵ be a compact set such that $|f(x)| \leq \epsilon$ for all $x \notin K_\epsilon$. Becasue X is locally compact, it is possible to find an open set U containing $K\epsilon$ such that U has compact closure. Then by Urysohn's Lemma, there exists a continuous function φ with $K \prec \varphi \prec U$ such that. Then it is clear that $\varphi\ in \mathcal{C}_c(X)$, and $\|f\varphi - f\|_\infty \leq \epsilon$. ■

In this chapter of the notes, our goal is to study the spaces of continuous linear functional on these function spaces. We shall identify there dual spaces with certain spaces of measures, and these spaces of measures will be our primary focus. In particular, we shall be concerned with finding precise and concrete descriptions of these dual spaces, and with finding useful descriptions of the compact sets in them, where compact is defined with reference to the weak-$*$ topology.

To be successful in this program, we shall of course need to know something about the topology of the underlying space X; Theorem 1.2 already gives some hint of this. In particular, we shall need it to be sufficiently rich in *both* open and compact sets.

There is a crucial balance to be struck here: The more open sets a topology contains the fewer compact sets it contains and *vice-versa*. Hence only under certain topological conditions can we carry out our program. There are two generally useful topological frameworks in which the program can be completed in a fully satisfactory manner. The first is that in which X is a locally compact Hausdorff space, and the other is that in which X is a complete, separable metric space. Both are important, and somewhat complementary. We begin with the locally compact Hausdorff space theory.

2 The Riesz-Markov Theorem for locally compact Hausdorff spaces

Throughout this section, let $(X,\mathcal{U})$ be a locally compact Hausdorff space. The spaces $\mathcal{C}_c(X)$, $\mathcal{C}_0(X)$ and $\mathcal{C}_b(X)$ are more than topological spaces: They contain a distinguished *cone* of non-negative elements: We say that *a function f on X is non-negative* in case for each x, $f(x) \in \mathbb{R}$, and $(x) \geq 0$. In this case we write $f \geq 0$.

Let L be a linear functional on $\mathcal{C}_c(X)$. We say that *L is a positive linear functional* on $\mathcal{C}_c(X)$ in case

$$f \geq 0 \quad \Rightarrow \quad L(f) \geq 0\ .$$

There is a close connection between the topology on $\mathcal{C}_c(X)$ and the partial order structure on $\mathcal{C}_c(X)$ induced by its cone of positive elements.

2.1 THEOREM. *Let L be a positive linear functional on $\mathcal{C}_c(X)$. Then for each compact $K \subset X$, there exists a finite constant C_K such that*

$$|f| \prec K \quad \rightarrow \quad |L(f)| \leq C_K \|f\|_\infty\ .$$

Proof: By a basic lemma concerning LCH spaces, there exists an open set U with compact closure $\overline{U}$ such that

$$K \subset U \subset \overline{U}\ ,$$

and then by Urysohn's Lemma, there exists a continuous function φ on X such that

$$K \prec \varphi \prec U \ .$$

Then evidently $\|f\|_\infty \varphi - |f| \geq 0$, and hence $L(\|f\|_\infty \varphi - f) \geq 0$. Thus,

$$L(f) \leq L(\|f\|_\infty \varphi) = L(\varphi)\|f\|_\infty \ .$$

Thus,

$$\sup\{|L(f)| \ : \ |f| \prec K \ , \|f\|_\infty \leq 1 \ \} \leq L(\varphi)\|f\|_\infty \ .$$

We may take $C_K = L(\varphi)$, or, better yet,

$$C_K = \inf\{L(\varphi) \ : \ K \prec \varphi \prec X \ \} \ .$$

■

To construct an example of a positive linear functional on $\mathcal{C}_c(X)$, let μ be a Borel measure on X that is finite on every compact set $K \subset X$.

2.2 DEFINITION (Radon mesure)**.** A *Radon measure* on a locally compact Hausdorf space X is a positive Borel measure μ on X such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

2.3 DEFINITION (innner and outer regularity)**.** A Borel measure μ is *outer regular* in case for each Borel set E

$$\mu(E) = \inf\{ \ \mu(U) \ : \ E \subset U \ , \ U \text{ open} \ \} \ . \tag{2.1}$$

A Borel measure μ is *inner regular* in case for each Borel set E

$$\mu(E) = \sup\{ \ \mu(K) \ : \ K \subset E \ , \ K \text{ compact} \ \} \ . \tag{2.2}$$

A Borel measure μ is *inner regular for open sets* in case (2.2) hold for all *open* E.

As we shall see, on any locally compact Hausdorff space, Radon measures are automatically outer regular, and inner regular for open sets. These properties are sometimes included in the definition of Radon measures, and then the condition that X is a locally compact Hausdorff space can be dropped. However, since we are working in the setting of locally compact Hausdorff spaces, the regularity is a theorem, and not part of the definition.

2.4 THEOREM (Riesz-Markov Theorem)**.** *Let L be any positive linear functional on $\mathcal{C}_c(X)$. Then there exists a unique Radon measure μ such that*

$$L(f) = \int_X f \mathrm{d}\mu \quad \text{for all} \quad f \in \mathcal{C}_c(X) \ . \tag{2.3}$$

Moreover,

$$\mu(U) = \sup\{ \ L(f) \ : \ f \prec U \ , f \in \mathcal{C}_c(X) \ \} \tag{2.4}$$

for all open sets U, and

$$\mu(K) = \inf\{ \ L(f) \ : \ K \prec f \ , f \in \mathcal{C}_c(X) \ \} \tag{2.5}$$

for all compact sets K. Finally, μ is outer regular, and inner regular for open sets.

Proof: *Step 1: Use L to construct an outer measure μ^*.* We define a set function μ^* on arbitrary subsets E of X by

$$\mu^*(U) = \sup\{\ L(f)\ :\ f \prec U\ , f \in \mathcal{C}_c(X)\ \}$$

for open sets U, and then

$$\mu^*(E) = \inf\{\ \mu(U)\ :\ E \subset U\ ,\ U \text{ open }\}\ .$$

It is clear the $\mu^*(\emptyset) = 0$, and that if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$. Therefore, to show that mu^* is an outer measure, we must show that for any sequence $\{E_n\}_{n\in\mathbb{N}}$ of subsets of X,

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)\ .$$

Let E denote the union $cup_{n=1}^{\infty} E_n$. It suffices to consider the case in which $\mu^*(E) < \infty$, in which case, $\mu^*(E_n) < \infty$ for all n.

Pick any $\epsilon > 0$. Then by construction, there exists an open set U_n with $E_n \subset U_n$, and $\mu^*(U_n) \leq \mu^*(E_n) + 2^{-n}\epsilon$. But then

$$E \subset U := \bigcup_{n=1}^{\infty} U_n\ ,$$

and

$$\sum_{n=1}^{\infty} \mu^*(U_n) \leq \sum_{n=1}^{\infty} \mu^*(U_n) + \epsilon\ .$$

It therefore suffices to prove that

$$\mu^*\left(U\right) \leq \sum_{n=1}^{\infty} \mu^*(U_n)\ . \tag{2.6}$$

To do this, choose a compactly supported f such that $f \prec U$ and

$$\mu^*(U) - \epsilon \leq L(f)\ . \tag{2.7}$$

Let K denote the support of f. Then $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of K, and so there exists a finite subcover, which we may take to be $\{U_1, \ldots, U_N\}$.

Let $\{h_1, \ldots, h_N\}$ be a partition of unity on K, subordinate to the open cover $\{U_1, \ldots, U_N\}$. Then

$$f = \sum_{n=1}^{N} fh_n \quad \text{and} \quad fh_n \prec U_n \quad n = 1, \ldots, N\ .$$

It follows that

$$L(f) = \sum_{n=1}^{N} L(fh_n) \leq \sum_{n=1}^{N} \mu^*(U_n) \leq \sum_{n=1}^{\infty} \mu^*(U_n)\ .$$

Combining this with (2.7), we obtain (2.6).

Step 2 : Show that every open set U belongs to the Caratheodory σ-algebra of the outer measure μ^*.

The Caratheodry σ-algebra of outer measure μ^* consists of the sets A such that for every subset E of X,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \ .$$

Since $E = (E \cap A) \cup (E \cap A^c)$, and μ^* is an outer measure, it is automatic that $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for any set A. Thus, it remains to be shown that if U is open, then

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \cap U^c) \tag{2.8}$$

for all $E \subset X$.

To do this, first suppose E is open. Then $E \cap U$ is open, and hence, for any $\epsilon > 0$, there exists an $f \prec E \cap U$ such that $L(f) \ge \mu^*(E \cap U) - \epsilon$. Now let K denote the support of f. Then $E \cap K^c$ is open, and so there exists some $g \prec E \cap K^c$ such that $L(g) \ge \mu^*(E \cap K^c) - \epsilon$. Then since $f + g \prec E$ (the both have compact, disjoint support contained in E),

$$\begin{aligned} \mu^*(E) &= L(f+g) = L(f) + L(g) \\ &\ge \mu^*(E \cap U) + \mu^*(E \cap K^c) - 2\epsilon \\ &\ge \mu^*(E \cap U) + \mu^*(E \cap U^c) - 2\epsilon \end{aligned}$$

where the last inequality hold due to the fact that $f \prec U$ mean $K \subset U$, and hence $K^c \subset U^c$. Since $\epsilon > 0$ is arbitrary, we have (2.8), with equality in fact, for all open sets E.

To prove it in general, choose an open set V with $E \subset V$ and $\mu^*(V) \le \mu^*(E) + \epsilon$. Then

$$\mu^*(E) + \epsilon \ge \mu^*(V) = \mu^*(V \cap U) + \mu^*(V \cap U^c) \ge \mu^*(E \cap U) + \mu^*(E \cap U^c) \ .$$

Thus, the Caratheodory σ-algebra of the outer measure μ^* contains all open sets, and therefore contains all Borel sets. We now let μ denote the restriction of μ^* to the Borel σ-algebra. The by Caratheodory's Theorem μ is a countably additive Borel measure, and it satisfies (2.4) by construction, and is outer regular by construction.

Step 3 : Prove that μ satisfies (2.5). As an imediate consequence, μ is then finite on all compact sets, and is therefore a Radon measure. To carry out this step, let be any K is any compact set, and suppose that f satisfies $K \prec f \prec X$. Pick any $\epsilon > 0$, and define $U_\epsilon = \{\ x\ :\ f(x) \ge 1 - \epsilon\ \}$ Then for all $g \prec U_\epsilon$, $(1-\epsilon)^{-1} \ge g$, and hence

$$(1-\epsilon)^{-1} L(f) \ge L(g) \ .$$

Since $G \prec U_\epsilon$ is arbitrary,

$$(1-\epsilon)^{-1} L(f) \ge \mu(U_\epsilon) \ge \mu(K) \ .$$

Since $\epsilon > 0$ is arbitrary,

$$\mu(K) \le L(f) \ . \tag{2.9}$$

To get the opposite inequality, we use the outer regularity: There exists an open set U such that $K \subset U$ and $\mu(U) \le \mu(K) + \epsilon$. By our basic lemma for locally compact Hausdorff spaces, there exists an f with $K \prec f \prec U$. Then $\mu(U) \ge L(f)$, and so

$$\mu(K) \ge \mu(U) - \epsilon \ge L(f) - \epsilon \ .$$

This gives us the opposite of (2.9), and hence proves (2.5).

Step 4 : Prove that for all $f \in \mathcal{C}_c$, $L(f) = \int_X f\mathrm{d}\mu$. For this, it suffices to suppose that $f \geq 0$, for we can separately consider the real and imaginary, and then the positive and negative parts. Moreover, since f is bounded, we may suppose that $0 \leq f \leq 1$ Pick some large integer N, and the define

$$f_j = \left(f - \frac{j-1}{N}\right)_+ \wedge \frac{1}{N} \qquad j = 1, \ldots, N \ .$$

Then

$$f = \sum_{j=1}^{N} f_j \ .$$

Let K_{j-1} denote the support of f_j, $j = 1, \ldots, N$, which is compact.

Note that $K_j \prec Nf_j$, and hence by (2.5),

$$\mu(K_j) \leq L(Nf_j).$$

Also, if U is any open set containing K_{j-1}, then $Nf_j \prec U$, and hence $L(Nf) \leq \mu(U)$. Then, by outer regularity, and since U is an arbitrary open set containing K_{j-1},

$$L(Nf_j) \leq \mu(K_{j-1}) \ .$$

Altogether,

$$\frac{1}{N}\mu(K_j) \leq L(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \ .$$

Next, note that since

$$1_{K_j}(x) \leq Nf_j(x) \leq 1_{K_{j-1}}(x)$$

for all x,

$$\frac{1}{N}\mu(K_j) \leq \int_X f_j \mathrm{d}\mu \leq \frac{1}{N}\mu(K_{j-1}) \ .$$

From here one readily concludes that

$$\left| L(f) - \int_X f\mathrm{d}\mu \right| \leq \frac{\mu(\mathrm{supp}(f)}{N} \ ,$$

which completes this step.

Step 5 : Prove that μ is inner regular for open sets. This is a very simple consequence of (2.5): If U is any open set, pick some $\epsilon > 0$ and some $f \prec U$ such that $L(f) \geq \mu(U) - \epsilon$. Let $K = \mathrm{supp}(f)$. Then by (2.5) there exists a g with $K \prec g$ such that $\mu(K) \geq L(g) - \epsilon$. But since $g = 1$ on the support of f, $g \geq f$, and so

$$\mu(K) \geq L(g) - \epsilon \geq L(f) - \epsilon \geq \mu(U) - 2\epsilon \ .$$

■

3 The Hahn-Saks Theorem Theorem

Let X denote a localy compact Hausdorff space that is σ-compact; i.e., such that there exists an increasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets with $X = \cup_{n=1}^{\infty} K_n$.

Let $\mathcal{M}(X)$ denote the space of *real* continuous linear fnctionals on $\mathcal{C}_0(X)$. That is,

$$\mathcal{M}(X) = (\mathcal{C}_0(X))^* \ .$$

Our first goal is to give a concrete description of $\mathcal{M}(X)$. As the notation may suggest, it will turn out to be a Banch space of measures – signed measures – on X.

We already know that *some* of the linear functionals in $\mathcal{M}(X)$ are measures. Indeed, let $L \in \mathcal{M}(X)$, and suppose that L is a positive linear functional on $\mathcal{C}_0(X)$ in the sense that

$$f \geq 0 \Rightarrow L(f) \geq 0 \ .$$

Then the restriction of L to $\mathcal{C}_c(X)$ is a positive linear functional on $\mathcal{C}_c(X)$, and so by the Riesz-Markov Theorem, there is a unique Radon measure μ_L so that $\mu_L(K) < \infty$ for all compact sets K and such that

$$L(f) = \int_X f \mathrm{d}\mu_L \tag{3.1}$$

for all $f \in \mathcal{C}_c(X)$.

Now let K_n be a sequence of compact sets in X with $\cup_n K_n = X$. For each n choose φ_n satisfying $K_n \prec \varphi_n \prec X$. Then

$$\begin{aligned}
\mu_L(X) &= \lim_{n\to\infty} \mu_L(K_n) \\
&\leq \lim_{n\to\infty} \int_X \varphi_n \mathrm{d}\mu_L \\
&\leq \lim_{n\to\infty} L(\varphi_n) \\
&\leq \lim_{n\to\infty} \|L\|_{\mathcal{M}} \|\varphi_n\|_\infty \\
&= \|L\|_{\mathcal{M}} \ .
\end{aligned}$$

Thus μ_L is not only a Radon measure, it is actually a *finite* Borel measure.

We now show that the general linear functional L on $\mathcal{C}_0(X)$ can be expressed in terms of finite Borel measures, though not necessarily positive. To do this, we shall decompose L into the difference of two positive linear functionals:

$$L = L_+ - L_- \ , \tag{3.2}$$

and then shall identify each of L_+ and L_- with finite Borel measures, as above.

Now it is completely clear how to take a function $f(x)$ apart into its paositive and negative parts

$$f(x) = f_+(x) - f_-(x) \ ,$$

but it is not so clear how this is to be done for a linear functional L. The way to do this was found by Hahn, and so it is called the *Hahn decomposition*. We now explain how this is done.

Let $L \in \mathcal{M}(X)$. For $f \in X$ with $f \geq 0$, define

$$L_+(f) = \sup\{\ L(h)\ :\ 0 \le h \le f\ \}\ . \tag{3.3}$$

We shall show that this gives us the "positive part" of L. Our first step toward goal is to prove the following:

3.1 LEMMA. *For all non–negative functions f and g in X, and all positive real numbers α,*

$$L_+(\alpha f + g) = \alpha L_+(f) + L_+(g)\ .$$

Proof: First consider the special case $g = 0$. It is clear from the definition that $L_+(\alpha f) = \alpha L_+(f)$. Thus, it suffices to show that

$$L_+(f+g) \ge L_+(f) + L_+(g) \qquad \text{and} \qquad L_+(f+g) \le L_+(f) + L_+(g)\ . \tag{3.4}$$

The first of the inequalities in (3.4) is very easy to prove: Given $\epsilon > 0$, the definition of L_+ ensures that there are functions $\tilde{h}_f$ and $\tilde{h}_g$ so that

$$0 \le \tilde{h}_f \le f \qquad \text{and} \qquad 0 \le \tilde{h}_g \le g$$

and

$$L(\tilde{h}_f) \ge L_+(f) - \frac{\epsilon}{2} \qquad \text{and} \qquad L(\tilde{h}_g) \ge L_+(g) - \frac{\epsilon}{2}\ .$$

Clealry $0 \le \tilde{f}_h + \tilde{f}_g \le f + g$, and so

$$L_+(f+g) \ge L(\tilde{h}_f + \tilde{h}_g) = L(\tilde{h}_f) + L(\tilde{h}_g) \ge L_+(f) + L_+(g) - \epsilon\ .$$

Since $\epsilon > 0$ is arbitray, we have the first inequality in (3.4).

The second of these inequality in (3.4) is the heart of the matter. Again fix any $\epsilon > 0$. By the definition of L_+, there is a function $h \in X$ with

$$0 \le h(x) \le f(x) + g(x) \tag{3.5}$$

for all x, and

$$L(h) \ge L_+(f+g) - \epsilon\ .$$

Define h_f by

$$h_f(x) = \min\{f(x), h(x)\}$$

and $h_g = h - g$.

For x such that $h_f(x) = f(x)$,

$$h_g(x) = h(x) - h_f(x) = h(x) - f(x) \le g(x)$$

by (3.8), while for x such that $h_f(x) = h(x)$, $h_g(x) = h(x) - h(x) = 0$. In either case,

$$0 \le h_f \le f \qquad \text{and} \qquad 0 \le h_g \le g\ .$$

Therefore, by the linearity of L and the definition (1) of L_+,

$$L(h) = L(h_f) + L(h_g) \le L_+(f) + L_+(g)\ .$$

This shows that $L_+(f+g) \le L_+(f) + L_+(g) + \epsilon$, and since $\epsilon > 0$ is arbitrary, the second inequality in (3.4) is proved. ■

We now wish to extend L_+ to all of X. Suppose that $f \in X$ and let

$$f = g - h \qquad \text{and} \qquad f = \tilde{g} - \tilde{h} \tag{3.6}$$

be two ways of writing f as the difference on non-negative elements of X. Then

$$g + \tilde{h} = \tilde{g} + h$$

and by this together with the lemma,

$$L_+(g) + L_+(\tilde{h}) = L_+(g + \tilde{h}) = L_+(\tilde{g} + h) = L_+(\tilde{g}) + L_+(h)\ .$$

Clearly then,

$$L_+(f) - L_+(g) = L_+(\tilde{f}) - L_+(\tilde{g})\ . \tag{3.7}$$

Therefore, we can extend L_+ to all of X by defining $L_+(f) = L_+(g) - L_+(h)$, where f and g are any non–negative functions in X such that $f = g - h$, because the reuslt is independent of the choice of g and h by the analysis that led from (3.6) to (3.7).

To be specific, we may as well take $g = f_+$ and $h = f_-$:

3.2 DEFINITION. L_+ is defined on X by

$$L_+(f) = L_+(f_+) - L_+(f_-)\ ,$$

and L_- is defined on X by

$$L_-(f) = L_+(f) - L(f)\ . \tag{3.8}$$

3.3 THEOREM. *Both L_+ and L_- are bounded linear functionals on X*

Proof We first show that L_+ is linear. Clearly it is homogenous, so it suffices to show that for all f_1 and f_2 in X, $L_+(f_1 + f_2) = L_+(f_1) + L_+(f_2)$. But

$$(f_1 + f_2)_+ - (f_1 + f_2)_- = ((f_1)_+ + (f_2)_+) - ((f_1)_- + (f_2)_-)$$

and so by the independence property established above, i.e., that (3.6) implies (3.7), and then the additivity of L_+ on positve functions,

$$\begin{aligned}
L_+(f_1 + f_2) &= L_+((f_1 + f_2)_+) - L_+((f_1 + f_2)_-) \\
&= L_+((f_1)_+ + (f_2)_+) - L_+((f_1)_- + (f_2)_-) \\
&= L_+((f_1)_+) + L_+((f_2)_+) - L_+((f_1)_-) - L_+((f_2)_-) \\
&= [L_+((f_1)_+) - L_+((f_1)_-)] - [L_+((f_2)_+) - L_+((f_2)_-)] \\
&= L_+(f_1) + L_+(f_2)\ .
\end{aligned}$$

This proves the linearity. Finally, for $f \ge 0$, it is clear that

$$0 \le L_+(f) \le \sup\{L(h) \mid 0 \le h \le f\} \le \sup\{\|L\|_{\mathcal{M}(X)}\|h\|_\infty \mid 0 \le h \le f\} \le \|L\|_{\mathcal{M}(X)}\|f\|_\infty\ .$$

Evidently then,

$$\begin{aligned} |L_+(f)| &\leq \max\{L_+(f_+)\ ,\ L_+(f_-)\} \\ &\leq \|L\|_{\mathcal{M}(X)} \max\{\|f_+\|_\infty\ ,\ \|f_-\|_\infty\} \\ &\leq \|L\|_{\mathcal{M}(X)}\|f\|_\infty\ . \end{aligned}$$

This shows that L_+ is bounded, and that in fact,

$$\|L_+\|_{\mathcal{M}(X)} \leq \|L\|_{\mathcal{M}(X)}\ .$$

Thus it is shown that L_+ is a bounded linear functional. Finally, since L_- is defined as the difference of two bounded linear functionals, it is clear that L_- is itself a bounded linear functional. ■

By the remarks made at the beginning of this section, we know that there are positive finite Borel measures μ_+ and μ_- so that for all $f \in X$

$$L_+(f) = \int_X f\mathrm{d}\mu_+ \qquad \text{and} \qquad L_-(f) = \int_X f\mathrm{d}\mu_-\ . \tag{3.9}$$

It follows that for all $f \in X$,

$$L(f) = \int_X f\mathrm{d}\mu_+ - \int_X f\mathrm{d}\mu_-\ . \tag{3.10}$$

We have now shown that every L in $\mathcal{M}(X)$ is represented by the difference of two finite Borel measures, as in (3.10). However, the particualr measures in (3.10) that we constructed using the Hahn decomposition technique are special. It turns out that:

• *The measures μ_+ and μ_- "live on separate subsets of X. That is, there is a Borel set A such that $\mu_-(A) = 0$ and $\mu_+(A^c) = 0$.*

• *The norm of L, $\|L\|_{\mathcal{M}}$, is given by the total masses of μ_+ and μ_- through*

$$\|L\|_{\mathcal{M}} = \mu_+(X) + \mu_-(X)\ . \tag{3.11}$$

These facts are quite useful. Before discussing them further, Let us define a term for the circumstance that two measures "live on separate subsets sets":

3.4 DEFINITION (Mutually Singular)**.** Two positive measures μ_1 and μ_2 on a sigma algebra $\mathcal{S}$ are *mutually singular* in case there is a measurable set A so that

$$\mu_1(A^c) = 0 \qquad \text{and} \qquad \mu_2(A) = 0\ . \tag{3.12}$$

3.5 LEMMA. *The two measures μ_+ and μ_- in (3.11) are mutually singular.*

Proof: Define the finite Borel mesure μ by $\mu = \mu_+ + \mu_-$. Then μ_+ and μ_- are absolutely continuous with respect to μ, and so there exist non-negative Borel functions g_+ and g_- in $L^1(X, \mathcal{B}, \mu)$ such that $\mu_+ = g_+\mu$ and $\mu_- = g_-\mu$.

Fix any $\delta > 0$. Let E denote the set

$$\{x \ : \ g_-(x) \geq g_+(x) \ \} \cap \{ \ x \ : \ g_+(x) > \delta \ \} \ .$$

Suppose $\mu(E) > 0$. Then, by inner regularity, there exists a compact set K such that $K \subset E$ and $\mu(K) > \mu(E)/2$. Fix $\epsilon > 0$ (and much smaller than $\mu(K)$). By outer regularity, there exist an open set U with $K \subset U$ and $\mu(K) \geq \mu(U) - \epsilon$. By Urysohn's Lemma, there exists a continuous function f with $K \prec f \prec U$. Then for any h with $0 \leq h \leq f$,

$$L(h) = L_+(h) - L_-(h) = \int_X hg_+ \mathrm{d}\mu - \int_X hg_- \mathrm{d}\mu = \int_K h(g_+ - g_-)\mathrm{d}\mu + \int_{U\backslash K} h(g_+ - g_-)\mathrm{d}\mu \ .$$

But on K, $g_+ - g_- \leq 0$, and

$$\int_{U\backslash K} h(g_+ - g_-)\mathrm{d}\mu \ \leq \ \int_{U\backslash K} (g_+ + g_-)\mathrm{d}\mu \ \leq \ \mu(U\backslash K) \ \leq \ \epsilon \ .$$

Therefore,

$$L(h) \leq \epsilon \qquad \text{for all} \qquad 0 \leq h \leq f \ .$$

By the definition of L_+, this means that $L_+(f) \leq \epsilon$. Then since $K \prec f$, $\mu_+(K) \leq \epsilon$. But since $g_+ > \delta$ on E and hence on K, $\mu(K) \leq \epsilon\delta$. Since $\epsilon > 0$ is arbitrary, $\mu(K) = 0$. Thus, it is impossible that $\mu(E) > 0$ for any $\delta > 0$, and hence for all x, $g_+(x)g_-(x) = 0$.

Therefore, define $A = \{x \ : \ g_+(x) > 0 \ \}$. Then $\mu_+(A^c) = 0$, and $\mu_-(A) = 0$. ■

We now show that there is only one way to write $L \in \mathcal{M}$ as the difference of two mutually singular Borel measures. In particular, there is only one way to represent L as the difference of two mutually singular Borel measures: the way given in (3.10) in terms of the measures μ_+ and μ_- provided by the Hahn decomposition.

3.6 LEMMA. *Let μ_1, μ_2, ν_1 and ν_2 be Borel measures such that for all continuous functions f,*

$$\int_X f\mathrm{d}\mu_1 - \int_X f\mathrm{d}\mu_2 = \int_X f\mathrm{d}\nu_1 - \int_X f\mathrm{d}\nu_2 \ . \tag{3.13}$$

Suppose also that μ_1 and μ_2 are mutualy singular, and that ν_1 and ν_2 are mutually singular as well. Then $\mu_1 = \nu_1$ and $\mu_2 = \nu_2$.

Proof: Let A be such that (3.12) holds, and let B be such that

$$\nu_1(B^c) = 0 \qquad \text{and} \qquad \nu_2(B) = 0 \ . \tag{3.14}$$

It follows directly from (3.12) and (3.14) that

$$\mu_2(A \cap B) = 0 \qquad \text{and} \qquad \nu_2(A \cap B) = 0 \ .$$

We now claim that

$$\mu_1(A^c \cup B^c) = 0 \qquad \text{and} \qquad \nu_1(A^c \cup B^c) = 0 \ . \tag{3.15}$$

Therefore, if we define $C = A \cap B$, then (3.12) and (3.14) both continue to hold true if we replace A and B respectively by C. To see the validity of (3.15), note that

$$\mu_1(A \cap B^c) + \nu_2(A \cap B^c) = 0 \ .$$

In particular, since $\mu_1(A^c) = 0$, $\mu_1(A^c \cup B^c) = 0$. In the same way, we see that $\nu_1(A^c \cup B^c) = 0$.

Now because of (3.13), for any Borel set E,

$$\mu_1(E) - \mu_2(E) = \nu_1(E) - \nu_2(E) \ .$$

But then,

$$\mu_1(E) = \mu_1(E \cap C) = \mu_1(E \cap C) - \mu_2(E \cap C) = \\ \nu_1(E \cap C) - \nu_2(E \cap C) = \nu_1(E \cap C) = \nu_1(E) \ .$$

Thus, $\mu_1 = \nu_1$. It now follows that $\mu_2 = \nu_2$. ■

Finally, we show how the fact that μ_+ and μ_- are mutually singular implies that (3.11) holds. As usual, we prove this equality using two inequalities. The first is easy, and does not use the fact that μ_+ and μ_- are mutually singular.

$$\begin{aligned} |L(f)| &= \left| \int_X f \mathrm{d}\mu_+ - \int_X f \mathrm{d}\mu_- \right| \\ &\leq \int_X |f| \mathrm{d}\mu_+ + \int_X |f| \mathrm{d}\mu_- \\ &\leq (\mu_+(X) + \mu_-(X)) \, \|f\|_\infty \ . \end{aligned}$$

This shows that

$$\|L\|_{\mathcal{M}(X)} \leq \mu_+(X) + \mu_-(X) \ .$$

For the other bound, let A be a Borel set such that $\mu_+(A^c) = \mu_-(A) = 0$, and let

$$f = 1_A - 1_{A^c} \ .$$

Evidently, $\|f\|_\infty = 1$. Now let ν be the measure

$$\mu = \mu_+ + \mu_- \ .$$

We know that we can approximate any bounded measurable function arbitrarily closely in the $L^1(X, \mu)$ norm by a continuous function without increasing the supremum norm. Thus, for any $\epsilon > 0$, there is a continuous function $\tilde{f}$ such that $\|\tilde{f} - f\|_{L^1(X,\mu)} \leq \epsilon$ and $\|\tilde{f}\|_\infty \leq 1$. Then

$$\begin{aligned} |L(\tilde{f})| &= \left| \int_X \tilde{f} \mathrm{d}\mu_+ - \int_X \tilde{f} \mathrm{d}\mu_- \right| \\ &\geq \left| \int_X f \mathrm{d}\mu_+ - \int_X f \mathrm{d}\mu_- \right| - \int_X |f - \tilde{f}| \mathrm{d}\nu \\ &= \mu_+(A) + \mu_-(A^c) - \epsilon \\ &\geq \mu_+(X) + \mu_-(X) - \epsilon \ . \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have from this and (15) that $\|L\|_{\mathcal{M}(X)} = \mu_+(X) + \mu_-(X)$.

Summarizing, we have proved the following theorem:

3.7 THEOREM (Hshn-Saks Theorem)**.** *Every continuous linear functional L on $\mathcal{C}_0(X)$ is of the form given in (11) where the measures μ_+ and μ_- are mutually singular finite Borel measures. Moreover,*

$$\|L\|_{\mathcal{M}(X)} = \mu_+(X) + \mu_-(X) \ .$$

We can deduce a number of consequences of this analysis that deserve further discussion. We begin with a deinition:

3.8 DEFINITION. A *signed measure* on X is a real valued function μ on the Borel σ–algebra of X such that there exist two positive finite borel measures μ_1 and μ_2 such that for all Borel sets E,

$$\mu(E) = \mu_1(E) - \mu_2(E) \ . \tag{24}$$

The set of signed measures is evidently a real vector space. (Complex measures are definied in the analougous way, and would constitute a complex measure space.) If f is a bounded measurable function, we define the integral $\int_X f\mathrm{d}\mu$ by

$$\int_X f\mathrm{d}\mu = \int_X f\mathrm{d}\mu_1 - \int_X f\mathrm{d}\mu_2 \ . \tag{25}$$

We get a continuous linear functional L on $\mathcal{C}_0(X)$ from

$$L(f) = \int_X f\mathrm{d}\mu. \tag{3.16}$$

The Hahn-Saks Theorem then gives us the existence of uniquely determined positive Borel measures μ_+ and μ_- that are mutually singular – i.e., supported on disjoint sets – and such that

$$\mu(E) = \mu_+(E) - \mu_-(E) \tag{3.17}$$

for all Borel sets E in X.

3.9 DEFINITION. For any signed measure μ, the positve measure $|\mu|$ given by

$$|\mu| = \mu_+ + \mu_-$$

is called the *total variation measure* of μ, and and the number

$$\|\mu\|_{\mathrm{TV}} = \mu_+(X) + \mu_-(X)$$

is called the *total variation norm* of μ.

It is easy to see from our analysis above that

$$\|\mu\|_{\mathrm{TV}} = \sup\left\{ \int_X f\mathrm{d}\mu \ \middle|\ f \in C_0(X) \ , \ -1 \le f \le 1 \right\} \ , \tag{3.18}$$

and from this the Minkowski inequality is easily seen to hold, so that $\|\cdot\|_{\mathrm{TV}}$ is actually a norm, as the name indicates, and in fact, $\|\cdot\|_{\mathrm{TV}} = \|\cdot\|_{\mathcal{M}}$. We will use these notations interchangably.

We know that the dual of a Banach space is complete in the dual norm, and so $\mathcal{M}(X)$ is complete in the total variation norm. Moreover $\mathcal{C}_0(X)$ and $\mathcal{M}(X)$ are a dual pair of Banach spaces with the bilnear form $\langle g, \mu \rangle$ defined on $\mathcal{C}_0(X) \times \mathcal{M}(X)$ by

$$\langle g, \mu \rangle = \int_X g \mathrm{d}\mu \ .$$

The fact that $\|\mu\|_{\mathcal{M}} = \sup\{ \ \langle g, \mu \rangle \ : \ \|g\|_\infty \leq 1 \ \}$ is true by definition, and then the fact that $\|\mu\|_{\mathcal{M}} = \|\mu\|_{\mathrm{TV}}$ is given by (3.18).

To see that

$$\|g\|_\infty = \sup\{ \ \langle g, \mu \rangle \ : \ \|\mu\|_{\mathcal{M}} \leq 1 \ \} \ ,$$

observe that there is an $x_0 \in X$ so that $|g(x_0)| = \|g\|_\infty$. Let μ_0 be the signed Borel measure defined by

$$\mu_0(E) = \begin{cases} \mathrm{sgn}(g(x_0) & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \in E^c \end{cases}$$

Then

$$\|g\|_\infty = \int_X g \mathrm{d}\mu_0$$

and $\|\mu_0\|_{\mathcal{M}} = 1$. Hence $\mathcal{C}_0(X)$ and $\mathcal{M}(X)$ are in fact a dual pair of Banach spaces.

However, it is not in general the case that $\mathcal{C}_0(X)$ is the dual of $\mathcal{M}(X)$. For example, let E be any Borel set such that 1_E is not continuous – this is the usual case.

Define a linear functional on $\mathcal{M}$ by

$$\Lambda(\mu) = \mu(E) = \int_X 1_E \mathrm{d}\mu \ .$$

Clealry Λ is linear and

$$|\Lambda(E)| \leq |\mu|(E) \leq |\mu|(X) = \|\mu\|_{\mathcal{M}} \ ,$$

so Λ is indeed bounded, and so is an element of $(\mathcal{M}(X))^*$. But if there were a function $g \in \mathcal{C}_0(X)$ for which

$$\Lambda(\mu) = \int_X g \mathrm{d}\mu \ ,$$

we would have

$$\int_X (1_E - g) \mathrm{d}\mu = 0$$

But this is impossible since g cannot equal 0 or 1 everywhere – no continuous function can. Taking μ to be the Borel measure that is a unit mass concentrated at some point x where $0 < g(x) < 1$, we get a contradition.

This is very similar to the situation we encountered with the dual pair (L^1, L^∞), and this similalirty is worth exploring further.

Let λ be a fixed positive Borel measure measure on X. We say that a signed measure μ in $\mathcal{M}$ is absolutely continuous with respect to λ in case $|\mu|$ is absolutely continuous with respect to λ in the sense already defined for positive measures. Evidently, this amounts to the requirement that both μ_+ and μ_- are absolutely continuous with respect to λ.

Clearly the subset of $\mathcal{M}$ consisting of signed measures that are absolutely continuous with respect to λ is a subspace. Using the Radon-Nikodymn Theorem, we can identify this subspace with $L^1(X, \mathcal{B}, \lambda)$ as follows. Given $f \in L^1(X, \mathcal{B}, \lambda)$, define a signed measure μ_f by

$$\mu_f(E) = \int_X 1_E f \mathrm{d}\lambda$$

for all Borel sets E. Evidently μ_{f_+} and μ_{f_-} are mutually singular and $\mu_f = \mu_{f_+} - \mu_{f_-}$. It follows that

$$|\mu_f| = \mu_{f_+} + \mu_{f_-}$$

and that

$$\begin{aligned} \|\mu_f\|_{\mathcal{M}} &= \mu_{f_+}(X) + \mu_{f_-}(X) \\ &= \int_X (f_+ + f_-)\mathrm{d}\lambda \\ &= \|f\|_{L^1(X,\mathcal{B},\lambda)} \ . \end{aligned}$$

The mapping

$$f \mapsto \mu_f$$

from $L^1(X, \mathcal{B}, \lambda)$ to $\mathcal{M}$ is clearly linear, and by the above, it is an isometry: a norm preserving, and hence distance preserving, map. The Radon–Nikodymn Theorem says that a signed measure μ is in the image of this map if and only if it is absolutely continuous with respect to λ.

Although this subspace is an isometric copy of $L^1(X, \mathcal{B}, \lambda)$, it is not a a closed subspace in the weak $\mathcal{C}_0(X)$ toplogy on $\mathcal{M}(X)$. For example, let $X = [0, 1]$, and let λ denote Lebesgue measure on X. For each positive integer n, define the function f_n in $L^1(X, \mathcal{B}, \lambda)$ by

$$f_n(x) = \begin{cases} n & \text{if } 0 \le x \le 1/n \\ 0 & \text{if } 1/n < x \le 1 \ . \end{cases}$$

This sequence is not a Cauchy sequence in $L^1(X, \mathcal{B}, \lambda)$, and therefore it is not convergent. Indeed, an easy computation shows that for $n > m$,

$$\|f_n - f_m\|_{L^1(X,\mathcal{B},\lambda)} = m\left(\frac{1}{m} - \frac{1}{n}\right) + (n-m)\frac{1}{n} \ .$$

Evidently then,

$$\lim_{n\to\infty} \|f_n - f_m\|_{L^1(X,\mathcal{B},\lambda)} = 1$$

for all n, so this sequence if far from being a Cauchy sequence. Becasue the map $f \mapsto \mu_f$ is an isometry,

$$\lim_{n\to\infty} \|\mu_{f_n} - \mu_{f_m}\|_{\mathcal{M}} = 1$$

and the sequence $\{\mu_{f_n}\}$ is not Cauchy, and hence not convergent, in $\mathcal{M}(X)$.

However, let δ_0 be the Borel measure on X defined by

$$\delta_0(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \in E^c \end{cases} \ .$$

It is clear that for all $g \in \mathcal{C}_0(X)$,

$$\lim_{n\to\infty} \int_X g f_n \mathrm{d}\lambda = g(0) = \int_X g \mathrm{d}\delta_0 \ .$$

This means that in the weak $\mathcal{C}_0(X)$ toplogy on $\mathcal{M}(X)$,

$$\lim_{n\to\infty} \mu_{f_n} = \delta_0 \ .$$

This example shows that the weak $\mathcal{C}_0(X)$ toplogy on $\mathcal{M}(X)$ is the "right" topology for working with the intuition that in some sense we must have $\lim_{n\to\infty} \mu_{f_n} = \delta_0$ since the measures μ_{f_n} are concentrating ever more tightly around $x = 0$. In this topolgy, the statement $\lim_{n\to\infty} \mu_{f_n} = \delta_0$ is true; in the norm topology it is not. Also, you see that while $\{f_n\}$ is "trying to converge to something", that "something" is just not in $L^1(X, \mathcal{B}, \lambda)$, so this would be the wrong space in which to look for the limit. The right space is $\mathcal{M}(X)$, and the right topology is the weak $\mathcal{C}_0(X)$ toplogy. This toplogy is quite frequently used, and so there are briefer terms of refeence for it:

3.10 DEFINITION. The weak $\mathcal{C}_0(X)$ toplogy on $\mathcal{M}(X)$ is called the *vauge topology* or the *weak–$*$* toplogy.

The term "vauge topology" sounds perjorative, which is unfortunate since for many questions, as explained above, this is the "right" toplogy. What people had in mind was this: Since $\mathcal{C}_0(X)$ is only a subset of the dual of $\mathcal{M}(X)$, the weak $\mathcal{C}_0(X)$ topology is weaker than what would usually called the "weak topology", namely the $(\mathcal{M}(X))^*$ weak topology on $\mathcal{M}(X)$. The term "vague", meaning "wave" in French, is meant to indicate "weaker than weak". However, $(\mathcal{M}(X))^*$ is not a nice Banach space, and the $(\mathcal{M}(X))^*$ weak topology on $\mathcal{M}(X)$ is essentially useless, so a comparisson with it is not very meaningful.

We close this section with a nice characterization of weak convergence in $\mathcal{C}_0(X)$. Recal, that since no other space is specifoied, this means the $\mathcal{M}(X)$ weak convergence. **Theroem 4 (Weak Convergence in $\mathcal{C}_0(X)$ and Pointwise Convergence)** *A sequence of functions $\{f_n\}$ in $\mathcal{C}_0(X)$ converges weakly if and only*

$$\sup_n \{\|f_n\|_\infty\} < \infty$$

and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$.

Proof: Suppose that $\{f_n\}$ converges weakly to f in $\mathcal{C}_0(X)$. Let μ be the point mass at x. Then

$$f(x) = \int_X f \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \lim_{n\to\infty} f_n(x) \ .$$

Moreover, the uniform boundedness principle implies that if $\{f_n\}$ converges weakly to f in $\mathcal{C}_0(X)$, then $\sup_n\{\|f_n\|_\infty\} < \infty$.

On the other hand, if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$, and if $|f_n(x)| \leq C \leq \infty$ for all n and x, then

$$\lim_{n\to\infty} \int_X |f_n - f| \mathrm{d}|\mu| = 0$$

for any finite Borel measure μ by dominated convergence since $2C$ is in $L^1(X, \mathcal{B}, |\mu|)$.