Exercise Set 3 for Real Analysis II, Spring 2009

Feb 20, 2009

1. When $L^2(X, \mathcal{M}, \mu)$ is infinite dimensional, the weak toplogoy is strictly weaker than the strong; i.e., norm, topology. Therefore, there exist sets that are closed in the strong topology, but not in the weak topology. However for one important class of sets – convex sets – this is not the case: Every norm closed convex set is weakly closed. The purpose of this exercise is to prove this.

(a) Let K be a non-empty strongly closed convex set in $L^2(X, \mathcal{M}, \mu)$. Suppose that there exists some f in $L^2(X, \mathcal{M}, \mu)$ that belongs to the weak closure of K, but not to K. We wish to show that this is impossible.

Assuming that such an f exists, let g be the unique element of K satisfying

$$\|f - g\|_2^2 \leq \|f - h\|_2^2 \qquad \text{for all} \qquad h \in K \ .$$

Define a linear bounded functional φ on $L^2(X, \mathcal{M}, \mu)$ by

$$\varphi(h) = \int_X h(g - f)\mathrm{d}\mu \ .$$

Show that

$$\varphi(h) \geq \varphi(g) \qquad \text{for all} \qquad h \in K \ .$$

(b) Show that

$$\varphi(g) = \varphi(f) + \|f - g\|_2^2 \ .$$

(c) Combining parts **(a)** and **(b)**, show that $\varphi(h) \geq \varphi(f) + \|f - g\|_2^2$ for all $h \in K$, and use this to conclude that f is not a weak limit point of K, contrary to the hypothesis on f that was made in part **(a)**.

(d) Give an example of a set in some $L^2(X, \mathcal{M}, \mu)$ space that is strongly closed but not weakly closed. By the first parts of this exercise, the set in your example must be non-convex!

2. We have seen that if $L^2(X, \mathcal{M}, \mu)$ is separable, then the relative weak topology on the unit ball (and hence any bounded set) is metrizable. In fact, we have written down a metric (defined on the whole space) such that the corresponding topology on an any bounded set agrees with the relative weak topology. The construction of this metric made explicit use of a dense sequence in $L^2(X, \mathcal{M}, \mu)$, and obviously this only exists in the separable case.

Hence we come to the question: Is our hypothesis that $L^2(X, \mathcal{M}, \mu)$ is separable a natural hypothesis? One might hope that one could find another proof that did not use this hypothesis. In this exercise, we shall settle this question.

(a) Show that a compact metric space is always separable (with respect to the metric topology). You can more or less extract this from our proof that for metric spaces, compactness is the same as sequential compactness.

(b) By the Banach-Alaoglu Theorem (essentially a direct consequence of the Tychonov Theorem), for *any* $(X\mathcal{M},\mu)$, the unit ball in $L^2(X,\mathcal{M},\mu)$ is weakly compact. Suppose that relative weak topology on the unit ball of $L^2(X,\mathcal{M},\mu)$ is metrizable. Show that in this case, $L^2(X,\mathcal{M},\mu)$ is separable with respect to the weak topology.

(c) Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a weakly dense sequence in $L^2(X,\mathcal{M},\mu)$. That is, the smallest weakly closed subset of $L^2(X,\mathcal{M},\mu)$ that contains $\{f_n\}_{n\in\mathbb{N}}$ is $L^2(X,\mathcal{M},\mu)$ itself. Let V be the smallest strongly closed subspace of $L^2(X,\mathcal{M},\mu)$ that contains $\{f_n\}_{n\in\mathbb{N}}$. Show that V is separable in the relative strong topology.

(d) Continuing in the notation of part **(c)**, if $V = L^2(X,\mathcal{M},\mu)$, then we have that $L^2(X,\mathcal{M},\mu)$ is strongly separable, and hence the unit ball is weakly metrizable *only if* $L^2(X,\mathcal{M},\mu)$ is strongly separable. Use the main result of Exercise **1** to show that V cannot be a proper subspace of $L^2(X,\mathcal{M},\mu)$, and that weak separability and strong separability are the same thing after all.

(e) Put the pieces together to show that the relative weak topology on the the unit ball in $L^2(X,\mathcal{M},\mu)$ is metrizable if and only if $L^2(X,\mathcal{M},\mu)$ is separable.