Midterm for Math 502, Real Analysis II, Spring 2009

Mar 13, 2009

NAME:

Instructions: Solve any three of the following five problems. Before turning in your work, circle the numbers of the three that you wish to have graded:

1 2 3 4 5

Unless three numbers in this line are circled, I will grade the first three problems.

1. For any measure space $(X, \mathcal{M}, \mu)$ and any $1 \leq p < q \leq \infty$, show that there exists a function f in L^p that is not in L^q if and only if X contains sets of arbitrarily small positive measure. Also show that there exists a function f in L^q that is not in L^p if and only if X contains sets of arbitrarily large finite measure.

2. Let X be a Banach space. Let C be a closed convex subset of X that has non-empty interior. Show that the interior of C is also convex, and that C is the closure of its interior.

3. For $(\mathbb{R}, \mathcal{B}, \mu)$ with $\mathcal{B}$ being the Borel sigma algebra and μ being Lebesgue measure, let K by the subset of L^2 given by

$$K := \left\{ f \ : \ \int_{\mathbb{R}} (|f(x)| + |f(x)|^2) \mathrm{d}\mu \leq 1 \right\} .$$

Show that K is weakly closed and weakly compact in L^2.

4. **(a)** Show that in any infinite dimensional Banach space X, there exists an infinite sequence $\{x_n\}_{n\in\mathbb{N}}$ of unit vectors such that $\|x_m - x_n\| > 1/2$ for all $m \neq n$.

(b) Show that in any infinite dimensional Banach space X, no closed ball (of positive radius) is compact.

(c) Show that in any infinite dimensional Banach space X, every strongly compact set is nowhere dense.

5. Let X be any Banach space. Show that if X^* is separable, then X is also separable, but not conversely. (For the "not conversely part", you may use an example. For the first part, one way to proceed starts by considering a dense sequence $\{\varphi_n\}_{n\in\mathbb{N}} \in X^*$, and then for each n, choosing a sequence of unit vectors $\{x_{n,m}\}_{m\in\mathbb{N}}$ such that $|\varphi_n(x_{n,m})| \geq \|\varphi_n\|(1 - 1/m)$.)