Notes on Harmonic Analysis on $\mathbb{R}^n$

Eric A. Carlen[1]
Rutgers University

April 27, 2009

Contents

1 Introduction

Lately we have been studying various aspects of real analysis of functions and measures on fairly general topological spaces – epsecial locally compact Hausdorff spaces and complete, separable metric spaces. $\mathbb{R}^n$ is both of these and more: It is also an abelian group. This group structure adds a special flavor to the analysis of functions and measures on $\mathbb{R}^n$.

Let f be any measurable function on $\mathbb{R}^n$, and fix any $y \in \mathbb{R}^n$. Then since the map $x\ mapsto x - y$ is clearly continuous from $\mathbb{R}^n$ to $\mathbb{R}^n$, and since the composition of measurable functions with continuous functions results in measurable functions, the function $\tau_y f$ defined by

$$\tau_y f(x) = f(x - y) \tag{1.1}$$

Throughout this section μ shall denote Lebesgue measure on $\mathbb{R}^n$, and for $!1 \leq p \leq \infty$, L^p shall denote $L^p(\mathbb{R}^n, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra.

Since Lebesgue measure is translation invariant; i.e., for all $E \in \mathcal{B}$, and all $y \in \mathbb{R}^n$,

$$\mu(\tau_y^{-1}(E) = \mu(E) \ ,$$

and consequently, for all $1 \leq p \leq \infty$, and all $f \in L^p$,

$$\|\tau_y f\|_p = \|f\|_p \ . \tag{1.2}$$

The basis of everything that follows in this section of notes is the following fundamental theorem:

1.1 THEOREM (Continuity of translation). *For all* $1 \le p < \infty$*, and all* $f \in L^p$*,*

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0 \ . \tag{1.3}$$

That is, for all $1 \le p < \infty$*, and all* $f \in L^p$*, the map* $y \mapsto \tau_y f$ *is continuous from* $\mathbb{R}^n$ *to* L^p*. For* $p = \infty$*, this map is not continuous except in trivial cases such as when* f *is constant.*

Proof: Since $\mathbb{R}^n$ is a locally compact Hausdorff space, and $1 \le p < \infty$, we know that for any $\epsilon > 0$, there exists a $g \in \mathcal{C}_c(\mathbb{R}^n)$ such that $\|g - f\|_p < \epsilon$. But then by Minkowski's inequality,

$$\begin{aligned} \|\tau_y f - f\|_p &\le \|\tau_y f - \tau_y g\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &\le \epsilon + \|\tau_y g - g\|_p + \epsilon \ , \end{aligned} \tag{1.4}$$

where we have used the fact that τ_y is linear and (1.2) to conclude

$$\|\tau_y f - \tau_y g\|_p = \|\tau_y (f - g)\|_p = \|f - g\|_p \le \epsilon \ .$$

Next since $g \in in\mathcal{C}_c(\mathbb{R}^n)$, there is some $R > 0$ such that for all $|y| < 1$ the support of $\tau_y g$ lies in $B_R(0)$, the centered ball of radius R in $\mathbb{R}^n$. Furthermore, since every continuous compactly supported function on $\mathbb{R}^n$ is uniformly continuous, there is a $\delta > 0$ so that

$$|y| < \delta \Rightarrow |\tau_y g(x) - g(x)| \le \frac{\epsilon}{\mu(B_R(0))}$$

uniformly in x. It then follows that

$$|y| < \delta \Rightarrow \|\tau_y g - g\| \le \epsilon \ .$$

Combining this with (1.4), we have $\|\tau_y f - f\|_p \le 3\epsilon$ whenever $|y| \le \delta$.

To see that $y \mapsto \tau_f f$ is not generally continuous in L^∞, consider the case in which f is the charactersitic function of the unit ball in $\mathbb{R}^n$. Then it is clear that for all $y \ne 0$, $\|\tau_y f - f\|_\infty = 1$. ■

2 Convolutions

2.1 The convolution of functions in L^p spaces

For $f \in L^\infty$ and $g \in L^1$, define $f * g$, the convolution of f and g, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \mathrm{d}\mu(y) \ . \tag{2.1}$$

for each x, the integrand is clearly integrable as a function of y, and we can do the integral pointwise, for each x. It is clear that the result is abounded function of x: We have $\|f * g\|_\infty \le \|f\|_\infty \|g\|_1$

Notice that (2.1) can be written, formally at least, as

$$f * g(x) = \int_{\mathbb{R}^n} \tau_y f(x) g(y) \mathrm{d}\mu(y) \ .$$

When g is non-negative and satisfies $\int_{\mathbb{R}^n} g\mathrm{d}\mu = 1$, this allows us to think of $f * g$ as an "average of translates of f".

It turns out that $f * g$ is well defined for $g \in L^1$ and $f \in L^p$ for *any* $1 \leq p \leq \infty$. This is most clear for $f \in L^1$. Then by Fubini's Theorem, for *almost every* x, the integrand in (2.1) is integrable, so that $f * g(x)$ is defined almost everywhere and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

with equality holding in case f and g are non-negative. Also, making several simple changes of variable,

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} f(-y)g(y+x)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} g(x-y)f(y)\mathrm{d}\mu(y) - g * f(x) \ .$$

That is, the convolution product is commutative. This product structure makes L^1 a commutative algebra. However, our main interest here is not so much with the convolution algebera on L^1 as it is in the extension of the convolution product to f and g in other L^p and L^q spaces for which the integral in (2.1) is not integrable (almost everywhere) for such obvious reasons as it is when $g \in L^1$ and $f \in L^\infty$ or $F \in L^1$.

In what follows p' shall always denote the conjugate index to p; that is,

$$\frac{1}{p} + \frac{1}{p'} = 1 \ .$$

For all $h \in L^{p'}$ and all $f \in L^p$, $1 \leq p < \infty$, the map

$$y \mapsto \int_{\mathbb{R}^n} h(x)\tau_y f(x)\mathrm{d}\mu(x)$$

is continuous by Theorem 1.1), and therefore measurable. By Hölder';s inequality and (1.2), it is bounded by $\|h\|_{p'}\|f\|_p$. Therefore, for $g \in L^1$,

$$h \mapsto \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} h(x)\tau_y f(x)\mathrm{d}\mu(x)\right] g(y)\mathrm{d}\mu(y)$$

is a well-defined linear functional on $L^{p'}$, and moreover, it is bounded by $\|f\|_p\|g\|_1$.

By the Riesz Representation Theorem, there is then a unique element of L^p that we *define* to be $f * g$ such that

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} h(x)\tau_y f(x)\mathrm{d}\mu(x)\right] g(y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} h(x) f * g(x)\mathrm{d}\mu(x) \ ,$$

and

$$\|f * g\|_p \leq \|f\|_p\|g\|_1 \ . \tag{2.2}$$

If $f \in L^1 capL^p$ and $h \in L^\infty$
$capL^{p'}$, then we can apply Fubini's Theorem to conclude that

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} h(x)\tau_y f(x)\mathrm{d}\mu(x)\right] g(y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} h(x)\left[\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y)\right]\mathrm{d}\mu(x) \ ,$$

so in this case $f*g(x)$ is given *pointwise* by the integral $\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y)$, so that our notation is consistent with (refconv1). The definition that we have just made, using the Riesz Representation Theorem, extends the definition of $f*g$ from $f \in L^1 capL^p$, $g \in L^1$ to all of $f \in L^p$, $g \in L^1$. Moreover, we see that with this definition, $\|f*g\|_p \le \|f\|_p\|g\|_1$.

There is one more case in which we can easily make pointwise sense of the integral $\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y)$: Suppose $f \in L^p$ and $g \in L^{p;}$. Then by Hölder's inequality,

$$\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y) \le \left(\int_{\mathbb{R}^n} |f(x-y)|^p \mathrm{d}\mu(y)\right)^{1/p} \left(\int_{\mathbb{R}^n} |g(y)|^{p'} \mathrm{d}\mu(y)\right)^{1/p'} = \|f\|_p\|g\|_{p'} \ .$$

That is, when $f \in L^p$ and $g \in L^{p'}$, $f*g$ is defined pointwise by (2.1), and moreover,

$$\|f*g\|_\infty \le \|f\|_p\|g\|_{p'} \ . \tag{2.3}$$

We now fix some $f \in L^p$, $1 \le p < \infty$, and define a linear transformation T on $L^1 \cap L^\infty$ by

$$g \mapsto Tg = f*g \ .$$

We see from (2.2) and (2.3) that for any non-zero $g \in L^1 \cap L^\infty$,

$$\frac{\|Tg\|_p}{\|g\|_1} \le \|f\|_p \qquad \text{and} \qquad \frac{\|Tg\|_p}{\|g\|_{p'}} \le \|f\|_p \ . \tag{2.4}$$

In other words, the norm of T as an operator from L^1 to L^p, $\|T\|_{1\to p}$, is bounded by $\|f\|_p$, and the norm of T as an operator from $L^{p'}$ to L^∞, $\|T\|_{p'\to\infty}$, is also bounded by $\|f\|_p$.

We may now apply the Riesz-Thorin Theorem to conclude that T extends to be a bounded operator from L^q to L^r where for $0 \le t \le 1$,

$$\frac{1}{q} = t\frac{1}{p'} + (1-t)\frac{1}{1} \qquad \text{and} \qquad \frac{1}{r} = t\frac{1}{\infty} + (1-t)\frac{1}{p} \ . \tag{2.5}$$

The Riesz-Thorin Theorem says that then

$$\|T\|_{q\to r} \le \left(\|T\|_{p'\to\infty}\right)^t \left(\|T\|_{1\to p}\right)^{1-t} \le \|f\|_p^t\|f\|_p^{1-t} = \|f\|_p \ .$$

That is, for any $g \in L^1 \cap L^\infty$,

$$\|f*g\|_r \le \|g\|_q\|f\|_p \ , \tag{2.6}$$

with q and r given by (2.5) for some $0 \le t \le 1$. In fact, it is easy to eliminate t from (2.5): Subtracting, one finds

$$\frac{1}{q} - \frac{1}{r} = t\left(\frac{1}{p'} + \frac{1}{p}\right) + (1-t) - \frac{1}{p} = 1 - \frac{1}{p} \ .$$

Thus, (2.6) holds whenever p, q and r satisfy $1/p + 1/q = 1 + 1/r$. We now show that $\|f*g\|_r$ cannot be bounded by *any* multiple of $\|g\|_q\|f\|_p$ unless $1/p+1/q = 1+1/r$. To see this, we note that the dual formula for the L^r norm, (2.6) is equivalent to $\int_{\mathbb{R}^n} h(x)f*g(x)\mathrm{d}\mu(x) \le \|h\|_{r'}\|g\|_q\|f\|_p$ for all $h \in L^{r'}$. In other words,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)g(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \le \|h\|_{r'}\|g\|_q\|f\|_p \ , \tag{2.7}$$

Now fix $\lambda > 0$, and make the repalcements $h(x) \to h_\lambda(x) := h(x/\lambda)$, $f(x) \to f_\lambda(x) := f(x/\lambda)$ and $g(x) \to g_\lambda(x) := g(x/\lambda)$. Changing variables, one computes that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x/\lambda)f((x-y)/\lambda)g(y/\lambda)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \lambda^{2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)g(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y)\ .$$

Likewise

$$\left(\int_{\mathbb{R}^n} |h(x/\lambda)|^{r'}\mathrm{d}\mu(x)\right)^{1/r'} = \lambda^{n/r'}\|h\|_{r'}\ ,$$

with similar identities for f and g. The results is that if (2.7) holds for h_λ, f_λ and g_λ for all $\lambda > 0$, we have

$$\lambda^{2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)g(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \le \lambda^{n/p'+n/q+n/p}\|h\|_{r'}\|g\|_q\|f\|_p\ .$$

Clearly, the powers of λ on the two sides of this inequality must be equal, or else we can get a contradiction by taking either $\lambda \to 0$ (if the power on the left is the lesser) or $\lambda \to \infty$ (if the power on the left is the greater). Hence, the inequality (2.7), or equivalently, the inequality (2.6) cannot hold in general unless

$$2 = \frac{1}{r'} + \frac{1}{q} + \frac{1}{r}\ ,$$

which of course is equivalent to $1/p + 1/q = 1 + 1/r$.

Let us summarize our conclusions in a theorem:

2.1 THEOREM (Young's inequality)**.** *For all* $1 \le p, q, r \le \infty$ *with* $1/p + 1/q = 1 + 1/r$*, and all* $f \in L^p$*, the linear transformation*

$$g \mapsto f * g := \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y)$$

defined initially for $g \in L^1 \cap L^\infty$*, so that the integrand is integrable for each* x*, extends to be a contractive linear transformation form* L^q *to* L^r *so that in particular, (2.7) holds for all* $h \in L^{r'}$*,* $f \in L^p$ *and* $g \in L^q$*.*

2.2 Approximate identities

The following theorem expresses an important continuity property of convolutions, and has many applications:

2.2 THEOREM (Approximate identities)**.** *For any* $g \in L^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} g(x)\mathrm{d}\mu(x) = 1$*, and any* $t > 0$*, define* $g_t(x) = t^{-n}g(x/t)$*. Then, for all* $1 \le p < \infty$*, and all* $f \in L^p$*,*

$$\lim_{t\to 0}\|f - f * g_t\|_p = 0\ . \tag{2.8}$$

Proof: First suppose that g has compact support, so that for some $R < \infty$, $g(y) = 0$ for all $|y| > R$. Then note that, making the obvious change of variables,

$$f * g_t(x) = \int_{\mathbb{R}^n} t^{-n}f(x-y)g(y/t)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} f(x-ty)g(y)\mathrm{d}\mu(y)\ .$$

Also, since $\int_{\mathbb{R}^n} g(x)\mathrm{d}\mu(x) = 1$,

$$f(x) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}\mu(x) \ .$$

Therefore,

$$f * g_t(x) - f(x) = \int_{\mathbb{R}^n} [f(x-ty) - f(t)]\, g(y)\mathrm{d}\mu(y) \ .$$

By Theorem 1.1, for all $\epsilon > 0$, there exists a $\delta > 0$ so that

$$t|y| < \delta \Rightarrow \|\tau_{ty} f - f\|_p \leq \epsilon \ .$$

But since $g(y) > 0$ for $|y| > R$, whenerver $t < \delta/R$,

$$\left(\int_{\mathbb{R}^n} |[f(x-ty) - f(x)]g(y)|^p \mathrm{d}\mu(x)\right)^{1/p} \leq \|\tau_{ty} f - f\|_p |g(y)| \leq \epsilon |g(y)|$$

for all $y \in \mathbb{R}^n$. Then by Minkowski's inequalitiy,

$$\|f * g_t - f\|_p = \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} [f(x-ty) - f(x)]g(y)\mathrm{d}\mu(y)\right|^p \mathrm{d}\mu(x)\right)^{1/p} \leq$$
$$\left(\int_{\mathbb{R}^n} |[f(x-ty) - f(x)]g(y)|^p \mathrm{d}\mu(x)\right)^{1/p} \leq \epsilon \|g\|_1 \ . \tag{2.9}$$

Since $\epsilon > 0$ is arbitrary, this proves the theorem in the case that g has compact support. For general $g \in \mathrm{L}^1$, choose $\tilde{g} \in L^1$ so that $\|g - \tilde{g}\| < \epsilon$, $\int_{\mathbb{R}^n} \tilde{g}(x)\mathrm{d}\mu(x) = 1$, and $\tilde{g}$ has compact support. A simple computation shows that $\|\tilde{g}_t - g\|_1 = \|\tilde{g} - g\|_1 = 1$ for all $t > 0$, and so

$$\|f * g_t - f\|_p \leq \|f * g_t - f * \tilde{g}_t\|_p + \|f * \tilde{g}_t - f\|_p = \|f * (g_t - \tilde{g}_t)\|_p + \|f * \tilde{g}_t - f\|_p \ .$$

But $\|f * (g_t - \tilde{g}_t)\|_p \leq \|f\|_p \|g - \tilde{g}\|_1 \leq \|f\|_p \epsilon$, and by the first part of the proof, $\|f * \tilde{g}_t - f\|_p < \epsilon$ for all sufficiently small t. That is,

$$\|f * g_t - f\|_p \leq (\|f\|_p + 1)\epsilon$$

for all sufficiently small t. Since $\epsilon > 0$ is arbitrary, this proved the result. ∎

We will apply Theorem 2.2 many times. Our first application is to prove that for $1 \leq p < \infty$, the set $\mathcal{C}_c^\infty(\mathbb{R}^n)$ of compactly supported, infinitely differentiable functions is dense in L^p.

The first thing to do is to exhibit one such function. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(s) = \exp(1/(1-s^2))$ for $-1 \leq s \leq 1$, and $h(s) = 0$ otherwise. It is easy to check that h is infinitely differentiable at $s = \pm 1$, and it is obviously so for all other s. This gives us one non-zero function in $\mathcal{C}_c^\infty(\mathbb{R})$

We will now show that for ant $f \in L^p$, $1\ lep < \infty$,

For the n dimensional case, define

$$g(x) = C^n \prod_{j=1}^n h(x_j) \ ,$$

where C^n is a constant chosen to make $\int_{\mathbb{R}^n} g(x)\mathrm{d}\mu(x) = 1$

We will now use g to construct an approximate identity and show that for any $f \in L^p$, $1\ lep < \infty$, $f * g_t$ is infinitely differentiable. Then since by Theorem 2.2, $\lim_{t\to 0} f * g_t = f$ in the L^p norm, this will show that the infinitely differentiable functions are dense in L^p. Moreover, we already know how to approximate $f \in L0$ by a compactly supported function, and it is clear that if f is compactly supported, then so is $f * g_t$ since the support of g is also compact, and hence bounded. This will yields us the following theorem:

2.3 THEOREM. *For all* $1 \le p < \infty$, $\mathcal{C}_c^\infty(\mathbb{R})$ *is dense in* L^p.

Proof: As explained above, the essential point is to show that for all $f \in L^p$, $1 \le p < \infty$, $f * g_t(x)$ is infinitely differentiable as a function of x. (Note that since $g_t \in L^{p'}$, the integral defining $f * g_t(x)$ is converges for every x, and not only almost every x.)

To prove the differentiability, fix $f \in L^p$, and $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}^n} g(x)\mathrm{d}\mu(x) = 1$. Then fix any $z \in \mathbb{R}^n$, any $t > 0$, and any $s \in \mathbb{R}$. Define

$$G_s(w) = \frac{[g_t(x-y+sz) - g_t(x-y)]}{s} \ ,$$

so that

$$\frac{f * g_t(x+sz) - f * g_t(x)}{s} = f * G_s(x) \ . \tag{2.10}$$

We now claim that

$$G(w) := \sup_{|s|\le 1} |G_s(w)| \in L^1 \cap L^\infty \ .$$

To see this, observe that by the Mean Value Theorem gives us

$$\int_{\mathbb{R}^n} \frac{[g_t(x-y+sz) - g_t(x-y)]}{s} \int_{\mathbb{R}^n} z \cdot \nabla g_t(x-y+c(s,x-y,z)z) \ , \tag{2.11}$$

for some $c(s, x-y, z)$ between 0 and s. A simple argument using the fact that $g \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ leads to the conclusion that for fixed x and z, and all $|s| < 1$, the right hand side is bounded uniformly in s; i.e., $G(w) \in L^\infty$.

Next, since $g_t \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, there is some fixed $R < \infty$ so that the ball of radius R contains the support of the right hand side for each $|s| < 1$. The uniform L^1 bound then follows from this uniform bound on the support, and and the uniform L^∞ bound derived above.

Then since $L^1 \cap L^\infty \subset L^{p'}$, $G(w) \in L^{p'}$, we have

$$|G_s(x-y)f(y)| \le |G(x-y)||f(y)| \ ,$$

and the right hand side is integrable in y. Therefore, by the Dominated Convergence Theorem,

$$\lim_{s\to 0} G_s * f = \int_{\mathbb{R}^n} \left(\lim_{s\to 0} G_s(x-y)\right) f(y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} z \cdot \nabla g_t(x-y) f(y)\mathrm{d}\mu(y) \ .$$

This shows that $x \mapsto f * g_t(x)$ is differentiable. Moreover, the function $w \mapsto z \cdot \nabla g_t(w)$ is again in $\mathcal{C}_c^\infty(\mathbb{R}^n)$, so the argument can be repeated to see that $x \mapsto f * g_t(x)$ is twice differentiable. Continuing, one concludes that $x \mapsto f * g_t(x)$ is infinitely differentiable. ∎

3 The Fourier transform

3.1 The Fourier transform on L^1.

3.1 DEFINITION (The Fourier transform on L^1)**.** For $f \in L^1$, define the function $\widehat{f}$ on $\mathbb{R}^n$ by

$$\widehat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi k\cdot x} f(x) \mathrm{d}\mu(x) \ , \tag{3.1}$$

where $k \cdot x$ denote the usual inner product of k and x in $\mathbb{R}^n$. The map $\mathcal{F} : f \to \widehat{f}$ is called the *L^1-Fourier transform.*

The integral defining $\widehat{f}(x)$ is obviously convergent for each x. and it is also clear that for each k, $|\widehat{f}(k)| \leq \|f\|_1$, so that $\|\widehat{f}\|_\infty \leq \|f\|_1$. A little less obvious is the fact that $k \mapsto \widehat{f}(k)$, and that is continuous and that $\lim_{k\to\infty} \widehat{f}(k) = 0$, so that $\widehat{f} \in \mathcal{C}_0(\mathbb{R}^n)$.

3.2 THEOREM (Riemann-Lebesgue Theorem)**.** *The map $\mathcal{F} : f \to \widehat{f}$ is a contraction from L^1 into $\mathcal{C}_0(\mathbb{R}^n)$.*

Proof: $\mathcal{F}$ is clearly linear, and we have already observed that it is bounded with norm one, so it is a contraction.

To see that $k \mapsto \widehat{f}(k)$ is continuous, observe that for any $k, h \in \mathbb{R}^n$,

$$\widehat{f}(k+h) - \widehat{f}(k) = \int_{\mathbb{R}^n} \left[e^{-2\pi(k+h)\cdot x} - e^{-2\pi k\cdot x}\right] f(x)\mathrm{d}\mu(x)$$

The integrand is dominated by $2|f(x)|$ for all $k, h \in \mathbb{R}^n$, and so by the Dominated Convergence Theorem,

$$\lim_{h\to 0}(\widehat{f}(k+h) - \widehat{f}(k)) = \int_{\mathbb{R}^n} \lim_{h\to 0} \left[e^{-2\pi(k+h)\cdot x} - e^{-2\pi k\cdot x}\right] f(x)\mathrm{d}\mu(x) = 0 \ .$$

This proves the continuity.

To show that $\lim_{k\to\infty} \widehat{f}(k) = 0$, we shall apply Theorem 1.1.

Note that since $e^{-i\pi=-1}$,

$$e^{-2\pi k\cdot x} = -e^{-2\pi k\cdot(x+k/(2|k|^2)} \ .$$

Therefore,

$$\begin{aligned} \widehat{f}(k) &= \frac{1}{2}\int_{\mathbb{R}^n} \left[e^{-2\pi k\cdot x} - e^{-2\pi k\cdot(x+k/(2|k|^2)}\right] f(x)\mathrm{d}\mu(x) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} e^{-2\pi k\cdot x} \left[f(x) - f(x - k/(2|k|^2))\right] \mathrm{d}\mu(x) \ . \end{aligned}$$

Therefore,

$$|\widehat{f}(k)| \leq \frac{1}{2}\|f - \tau_{k/(2|k|^2)} f\|_1 \ .$$

But by Theorem 1.1, $\lim_{k\to\infty} \|f - \tau_{k/(2|k|^2)} f\|_1 = 0$. ■

The proof yields useful information that is not recorded in Theorem 3.2. It shows that the *rate* at which $|\widehat{f}(k)| \to 0$ and $k \to \infty$ depends quantitatively on how fast $\|f - \tau_y f\|_1 \to 0$ and $y \to 0$.

We next observe that the Fourier transform carries the convolution product in L^1 over into pointwise multiplication in $\mathcal{C}_0(\mathbb{R}^n)$.

3.3 THEOREM (The Fourier transform and convolution). *For all* $f, g \in L^1$, $\widehat{f * g} = \widehat{f}\widehat{g}$.

Proof: By definition, and then Fubini's Theorem,

$$\begin{aligned}
\widehat{f * g}(k) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}\mu(y) \right) e^{-2\pi k\cdot x}\mathrm{d}\mu(x) \\
&= \int_{\mathbb{R}^{2n}} f(x-y)g(y)e^{-2\pi k\cdot x}\mathrm{d}\mu(x,y) \\
&= \int_{\mathbb{R}^{2n}} f(x-y)g(y)e^{-2\pi k\cdot(x-y)}e^{-2\pi k\cdot y}\mathrm{d}\mu(x,y) \\
&= \int_{\mathbb{R}^{2n}} f(z)g(y)e^{-2\pi k\cdot(z)}e^{-2\pi k\cdot y}\mathrm{d}\mu(z,y) \\
&= \left(\int_{\mathbb{R}^n} f(z)e^{-2\pi k\cdot z}\mathrm{d}\mu(z) \right) \left(\int_{\mathbb{R}^n} g(y)e^{-2\pi k\cdot y}dd\mu(y) \right) = \widehat{f}(k)\widehat{g}(k) \ . \qquad (3.2)
\end{aligned}$$

In more detail, the first equality follws from the definitions, the second from Fubini's Theorem, the third from simple manipulation, the fourth from the fact that the transformation $(x, y) \mapsto (z, y) := (x-y, y)$ preserves Lebesgue measure as it has unit Jacobian, and the fifth is from Fubini's Theorem once more. ■

We close this section with a lemma that will prove useful in the next section.

3.4 LEMMA (Translations and the Fourier Transform). *For all* $y \in \mathbb{R}^n$*, let* M_y *denote the multiplication operator on* L^p*,* $1 \le p \le \infty$*, given by*

$$M_y f(x) := e^{2\pi i x\cdot y} f(x) \ ,$$

which is a contraction on L^p *for all* $1 \le p \le \infty$*. Then For all* $y \in \mathbb{R}^n$ *and all* $f \in L^1$*,*

$$\widehat{\tau_y f} = M_{-y} \ , \qquad (3.3)$$

and

$$\widehat{M_y f} = \tau_y f \ . \qquad (3.4)$$

Proof: By definition, and then an obvious change of variables,

$$\widehat{\tau_y f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} f(x-y)\mathrm{d}\mu(x) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot(x+y)} f(x)\mathrm{d}\mu(x) = e^{-2\pi i k\cdot y}\widehat{f}(k) \ .$$

The proof of (3.4) is similar. ■

3.2 The Fourier transform on L^2.

For a general $f \in L^2$, there is no reason to expect $e^{-2\pi k\cdot x}f(x)$ to be integrable in x. Therefore, one cannot directly use (3.1) to define $\widehat{f}(k)$ for general $f \in L^2$, or any other L^p, $p > 1$ for that matter.

However, the Fourier transform $\mathcal{F}$ is defined by (3.1) on $L^1 \cap L^2$, which is dense in L^2. As we shall now show, $\mathcal{F}$ is an L^2 isometry on $L^1 \cap L^2$; that is for all $f \in L^1 \cap L^2$,

$$\|\widehat{f}\|_2 = \|f\|_2 \ . \qquad (3.5)$$

In particular, the linear transformation $\mathcal{F} : L^1 \cap L^2 \to L^2$ is bounded, and therefore continuous, and so by closure extends to a map, still denoted $\mathcal{F}$:, from L^2 to L^2.

Indeed, suppose that (3.5) is proved for all $f \in L^1 \cap L^2$. Fix any $f \in L^2$, and let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence in $L^1 \cap L^2$ with $\lim_{n\to\infty} \|f_n - f\|_2 = 0$. Then $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2, so that $\lim_{m,n\to\infty} \|f_n - f_m\|_2 = 0$. But then by (3.5), $\lim_{m,n\to\infty} \|\widehat{f_n} - \widehat{f_m}\|_2 = 0$, so that $\{\widehat{f_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2. By the Riesz-Fischer Theorem, this sequence has a limit in L^2, and so we define $\widehat{f}$ to be the limit. That is:

$$\widehat{f} = \lim_{n\to\infty} \widehat{f_n} \ . \tag{3.6}$$

To see that this definition of $\widehat{f}$ depends only on f, and not on the particular sequence $\{f_n\}_{n\in\mathbb{N}}$ that we used to approximate f, let $\{g_n\}_{n\in\mathbb{N}}$ be any other sequence in $L^1 \cap L^2$ with $\lim_{n\to\infty} \|g_n - f\|_2 = 0$. Then as above, $\{\widehat{g_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2, and has a limit $\widehat{g}$. We claim that $\widehat{g} = \widehat{f}$. To see this, pick n large enough that

$$\|\widehat{f} - \widehat{f_n}\|_2 < \epsilon \qquad \|f - f_n\|_2 < \epsilon \qquad \|\widehat{g} - \widehat{g_n}\|_2 < \epsilon \qquad \text{and} \qquad \|f - g_n\|_2 < \epsilon \ .$$

Then

$$\|\widehat{f} - whg\|_2 \le \|\widehat{f} - \widehat{f_n}\|_2 + \|\widehat{f_n} - \widehat{g_n}\|_2 + \|\widehat{g_n} - \widehat{g}\|_2 \le 2\epsilon + \|\widehat{f_n} - \widehat{g_n}\|_2 \ .$$

But by (3.5) again,

$$\|\widehat{f_n} - \widehat{g_n}\|_2 = \|f_n - g_n\|_2 \le \|f_n - f\|_2 + \|f - g_n\|_2 \le 2\epsilon \ .$$

Altogether, $\|\widehat{f} - whg\|_2 \le 4\epsilon$, and since $\epsilon > 0$ is arbitrary, this means that $\widehat{g} = \widehat{f}$. In summary, once we have shown (3.5), we will know that the left hand side of (3.6) is independent of the choice of the particular sequence $\{f_n\}_{n\in\mathbb{N}}$ used to approximate f, and therefore depends only on f itself. Thus, we can use (3.6) to define the Fourier transform on all of L^2. It is clear that this extension is linear. Finally note that

$$\|\widehat{f}\|_2 = \lim_{n\to\infty} \|\widehat{f_n}\|_2 = \lim_{n\to\infty} \|f_n\|_2 = \|f\|_2 \ ,$$

so that (3.5) will then holds for *all* $f \in L^2$.

The key to proving (3.5) is the following simple computation, together with the thoerem on approximate identities.

3.5 LEMMA. *Let* $g(x) = e^{-\pi|x|^2}$ *and, for* $t > 0$, $g_t(x) = t^{-n}g(x/t)$. *Then for each* $t > 0$,

$$\widehat{g_t}(k) = e^{-t|\pi|k|^2} \ . \tag{3.7}$$

3.6 Remark. Note that for an $f \in L^1$, $\widehat{f}(0) = \int_{\mathbb{R}^n} f(x)\mathrm{d}\mu(x)$, and so as a special case of (3.7), we have that $\int_{\mathbb{R}^n} g_t(x)\mathrm{d}\mu(x) = 1$ for all t. In particular, by the Theorem 2.2, for all $f \in L^p$, $1 \le p < \infty$, $\lim_{t\to 0} f \mapsto f * g_t = f$ in L^p.

Proof: Since $g_t \in L^1$, we have

$$\widehat{g_t}(k) = \int_{\mathbb{R}^n} t^{-n} e^{-\pi|x/t|^2} e^{-2\pi i k\cdot x}\mathrm{d}\mu(x) = \int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-2\pi i k\cdot tx}\mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \prod_{j=1}^n e^{-\pi|x_j|^2 - 2\pi i t k\cdot x_j}\mathrm{d}\mu(x) \ .$$

But clearly,

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{-\pi|x_j|^2 - 2\pi i t k\cdot x_j} \mathrm{d}\mu(x) = \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} e^{-\pi|x_j|^2 - 2\pi i t k\cdot x_j} \mathrm{d}x_j \right) .$$

Now, completing the square,

$$\int_{-\infty}^{\infty} e^{-\pi|x|^2 - 2\pi t k\cdot x} \mathrm{d}x = e^{-t\pi|k|^2} \int_{-\infty}^{\infty} e^{-\pi|x - itk|^2} \mathrm{d}x .$$

It remains to show that with $\varphi(k)$ defined by $\varphi(k) := \int_{-\infty}^{\infty} e^{-\pi|x-itk|^2}\mathrm{d}x$, $\varphi(k) = 1$ for all k. For $k = 0$, this reduces to the familiar, and readily checked, fact that $\int_{-\infty}^{\infty} e^{-\pi|x|^2}\mathrm{d}x = 1$. Hence it remains to show that $\varphi(k)$ is constant. To do this, we differentiate. An easy argument like the one used in the proof of Theorem 2.3, but simpler, shows that one may differentiate under the integral sign and then:

$$\frac{\mathrm{d}}{\mathrm{d}k}\varphi(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}k} e^{-\pi|x-itk|^2} \mathrm{d}x = -\frac{1}{it} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} e^{-\pi|x-itk|^2} \mathrm{d}x = -\frac{1}{it} e^{-\pi|x-itk|^2} \Big|_{-\infty}^{+\infty} = 0 .$$

This proves (3.7). ■

3.7 LEMMA. *For all $f \in L^1 \cap L^2$, $\|\widehat{f}\|_2 = \|f\|_2$.*

Proof: By the the Riemann-Lebesgue Theorem, when $f \in L^1 \cap L^2$, $k \mapsto |\widehat{f}(k)|^2$ is a bounded, continuous function on $\mathbb{R}^n$, Thus, the intergral $\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 \mathrm{d}\mu(k)$ is well defined, though for all we know right now, is possibly $+\infty$. In any case, we have, my the Monotone convergence Theorem

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 \mathrm{d}\mu(k) = \lim_{t\to 0} \int_{\mathbb{R}^n} |\widehat{f}(k)|^2 e^{-\pi t|k|^2} \mathrm{d}\mu(k) . \tag{3.8}$$

Then, by the definition of the Fourier transform,

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 e^{-\pi t|k|^2} \mathrm{d}\mu(k) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{f}(x) f(y) e^{-2\pi k\cdot(y-x)} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \right] e^{-\pi t|k|^2} \mathrm{d}\mu(k) .$$

Since $\overline{f}(x)f(y)e^{-\pi t|k|^2}$ integrable on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, we may apply Fubini's Theorem to conclude

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{f}(x) f(y) e^{-2\pi k\cdot(y-x)} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \right] e^{-\pi t|k|^2} \mathrm{d}\mu(k) =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{f}(x) f(y) \left[\int_{\mathbb{R}^n} e^{-2\pi k\cdot(y-x)} e^{-\pi t|k|^2} \mathrm{d}\mu(k) \right] \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

Then by Lemma 3.5,

$$\int_{\mathbb{R}^n} e^{-2\pi k\cdot(y-x)} e^{-\pi t|k|^2} \mathrm{d}\mu(k) = t^{-n} e^{-\pi|(x-y)/t|^2} .$$

Combining this with the last two calculations yields

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 e^{-\pi t|k|^2} \mathrm{d}\mu(k) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{f}(x) f(y) t^{-n} e^{-\pi|(x-y)/t|^2} \mathrm{d}\mu(x)\mathrm{d}\mu(y) .$$

Applying Fubini's Theorem one more, we do the integral in y and obtain

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 e^{-\pi t|k|^2} \mathrm{d}\mu(k) = \langle f, f * g_t \rangle ,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in L^2, and $g_t(z) = t^{-n}e^{-\pi|z/t|^2}$. But by Theorem 2.2, $\lim_{t\to 0} f*g_t = f$ in L^2, and so by (3.8),

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 \mathrm{d}\mu(k) = \lim_{t\to 0}\langle f, f*g_t\rangle = \langle f,f\rangle = \|f\|_2^2\ .$$

■

The main result of this section is the following thoerem, which says that not only is $\mathcal{F} : f \mapsto \widehat{f}$ an isometry from L^2 into L^2, its range is actually all of L^2, so that $\mathcal{F} : f \mapsto \widehat{f}$ is a unitary transformation of L^2 onto itself.

3.8 THEOREM (Plancherel's Theorem)**.** *The L^2 Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ is a unitary transformation of L^2 onto L^2. Furthermore, for all $f,g \in L^2$,*

$$\langle\widehat{f},\widehat{g}\rangle = \langle f,f\rangle\ . \tag{3.9}$$

Proof: Since we have already proved that $\mathcal{F} : f \mapsto \widehat{f}$ an isometry from L^2 into L^2, once we show that the range of $\mathcal{F}$ is all of L^2, we will have shown that $\mathcal{F}$ is an isometry of L^2 onto itself, and hence unitary.

Therefore, let V denote the range of $\mathcal{F}$. We first note that V is a closed subspace of L^2: If g lies in the closure of V, then by definition, there is a sequnce $\{f_n\}_{n\in\mathbb{N}}$ in Ł^2 so that $\lim_{n\to\infty}\widehat{f}_n = g$. But then the sequence $\{\widehat{f_n}\}_{n\in\mathbb{N}}$, being convergent, is Cauchy. By the isometry property of $\mathcal{F}$, $\{f_n\}_{n\in\mathbb{N}}$ is also Cauchy, and therefore has a limit f. But then $\widehat{f} = \lim_{n\to\infty}\widehat{f_n}$, and so $g = \widehat{f}$. Thereofre, g lies in the range of $\mathcal{F}$; i.e., $g \in V$. This proves that V is closed.

Now, if V is a proper subspace of L^2, there exists a non-zero continuous linear functional on L^2 that is zero on all of V. By the Riesz Representation Theorem, this means that there is a non-zero $h \in L^2$ so that $\langle h,g\rangle = 0$ for all $g\ inV$. Therefore, to prove that $V = L^2$, we need only show that if $h \in L^2$ is orthgonal to every vecotr in V, then $h = 0$.

To see this, note that by Lemma 3.5, with $g_t(k) := t^{-n}e^{-\pi|k/t|^2}$, $g_t \in V$ for all $t > 0$, and then by Lemma 3.4, $\tau_y g_t \in V$ for each $y \in \mathbb{R}^n$ and all $t > 0$.

It follows that $\langle h,\tau_y g_t\rangle = 0$ for each $t > 0$, and each $y \in \mathbb{R}^n$. But $\langle h,\tau_y g_t\rangle = h*g_t(y)$, so that $h*g_t = 0$. But by Theorem 2.2, $h = \lim_{t\to 0} h*g_t$. Thus, $h = 0$. This proves that $\mathcal{F}$ maps L^2 onto L^2, and completes the proof that $\mathcal{F}$ is unitary.

Next, for any $f,g \in L^2$, note that

$$\mathcal{R}e(\langle f,g\rangle) = \frac{\|f+g\|_2^2 + \|f-g\|_2^2}{4}\ .$$

It follows from this, the linearity of $\mathcal{F}$, and the isometry property that $\mathcal{R}(\langle\widehat{f},\widehat{g}\rangle) = \mathcal{R}(\langle f,g\rangle)$. Since $\mathcal{I}m(\langle f,g\rangle) = -\mathcal{R}e(\langle f,ig\rangle)$, (3.9) then follows. ■

The inverse of a unitary operator is its Hermitian adjoint. so to compute a formula for the inverse of the Fourier transform, it suffices to compute the adjoint $\mathcal{F}^*$ of $\mathcal{F}$. To do this, let $f,g \in L^1\cap L^2$. Then by the definitions and Fubini's Theorem,

$$\langle f,\mathcal{F}g\rangle = \int_{\mathbb{R}^n}\overline{f}(k)\left(\int_{\mathbb{R}^n} e^{-2\pi ik\cdot x}g(x)\mathrm{d}\mu(x)\right)\mathrm{d}\mu(k) = \int_{\mathbb{R}^n}\left(\overline{\int_{\mathbb{R}^n} f(k)e^{-2\pi ik\cdot x}g(x)\mathrm{d}\mu(k)}\right)g(x)\mathrm{d}\mu(k)\ .$$

Thus, if we define $\mathcal{F}^* f$ by

$$\mathcal{F}^* f(x) = \int_{\mathbb{R}^n} f(k) e^{2\pi i k \cdot x} g(x) \mathrm{d}\mu(k) \ ,$$

we have

$$\langle f, \mathcal{F} g \rangle = \langle \mathcal{F}^* f, g \rangle \ ,$$

and it follows that $f \mapsto \mathcal{F}^* f$ is indeed the Hermitian adjoint, and hence the inverse, to $\mathcal{F}$.

We can now give a simple but significant reinterpretation of Lemma 3.4. For any $f \in L^1 \cap L^2$, and any $y \in \mathbb{R}^n$, we have that $\mathcal{F}(\tau_y f) = M_- y(\mathcal{F} f)$ and hence

$$\tau_y f = \mathcal{F}^*(M_- y(\mathcal{F} f)) \ .$$

That is

$$\tau_y = \mathcal{F}^* M_- y \mathcal{F} \ ,$$

which says that the unitary transformation $\mathcal{F}$ *diagonalizes* the translation operator τ_y.

Now for $f \in C_c^\infty(\mathbb{R}^n)$, it is certainly the case that for any $y \in \mathbb{R}^n$,

$$\lim_{s \to 0} \frac{1}{s}(I - \tau_{sy}) f(x) \ y \cdot \nabla f(x) \ .$$

By the above, for all $s \neq 0$

$$\mathcal{F}^* \frac{1}{s}(I - \tau_{sy}) \mathcal{F} = \frac{1}{s}(I - M_{-sy}) \ .$$

Thus, the Fourier transform diagonalizes the approximate differentiation operator $\frac{1}{s}(I - \tau_{sy})$, and by taking limits, diagonalizes the differentiaion operator. Developing this idea would take us into the study of unbounded operators, so we will not develop it here, but it is the basis of many applications of the Fourier transform in the study of partial differential equations.

3.3 The Fourier transform in L^p, $1 < p < 2$

We have seen that the Fourier transform $\mathcal{F}$ defined on $L^1 \cap L^2$ by the integral formula (3.1) satisfies the bounds $\|\mathcal{F}\|_{1 \to \infty} = 1$ and $\|\mathcal{F}\|_{2 \to 2} = 1$. We now apply the Riesz-Thorin inequality to prove the Hausdorff-Young Theorem:

3.9 THEOREM (Hausdorff-Young Theorem)**.** *For all $1 < p < 2$, the Fourier transform $\mathcal{F}$ defined on $L^1 \cap L^2$ extend to a contraction from L^p to $L^{p'}$. That is, for all $f \in L^1 \cap L^2$ and all $1 < p < 2$,*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p \ .$$

Proof: Since $1 < p < 2$, there is some $0 < t < 1$ such that

$$\frac{1}{p} = t \frac{1}{1} + (1 - t) \frac{1}{2} \ .$$

Now define q by

$$\frac{1}{q} = t \frac{1}{\infty} + (1 - t) \frac{1}{2} \ .$$

solving for q, one finds $q = p'$. Then by the Riesz-Thorin Theorem,

$$\|\mathcal{F}\|_{p \to p'} \leq (\|\mathcal{F}\|_{1 \to \infty})^t (\|\mathcal{F}\|_{2 \to 2})^{1-t} = 1^t 1^{1-t} = 1 \ .$$

■