# Midterm Exercises for Real Analysis II, Spring 2009

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**1.** For $1 \leq p \leq \infty$, let $\ell_p$ denote $L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ where $\mu$ is counting measure on the natural numbers $\mathbb{N}$, as usual. Let $c_0$ denote the subspace of $\ell_\infty$ consisting of sequences functions $f$ from $n$ to $\mathbb{C}$ such that $\lim_{n\to\infty} f(n) = 0$.

Show that for $1 \leq p < \infty$ $\ell_p$ is separable, as is $c_0$, but that $\ell_\infty$ is not separable.

**2.** For any measure space $(X, \mathcal{M}, \mu)$, and any $1 \leq p < q \leq \infty$, let

$$X = L^p(X, \mathcal{M}, \mu) \cap L^q(X, \mathcal{M}, \mu) \ .$$

For $f \in X$, define

$$|\!|\!| f |\!|\!|_{p,q} = \max\{\ \|f\|_p, \|f\|_q\ \}\ .$$

Show that $|\!|\!| \cdot |\!|\!|$ is a norm on $X$, and that equipped with this norm, $X$ is a Banach space.

**3.** For any measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$, define $K \subset L^2(X, \mathcal{M}, \mu)$ by

$$K = \left\{ f \in L^2(X, \mathcal{M}, \mu)\ :\ 0 \leq f(x) \leq 1\ a.e. \quad \text{and} \quad \int_X f(x)\mathrm{d}\mu = \frac{\mu(X)}{2}\ \right\}\ .$$

Show that for each $g \in L^2(X, \mathcal{M}, \mu)$ there exists a unique $f_g$ in $K$ such that

$$\|f_g - g\|_2 \leq \|f - g\|_2$$

for all $f \in K$.

**4.** Consider any measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$, Let $V$ be a closed subspace of $L^1(X, \mathcal{M}, \mu)$ such that for each $\in V$, there is a $p > 1$ (possibly depending on $f$) so that $f \in L^1(X, \mathcal{M}, \mu)$. Show that in fact there is some $p_0 > 1$ so that $V \subset L^{p_o}(X, \mathcal{M}, \mu)$.

**5.** Construct an example of a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^1 \cap L_2$ of some measure space $(X, \mathcal{M}, \mu)$ such that $f_n$ converges weakly to zero in $L^2$, and converges strongly to zero in $L^{3/2}$, but does not converge to zero strongly in $L^2$.

**6.** Let $X$ be a Banach space. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of $X$ that is nested so that $A_{n+1} \subset A_n$. Let $d_n$ denote the *diameter* of $A_n$; i.e.,

$$d_n = \sup\{\|x - y\|\ :\ x, y \in A_n\ \}\ .$$

Show that

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset\ .$$

**7.** Let $\mathcal{B}$ be the $\sigma$–algebra of Borel sets, and let $\mu$ be Lebesgue measure. Show that there exists a non-zero continuous linear functional on $L^\infty(\mathbb{R}, \mathcal{B}, \mu)$ that is zero on all of $\mathcal{C}(\mathbb{R})$.

**8.** Let $(X, d)$ be a metric space.

**(a)** Show that if $(X, d)$ is not compact, there is sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ that has no convergent subsequences. construct a subsequences (again denoted $\{x_n\}_{n\in\mathbb{N}}$) of this sequence, and a monotone decreasing sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of *strictly* positive numbers such that for all $m \geq 1$,

$$d(x_m, x_n) \geq \epsilon_m \quad \text{for all } n > m\ .$$

That is, the sequence stays "uniformly away" from where it has already been.

**(b)** For the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ from part **(a)**, show that if $d(x, x_m) < \epsilon_m/3$, then for every $k < m$, $d(x, x_k) \geq 2\epsilon_k/3$.

**(c)** For the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ from part **(a)**, define the sequence $\{f_n\}$ of real valued functions on $X$ by

$$f_n(x) = \frac{3(\delta_n/3 - d(x, x_n))_+}{\epsilon_n}\ ,$$

where $0 < \delta_n \leq \epsilon_n$, and $a \mapsto (a)_+$ is the *positive part* function. That is, $(a)_+ = a$ for $a \geq 0$, and $(a)_+ = 0$ otherwise.

Show that the real valued function $f$ on $X$ defined by

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well defined (i.e., the sum converges to a real number), and that $f$ is continuous. Show, however, that if $X$ has no isolated points, then for an appropriate choice of the $\delta_n$'s, $f$ is not uniformly continuous.

**(d)** Show that if $(X, d)$ has no isolated points, then $(X, d)$ is compact if and only if every continuous real valued function on $(X, d)$ is uniformly continuous.

**9.** Let $X$ be a real Banach space, and let $C$ be the unit ball of $X^*$, equipped with the weak-$*$ topology. Show that if $f$ is any continuous function on $B$, then for every $\epsilon > 0$, there os a polynomial $p_\epsilon$ in $n$ complex variables, and $n$ elements $x_1, \ldots, x_n$ of $X$ such that for each $x^* \in X^*$,

$$|f(x^*) - p(x^*(x_1), \ldots, x^*(x_n))| \leq \epsilon\ .$$

**10.** Let $X$ be the space of all complex sequences $a = \{a_n\}_{n\in\mathbb{N}}$ with only finitely many non-zero terms. Let $X$ be equipped with the $\ell^\infty$ norm; i.e., $\|a\| = \sup_{n\in\mathbb{N}}\{\ |a_n|\ \}$. For all $m \in \mathbb{N}$, define the linear functional $f_m$ on $X$ by

$$f_m(a) = \sum_{n=1}^{m} a_n\ .$$

**(a)** Show that there is a constant $C_a$ depending only on $a$ so that

$$|f_n(a)| \leq C_a \qquad \text{for all} \quad n \in \mathbb{N}\ .$$

**(b)** Show that for all $n$, $\|f_n\| = n$.

**(c)** Explain why the results from **(a)** and **(b)** are not in conflict with the uniform roundedness principle.