Final Exam for Math 502, Real Analysis II, Spring 2009

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NAME:

Instructions: Solve any five of the following six problems. Only turn in your work on the five problems you want graded. List the five problems you are turning in after your name. There is no time limit. You may consult your notes and Folland's Text, but nothing else.

1. Let $X = L^\infty(\mathbb{R}, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra, and μ is Lebesgue measure. For each $n \in \mathbb{N}$, define the linear functional $\varphi_n \in X^*$ by

$$\varphi_n(f) = \frac{1}{2n}\int_{[-n,n]} f \mathrm{d}\mu$$

for all $f \in X$. Show that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ has a weak-$*$ cluster point $\varphi \in X^*$, and that there is no finite Borel measure ν such that $\varphi(f) = \int_{\mathbb{R}} f\mathrm{d}\nu$ for all $f \in X$.

2. (a) Let X and Y be Banach spaces. Let $T : X \to Y$ be a linear map $f \circ T \in X^*$ for all $f \in Y^*$. Prove that T is bounded.

(b) Let X be a separable Banach space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence that is dense in the unit ball of X. Let μ be counting meausre on $\mathbb{N}$, and define $T : L^1(\mu) \to X$ by $T(f) = \sum_{n=1}^\infty f(n)x_n$. Show that T is bounded, and is surjective; i.e., the range of T is X.

3. Let L^2 denote $L^2(\mathbb{R}, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra, and μ is Lebesgue measure. For $y \in \mathbb{R}$, let τ_y be the translation operator on X; i.e.,$\tau_y f(x) = f(x - y)$. For $h > 0$ define the operator D_h on L^2 by

$$D_h f(x) = \frac{1}{h}(I - \tau_h) \ ,$$

where I is the identity. Show that for all $f \in L^2$, $\lim_{h\to 0} D_h f$ exists in L^2 (strongly) if and only if $\int_{\mathbb{R}} |k|^2|\widehat{f}(k)|^2\mathrm{d}k < \infty$.

4. Let (X, d) be a metric space. Let $\mathcal{C}_b(X)$ denote the space of continuous real valued, uniformly bounded functions f on X. For each $f \in \mathcal{C}_b(X)$, and each $k \in \mathbb{N}$, define

$$f_k(x) = \inf\{\ f(y) + kd(x, y) \ :\ y \in X\ \} \ .$$

(a) Show that for each x $\{f_k(x)\}_{k\in\mathbb{N}}$ is a monotone non-decreasing sequence with

$$\inf_{y\in X} f(y) \le f_k(x) \le f(x) \qquad \text{and} \qquad \lim_{k\to\infty} f_k(x) = f(x) \ .$$

(b) for each k, f_k is Lipschitz continuous with Lipschitz constant k; i.e., for all $x, z \in X$,

$$|f_k(x) - f_k(z)| \leq kd(x,z) \ .$$

(c) Let $\mathrm{Lip}_b(X)$ be the set of bounded, continuous, real valued functions f on X such that f is k-Lipschitz for some finite k. Show that if X is compact, then $\mathrm{Lip}_b(X)$ is uniformly dense in $\mathcal{C}_b(X)$.

5. (a) Let X be a locally compact Hausdorff space, and let $\mathcal{M}(X)$ denote $C_0(X)^*$. Show that if $\lim_{n\to\infty} \mu_n = \mu$ in the weak-$*$ topology on $\mathcal{M}(X)$, and if $\lim_{n\to\infty} \|\mu_n\| = \|\mu\|$, then

$$\lim_{n\to\infty} \int_X f \mathrm{d}\mu_n = \int_X f \mathrm{d}\mu \ ,$$

and that the hypotheses that $\lim_{n\to\infty} \|\mu_n\| = \|\mu\|$ cannot be dropped.

(b) Specialize to the casr $X = \mathbb{R}$, and give an example in which $\lim_{n\to\infty} \mu_n = 0$ in the weak-$*$ topology on $\mathcal{M}(X)$, but $\lim_{n\to\infty} \|\mu_n\| \neq 0$.

6. Let g be a bounded continuous function on $\mathbb{R}$ supported by $[-1,1]$. Let K denote the set of bounded, measurable functions f on $\mathbb{R}$ with such that $|f(x)| \leq 1$ almost everywhere, and $|f(x)| = 0$ for almost every $|x| > 1$. Define the functional Φ on K by

$$\Phi(f) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(x-y) f(y) \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

where μ is Lebesgue measure. Prove that there exists some $f_0 \in K$ such that

$$\Phi(f_0) \leq \Phi(f)$$

for all $f \in K$.