Solutions to Selected Exercises for Math 501

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The exercsies come from Folland's text. The numbering follows the numbering in the text.

1 Solutions to Exercises from Chapter 1

3: Show that every σ-algebra $\mathcal{M}$ that contains infinitely many distinct sets has a cardinality that is at least as large as that of the continuum.

SOLUTION: Let $A \in \mathcal{M}$ be such that neither A nor A^c is empty. Define

$$\mathcal{M}_A = \{A \cap B \ : \ B \in \mathcal{M}\} \ .$$

Then $\mathcal{M}_A$ and $\mathcal{M}_{A^c}$ are both σ-algebras, and

$$\mathcal{M}_A = \mathcal{M}_A \cup \mathcal{M}_{A^c} \ .$$

If $\mathcal{M}_A$ contains inifnitely many distinct sets, then, by the pigeonhole priciple,at least one of $\mathcal{M}_A$ and $\mathcal{M}_{A^c}$ contains inifnitely many distinct sets.

We now construct an infinite strictly decrwasing nested family of sets in $\mathcal{M}$. First, pick $A \in \mathcal{M}$ such that neither A nor A^c is empty. Define E_1 to be A if $\mathcal{M}_A$ contains inifnitely many distinct sets, and otherwise take $E_1 = A^c$, in whihc case $\mathcal{M}_{A^c}$ contains inifnitely many distinct sets. Either way, $\mathcal{M}_{E_1}$ contains inifnitely many distinct sets.

Now with E_j defined for $j < n$, pick $A \ in \mathcal{M}_{E_{n-1}}$ such that neither A nor A^c is empty. Define E_n to be A if $\mathcal{M}_A$ contains inifnitely many distinct sets, and otherwise take $E_n = A^c$. Then, $\mathcal{M}_{E_n}$ contains inifnitely many distinct sets.

Clearly $\{E_n\}_{n\in N}$ is a sequence in $\mathcal{M}$ such that $E_{n+1} \subset E_n$ for all n, and the containment is strict: $E_n \cap E_{n+1}^c \neq \emptyset$. Now define $F_n = E_n \cap E_{n+1}^c$. This is evidently a sequence of infinitely many disjoint, non-empty sets in $\mathcal{M}$.

Now for each $x \in [0, 1]$, let $\{b_1(x), b_2(x), b_3(x), \dots\}$ be the sequence of bits in its canonical binary expansion. Define

$$A_x = \bigcup \{F_n \ : \ b_n(x) \neq 0 \ \}$$

Then, as a counable union of sets in $\mathcal{M}$, $A_x \in \mathcal{M}$, and clearly the map

$$x \mapsto A_x$$

is injective into $\mathcal{M}$. This proves that the cardinality of $\mathcal{M}$ is at least as large as that of the continuum.

REMARK The were a number of mistaken contruction of the infinite disjoint sequence of non-empty sets. Many attempts went roughly along the following lines:

Ler $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of infinitely many distinct sets in $\mathcal{M}$. Define $F_1 = E_1$, and for $n > 1$ define $F_n \in \mathcal{M}$ by

$$F_n = E_n \cap (F_1 \cup \cdots \cup F_{n-1})^c \ .$$

(Note that $\mathcal{M}$ is closed under the operations used to form F_n.)

The problem is that for all but finitely many n, $F_n = \emptyset$.For example, let $E_2 = E_1^c$. Then $F_1 = E_1$, $F_2 = E_1^c$, and $F_n = \emptyset$ for all $n \geq 3$. Also, one could have $E_m = X$, the whole spaces, for some m. Then $F_n = \emptyset$ for all $n > m$.

4: An algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing unions.

SOLUTION: All σ-algebras are algebra that are closed under countable union, and hence they are ertinaly closed under countable increasing unions. For the converse, let $\mathcal{A}$ be an algebra that is closed under countable increasing unions.

Algebras are always closed under complements, so we need only show that $\mathcal{A}$ under countable unions. Using the identity Given any sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$, define $B_n = \cup_{j=1}^n A_j$. Then for all n, $B_n \subset B_{n+1}$, and $B_n \in \mathcal{A}$ since any algebra is closed under finite unions. But

$$\bigcup_{j=1}^\infty A_j = \bigcup_{j=1}^\infty B_j \in \mathcal{A}$$

by the closure undert countable increasing unions. Hence $\mathcal{A}$ is closed under countalbe unions.

5: If $\mathcal{M} = \sigma(\mathcal{E})$, then $\mathcal{M}$ is the union of all of the σ-algebras generated by countable subsets of $\mathcal{E}$. countable increasing unions.

SOLUTION: Let $\mathcal{F}$ be a countable subset of $\mathcal{E}$. Then clearly,

$$\sigma(\mathcal{F}) \subset \sigma(\mathcal{E}) = \mathcal{M} \ .$$

Therefore, if we define

$$\mathcal{N} := \bigcup\{\sigma(\mathcal{F}) \ : \ \mathcal{F} \subset \mathcal{E} \text{ is countable } \} \ ,$$

we have $\mathcal{N} \subset \mathcal{M}$.

Note that $\mathcal{E} \subset \mathcal{N}$. Hence, if we show that $\mathcal{N}$ is a σ-algebra, it will follow that $\mathcal{M} \subset \mathcal{N}$. Together with what we have said above, this would prove that $\mathcal{N} = \mathcal{M}$.

If $A \in \mathcal{N}$, then $A \in \sigma(\mathcal{F})$ for some cuntable subset $\mathcal{F}$ of $\mathcal{E}$. But $\sigma(\mathcal{F})$ is closed undercomplementation, so

$$A^c \in \sigma(\mathcal{F}) \subset \mathcal{N} \ .$$

Thus, $\mathcal{N}$ is closed under complementation.

Next, let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{N}$. Then each $A_n \in \sigma(\mathcal{F}_\backslash)$ for some cuntable subset $\mathcal{F}_\backslash$ of $\mathcal{E}$. Define

$$\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n \ ,$$

whihci s a countable subset of $\mathcal{E}$ since countable unions of countable sets are countable. Thus,

$$A_n \in \sigma(\mathcal{F}) \subset \mathcal{N}$$

for all n, and since $\sigma(\mathcal{F})$ is closed under countable unions,

$$\bigcup_{n=1}^{\infty} A_n \in \sigma(\mathcal{F}) \subset \mathcal{N} \ .$$

Thus, $\mathcal{N}$ is closed under countable unions as well as complements, and hence is a σ-algebra.

2 Solutions to Exercises from Chapter 2

3: If $\{f_n\}$ is a sequence of complex valued measurable functions on $(X, \mathcal{M})$, then

$$E := \{x \ : \ \lim_{n\to\infty} f_n(x) \text{ exists } \}$$

is a measurable set.

SOLUTION: Suppose that f is real. Let $g_1(x) = \liminf_{n\to\infty} f_n(x)$ and let $g_2(x) = \limsup_{n\to\infty} f_n(x)$. These are measurable by proposition 2.7, and then $h = g_1 - g_2$ is measurable by Proposition 2.6. Then since

$$E = \{x : \ h(x) = 0\} = h^{-1}(\{0\}) \ ,$$

and since $\{0\}$ is a Borel subset of $\mathbb{R}$, $E \in \mathcal{M}$.

For f complex, apply the argument separately to the real and imaginary parts of f, and take the intersection of the two sets on which the separate limits exist.

4: If $f^{-1}((r,\infty))$ is measurable for each rational r, then f is measurable.

SOLUTION: It suffices to show that $f^{-1}((r,\infty))$ is measurable for each $a \in \mathbb{R}$. Let r_n be a sqnece of rational that decreases to a. Then

$$f^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f^{-1}((r_n,\infty)) \ ,$$

and the latter is a countable union of measurable sets.

5: If $X = A \cup B$ where A and B are measureable, the f is measaureable if and only if the restrictions of f to A and B are measureable.

SOLUTION: Assume f is real values, or else apply the following to the real and imaginary parts. For all $a \in \mathbb{R}$,

$$f^{-1}((a,\infty)) \cap A = (f|_A)^{-1}((a,\infty))$$

and likewise for B.

Thus is f is measuarable, the left side is measurable and hence $f|_A$ is measurale, and likewise for $f|_B$. On the other hand, if $f|_A$ is measurale, so is the left side, and likewise $f^{-1}((a,\infty)) \cap A$.

8: Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

SOLUTION: Suppose, without loss of generality, that f is monotone increasing, Then for all $a \in \mathbb{R}$, either $f^{-1}((a,\infty)) = \mathbb{R}$ or $\emptyset$, or is of the form $[b,\infty)$ or (b,∞) for some $b \in \mathbb{R}$. In any case, $f^{-1}((a,\infty)) \in \mathcal{B}_{\mathbb{R}}$ for all a, and since these rays generate $\mathcal{B}_{\mathbb{R}}$, f is Borel measurable.

11: Show that if $f : \mathbb{R}^n \to \mathbb{C}$ is separately continuous in each if $x_1, \ldots, x_n$, then f is Borel measurable.

SOLUTION: We proceed by induction on the dimension. The case $n = 1$ is trivial since for $n = 1$, separate continuity is continuity, which implies Borel measurability.

Now suppose the proposition is established for all $n \leq k$, and let $f : \mathbb{R} \times \mathbb{R}^k$ be a function such that for each $x \in \mathbb{R}$, $f(x, \cdot)$ is Borel measurable on $\mathbb{R}^k$, and such that for each $y \in \mathbb{R}^k$, $f(\cdot, y)$ is continuous on $\mathbb{R}$. By the inductive hypothesis, if $f : \mathbb{R}^{k+1} \to \mathbb{R}$ is separately continuous it satisfies this property, and so it suffices to show under these more general conditions that f is Borel measurable on $\mathbb{R}^{k+1}$.

For each integer i, define $a_i = i/n$, and define the functions

$$\phi_i(x) = \frac{x - a_i}{a_{i+1} - a_i} 1_{(a_i, a_{i+1}]}(x) \qquad \text{and} \qquad \psi_i(x) = \frac{x - a_{i+1}}{a_{i+1} - a_i} 1_{(a_i, a_{i+1}]}(x) \ .$$

These are both evidently the products of two Borel functions on $\mathbb{R}^{k+1}$, since the coordinate projection $(x, y) \mapsto x$ is continuous and hence Borel, we see that ϕ_i and ψ_i are Borel on $\mathbb{R}^{k+1}$ for each i.

Next, by continuity of coordinate projections and by hypothesis,

$$(x, y) \mapsto y \mapsto f(a_i, y)$$

is Borel on $\mathbb{R}^{k+1}$ for each i.

Now again using the fact that sums and products of Borel functions are Borel,

$$f_n(x, y) = \sum_{i \in \mathbb{Z}} [f(a_{i+1}, y)\phi_i(x) + f(a_i, y)\psi_i(x)]$$

is Borel measurable on $\mathbb{R}^{k+1}$. (Note that the sum converges since each summand is non-zero for exactly one value of x.)

Notice that for each x and y, $f_n(x, y)$ is a convex combination of $f(a_i, y)$ and $f(a_{i+1}, y)$ for the unique value of i such that $a_i < x \leq a_{i+1}$. Since $|a_i - x| + |a_{i+1} - x| = 1/n$, and since for each y, $f(\cdot, y)$ is continuous,

$$\lim_{n \to \infty} f_n(x, y) = f(x, y) \ .$$

Since pointwise limits of Borel functions are Borel, f is Borel.

13: Let $\{f_n\}$ be a sequence in $L^+(X, \mathcal{M}, \mu)$ such that $f_n \to f$ pointwise, and such that

$$\int_X f \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \mathrm{d}\mu < \infty \ .$$

Then

$$\int_E f \mathrm{d}\mu = \lim_{n \to \infty} \int_E f_n \mathrm{d}\mu$$

for all $E \in \mathcal{M}$. However, this need not be true if $\int_X f \mathrm{d}\mu = \infty$.

SOLUTION: Let $E \in \mathcal{M}$. By Fatou's Lemma,

$$\int_E f \mathrm{d}\mu = \int_X 1_E f \mathrm{d}\mu \leq \liminf \int_X 1_E f_n \mathrm{d}\mu = \liminf_{n \to \infty} \int_E f_n \mathrm{d}\mu \ .$$

Since $E \in \mathcal{M}$ is arbitrary, we also have that

$$\int_{E^c} f \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_{E^c} f_n \mathrm{d}\mu \ . \tag{2.1}$$

But this is the same as

$$\begin{aligned} \int_X f\mathrm{d}\mu - \int_E f\mathrm{d}\mu &\le \liminf_{n\to\infty}\left(\int_X f_n\mathrm{d}\mu - \int_E f_n\mathrm{d}\mu\right) \\ &= \int_X f\mathrm{d}\mu + \liminf_{n\to\infty}\left(-\int_E f_n\mathrm{d}\mu\right) \\ &= \int_X f\mathrm{d}\mu - \limsup_{n\to\infty}\int_E f_n\mathrm{d}\mu\ . \end{aligned}$$

where we used the fact that if $\{a_n\}$ is a convergence sequence in $\mathbb{R}$ with $\lim_{n\to\infty} a_n = a$, and $\{b_n\}$ is any sequence in $\mathbb{R}$, then $\liminf_{n\to\infty}(a_n + b_n) = a + \liminf_{n\to\infty} b_n$.

Now, since by hypothesis $\int_X f\mathrm{d}\mu$ is finite we may cancel it and conclude that

$$\limsup_{n\to\infty}\int_E f_n\mathrm{d}\mu \le \int_E f\mathrm{d}\mu\ .$$

Combining this with (2.1), we conclude that $\lim_{n\to\infty}\int_E f_n\mathrm{d}\mu = \int_E f\mathrm{d}\mu$.

To see that the condition $\int_X f\mathrm{d}\mu < \infty$ is necessary, consider $(X,\mathcal{M},\mu) = (\mathbb{R},\mathcal{L},m)$, and define

$$f_n(x) = \begin{cases} 1 & x \le 0 \\ 1/n & 0 < x \le n \\ 0 & x > n \end{cases}\ .$$

With $f = 1_{(-\infty,0]}$, $f_n \to f$ pointwise. Let $E = (0,\infty)$. Then $\int_E f\mathrm{d}m = 0$, while $\int_E f_n\mathrm{d}m = 1$ for all n.

19: Let $\{f_n\}$ be a sequence in $L^1(X,\mathcal{M},\mu)$ such that $f_n \to f$ uniformly. Then, if $\mu(X) < \infty$, $f \in L^1$ and $\lim_n \to \infty\|f_n - f\|_1 = 0$. However, if $\mu(X) = \infty$ this need not be the case.

SOLUTION: Since $f_n \to f$ uniformly, and hence pointwise, f is measurable. Now, suppose first that $\mu(X) < \infty$. Since $f_n \to f$ uniformly, for any $\epsilon > 0$, there is an N_ϵ so that for all $n \ge N_\epsilon$, $|f_n(x) - f(x)| < \epsilon$. But then for $n \ge N_\epsilon$

$$\int_X |f_n - f|\mathrm{d}\mu \le \int_X \epsilon\mathrm{d}\mu = \epsilon\mu(X). \tag{2.2}$$

Then since $|f| \le |f_n| + |f_n - f|$,

$$\int_X |f|\mathrm{d}\mu \le \int_X |f_{N_\epsilon}|\mathrm{d}\mu + \epsilon\mu(X) < \infty\ ,$$

and so $f \in L^1$.

Moreover, by (2.2), $\|f_n - f\|_1 \le \epsilon\mu(X)$ for all $n \ge N_\epsilon$, and since $\mu(X) < \infty$, this means that $\lim_{n\to\infty}\|f_n - f\|_1 = 0$.

Now let $f_n \in L^1(\mathbb{R},\mathcal{L},m)$ be given by

$$f_n = \frac{1}{n}1_{[0,n]}$$

and let $f(x) = 0$ for all x. Then for all $x \in \mathbb{R}$ and all $\epsilon > 0$, $|f_n(x) - f(x)| < \epsilon$ for all $n > 1/\epsilon$. Thus, $f_n \to f$ uniformly. But $\int_{\mathbb{R}} f_n\mathrm{d}m = 1$ for all n, and $\int_{\mathbb{R}} f\mathrm{d}m = 0$.

25: Let $f(x) = x^{-1/2}1_{(0,1)}$. Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of the rationals, and define

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - q_n)\ .$$

Then $g \in L^1$, and hence g is finite a.e., but for for all $\lambda \geq 1$, every open interval I,

$$m(I \cap \{x \ \ g(x) > \lambda\}) > 0 \ ,$$

so that g (and every function equal to it a.e.) is unbounded on every interval (and hence discontinuous at every point). Moreover, g^2 is finite a.e., but is not integrable on any interval.

SOLUTION: By direct computation, $\int_{\mathbb{R}} f \mathrm{d}m = 2$, and by the translation invariance of the Lebesgue integral, $\int_{\mathbb{R}} f(\cdot - q_n)\mathrm{d}m = 2$ for all n. It follows that for all $N \in \mathbb{N}$

$$\int_{\mathbb{R}} \left(\sum_{n=1}^{N} 2^{-n} f(\cdot - q_n) \right) \mathrm{d}m \leq 2 \ ,$$

and then by the Lebesgue Monotone Convergence Theorem,

$$\int_{\mathbb{R}} g \mathrm{d}m = \lim_{N\to\infty} \int_{\mathbb{R}} \left(\sum_{n=1}^{N} 2^{-n} f(\cdot - q_n) \right) \mathrm{d}m \leq 2 \ .$$

Hence $g \in L^1$, and thus is finite a.e.

Next, note that for all $\lambda \geq 1$

$$g(x) \geq 2^{-n} f(x - q_n) \geq \lambda \qquad \text{on} \quad (q_n, q_n + 2^{-n}/\lambda) \ .$$

If $I = (a,b)$ is any open interval, then for some n, $q_n \in I$, and then $m(I \cap (q_n, q_n + 2^{-n}/\lambda)) > 0$. On this set, $g \geq \lambda$. Hence $m(I \cap \{x \ \ g(x) \geq \lambda\}) > 0$ for all open intervals I.

If g were continuous at some x_0, necessarily g would be bounded on an interval about x_0. But we have shown that this is not the case.

Finally, since $f^2(\cdot - q_n)$ is not integrable on any open interval containing q_n, and since every open interval in $\mathbb{R}$ contains q_n for infinitely many n, and since $g^2 \geq 2^{-2n} f^2(\cdot - q_n)$ for all n, g^2 is not integrable on any open interval.

36: If a Cauchy sequence of indicator fuctions is in $L^1(X, \mathcal{M}, \mu)$, it convergs to an indicator function in $L^1(X, \mathcal{M}, \mu)$.

SOLUTION: Let $\{1_{E_n}\}_{n\in\mathbb{N}}$ be Cauchy sequence of indicator fuctions is in $L^1(X, \mathcal{M}, \mu)$. Then there is a subsequence $\{1_{E_{n_k}}\}_{k\in\mathbb{N}}$ that converges μ-*a.e.* and in L^1 to a measureable function f. Since $1_{E_{n_k}}(x) \in \{0,1\}$ for all k, $f(x) \in \{0,1\}$ on the set of convergence. Without loss of generality, we may redifine f to be zero on the null set where there is no convergence. Thus, $f(x) \in \{0,1\}$ for all x, so $f = 1_E$ where $E = F^{-1}(\{1\})$.

41: If $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, and $f_n \to f$ a.e. μ there exist measurable sets

$$E_1 \subset E_2 \subset E_3 \subset \ldots$$

with $\mu(\cap_{n=1}^{\infty} E_n^c) = 0$ and such that $f_n \to f$ uniformly on each E_n.

SOLUTION: Let A_n be a partition of X into measurable sets of finite measure. Let E_1 be a subset of A_1 with $\mu(A_1 \backslash E_1) < 1$ on which $f_n \to f$ uniformly. This set exists by Egoroff's Theorem.

Let $B_2 = A_2 \cup (A_1 \backslash E_1)$. Let E_2 be a subset of B_2 with $\mu(B_2 \backslash E_2) < 1/2$. Notice that

$$(B_2 \backslash E_2) \cup (E_1 \cup E_2) = A_1 \cup A_2 \ , \tag{2.3}$$

and hence

$$(A_1 \cup A_2)\backslash(E_1 \cup E_2) \subset B_2\backslash E_2 \ ,$$

and hence

$$\mu((A_1 \cup A_2)\backslash(E_1 \cup E_2)) < 1/2 \ .$$

Continuing, in this way, we inductively define

$$B_n = A_n \cup B_{n-1}$$

and then define E_n to be a subset of B_2 with $\mu(B_n\backslash E_n) < 1/2^n$ on which $f_n \to f$ uniformly. This set exists by Egoroff's Theorem. If we suppose that

$$(B_{n-1}\backslash E_{n-1}) \cup (\cup_{j=1}^{n-1} E_j) = (\cup_{j=1}^{n-1} A_j) \ , \tag{2.4}$$

which is true for $n = 3$ by (2.3), Then

$$\begin{aligned}(B_n\backslash E_n) \cup (\cup_{j=1}^{n} E_j) &= B_n \cup (\cup_{j=1}^{n-1} E_j) \\ &= A_n \cup (B_{n-1} \cup (\cup_{j=1}^{n-1} E_j)) \\ &= A_n \cup (\cup_{j=1}^{n-1} A_j) = \cup_{j=1}^{n-1} A_j \ .\end{aligned}$$

Thus, by induction, our construction yields (2.4) for all n. It follows that

$$\mu((\cup_{j=1}^{n-1} A_j)\backslash \cup_{j=1}^{n-1} E_j) \le \mu(B_n\backslash E_n) < 2^{-2} \ .$$

In particular, for all m, n

$$\mu(A_m \cap (\cap_{j=1}^{n} E_j^c)) < 2^{-n}$$

and so by continuity from above, using the fact that A_m has finite measure,

$$\mu(A_m \cap (\cap_{j=1}^{\infty} E_j^c)) = 0 \ .$$

Finally,

$$\cap_{j=1}^{\infty} E_j^c) = \cup_{m=11}^{\infty} [A_m \cap (\cap_{j=1}^{\infty} E_j^c)] \ .$$

and a countable union oset of measure zero is a set of measure zero. Hence

$$\mu(\cap_{j=1}^{\infty} E_j^c)) = 0$$

as was to be shown.