Notes on Lebesgue Measure on $\mathbb{R}^n$ and S^{n-1} for Math 501

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1 Construction of Lebesgue measure on $\mathbb{R}^n$

For $n \in \mathbb{N}$, define $\mathcal{E}_n$ to be the set of half open rectangles in $\mathbb{R}^n$; i.e., the sets of the form

$$\{x \in \mathbb{R}^n \ : \ \langle \mathbf{e}_j, x\rangle \in (a_j, b_j] \quad j = 1, \dots, n\ \} \tag{1.1}$$

where $\mathbf{e}_j$ is the jth standard basis vector in $\mathbb{R}^n$, and where $\langle x, y\rangle$ denotes the standard inner product in $\mathbb{R}^n$. Since $\mathcal{E}_n$ is an elementary family, $\mathcal{A}_n$, the set of all finite disjoint unions of sets in $\mathcal{E}_n$ is an algebra.

Now let $A \in \mathcal{A}_n$. By definition, A is the disjoint of finitely many half-open rectangles $E_1, \dots, E_N$. We then define

$$\rho_n(A) = \sum_{j=1}^N \rho_n(E_j)\ ,$$

and note that there is no ambiguity stemming from the fact that A can be written in more than one way as a finite disjoint union of half-open rectangles: Considering any common refinement of two such partitions of A into half-open rectangles, we see that they yield the same value for $\rho(A)$.

1.1 THEOREM. *For each $n \in \mathbb{N}$, ρ_n is a semi-finite premeasure on $\mathcal{A}_n$ that is continuous at the empty set.*

Proof. The fact the ρ_n is semifinte is clear. We prove that it is continuous at the empty set by induction on n, noting that we have already proved this for $n = 1$.

For any $E \subset \mathbb{R}^n$, and any $t \in \mathbb{R}$, define

$$E_t = \{\ x \in E \ : \ \langle \mathbf{e}_1, x\rangle = t\ \}\ .$$

Notice that if $E \in \mathcal{E}_n$, then for each t, we may identify E_t with a rectangle in $\mathcal{E}_{n-1}$ (by deleting the first coordinate), and then we have, for E given by (1.1),

$$\rho_{n-1}(E_t) = \begin{cases} \prod_{j=2}^n (b_j - a_j) & t \in (a_1, b_1] \\ 0 & t \notin (a_1, b_1]\ . \end{cases} \tag{1.2}$$

so that

$$\rho(n)(E) = \int_{\mathbb{R}} \rho_{n-1}(E_t) \mathrm{d}m(t)\ .$$

where m denotes Lebesgue measure on $\mathbb{R}$.

Now suppose that $A \in \mathcal{A}_n$ is the disjoint union of $E_1, \ldots, E_N$, where each $E_j \in \mathcal{E}_n$. Then for each t, A_t is the disjoint union of $(E_1)_t, \ldots, (E_N)_t$, and

$$\rho_{n-1}(A_t) = \sum_{j=1}^{N} \rho_{n-1}((E_j)_t) \ .$$

By (1.2), $\rho_{n-1}(A_t)$ is a measurable function of t, and then

$$\rho_n(A) = \sum_{j=1}^{N} \rho_n(E_j) = \sum_{j=1}^{N} \left[\int_{\mathbb{R}} \rho_{n-1}((E_j)_t) \mathrm{d}m(t) \right] = \int_{\mathbb{R}} \rho_{n-1}(A_t) \mathrm{d}m(t) \ .$$

Now let $\{A_j\}$ be any decreasing sequence of sets in $\mathcal{A}_n$ such that $\rho_n(A_1) < \infty$ and $\cap_{j=1}^{\infty} A_j = \emptyset$. Then for each t, $\cap_{j=1}^{\infty} (A_j)_t = \emptyset$. By the inductive hypothesis $\lim_{j\to\infty} \rho_{n-1}((A_j)_t) = 0$. Then since $(A_j)_t \subset (A_1)_t$ for all t,

$$\rho_{n-1}((A_j)_t) \leq \rho_{n-1}((A_1)_t) \qquad \text{and} \qquad \int_{\mathbb{R}} \rho_{n-1}((A_1)_t) \mathrm{d}m(t) = \rho_n(A_1) < \infty \ ,$$

we may apply the Lebesgue Dominated Convergence Theorem with $\rho_n(A_1) < \infty$ being the dominating function to show that

$$\lim_{j\to\infty} \rho_n(A_j) = \lim_{j\to\infty} \int_{\mathbb{R}} \rho_{n-1}((A_j)_t) \mathrm{d}m(t) = \int_{\mathbb{R}} \left[\lim_{j\to\infty} \rho_{n-1}((A_j)_t) \right] \mathrm{d}m(t) = 0 \ .$$

This proves continuity a the empty set. □

1.2 DEFINITION (Lebegue outer measure and measure)**.** The outer measure μ_n^* on $\mathbb{R}^n$ defined by

$$\mu_n^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \rho_n(A_j) \ : \ \{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}_n \quad \text{and} \quad E \subset \bigcup_{j=1}^{\infty} A_j \right\} \tag{1.3}$$

is the *Lebesgue outer measure on* $\mathbb{R}^n$. Its Caratheodory σ-algebra is the σ-algebra $\mathcal{L}_n$ of *Lebesgue measurable subsets of* $\mathbb{R}^n$, and the restriction m_n of μ_n^* restriction to $\mathcal{L}_n$ is Lebesgue measure on $\mathbb{R}^n$.

1.3 Remark. By decomposing each A_j into a finite union of half open rectangles, we see that we do not raise the infimum if we further require that each A_j in (1.3) is a half open rectangle. Next, slightly enlarging each of these, we see that we do not change the infimum if we further require that E is contained in the union of the interiors of the A_j. Extending the definition of ρ_n to open rectangles in the obvious way, we then have the alternate formula

$$\mu_n^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \rho_n(A_j) \ : \ \text{each } A_j \text{ an open rectangle and} \quad E \subset \bigcup_{j=1}^{\infty} A_j \right\} \tag{1.4}$$

Because ρ_n is semifinte and continuous at the empty set, the restriction of μ_n^* to $\mathcal{A}_n$ agrees with ρ_n. We also know that $\sigma(\mathcal{A}_n) \subset \mathcal{L}_n$.

1.4 PROPOSITION. *$\sigma(\mathcal{A}_n) = \mathcal{B}_n$, the Borel σ-algebra on $\mathbb{R}^n$.*

Proof. Since every open rectangle in $\mathbb{R}^n$ is a countable union of half-open rectangles in $\mathbb{R}^n$, $\sigma(\mathcal{A}_n)$ contains all open rectangles in $\mathbb{R}^n$. Since every open set in $\mathbb{R}^n$ is a countable union of open rectangles, $\sigma(\mathcal{A}_n)$ contains all open sets, and hence contains $\mathcal{B}_n$. Conversely, since each half open rectangle is a Borel set, $\sigma(\mathcal{A}_n) \subset \mathcal{B}_n$. □

Now let $E \in \mathcal{L}_n$, and $\epsilon > 0$. Then there is a sequence of open rectangles $\{A_j\}$ such that with $U := \cup_{j=1}^{\infty} A_j$, so that U is open,

$$E \subset U \qquad \text{and} \quad m_n(U) \le \sum_{j=1}^{\infty} m_n(A_j) = \sum_{j=1}^{\infty} \rho_n(A_j) \le m_n(E) + \epsilon \ .$$

This shows that m_n is *outer regular* on $\mathcal{L}$.

We next show that m_n is *inner regular*: Let $C_N = \{\ x \ :\ \langle \mathbf{e}_j, x\rangle \in [-N, N]\ ,\ j = 1, \dots n\ \}$ be the centered closed cube of side length $2N$.

For any $E \in \mathcal{L}_n$, and any $\epsilon > 0$, let U be an open set containing $C_N \cap E^c$ such that $m_n(U) \le m_n(C_N \cap E^c) + \epsilon$. Define $K_N := C_N \cap U^c$. Then K_N is compact and contained in $E \cap C_N$. Then

$$\begin{aligned} m_n(C_N) &= m_n(K_N) + m_n(C_N \cap U) \\ &\le m_n(K_N) + m_n(C_N \cap E^c) + \epsilon \ . \end{aligned}$$

Therefore,

$$m_n(K_N) \ge m_n(C_N) - m_n(C_N \cap E^c) - \epsilon = m_n(C_N \cap E) - \epsilon \ .$$

Now suppose $m_n(E) < \infty$. Then by continuity from below, there exists N so that $m_n(C_N \cap E) \ge m_n(E) - \epsilon$. Then with K_N as above, $m_n(K_N) \ge m_n(E) - 2\epsilon$. On the other hand, if $m_n(E) = \infty$, then for each k, there is an N so that $m_n(C_N \cap E) \ge k + 1$, and then, taking $\epsilon = 1$, $m_n(K_N) \ge k$. Either way, we have that

$$m_n(E) = \sup\{\ m_n(K) \ :\ K \text{ compact}, K \subset E\ \} \ .$$

This shows that m is *inner regular* on $\mathcal{L}$. We summarize and extend:

1.5 THEOREM. *Lebesgue measure m_n is inner and outer regular on the Lebesgue measurable subsets of $\mathbb{R}^n$. Moreover, for every E in $\mathcal{L}_n$, there are Borel sets F and G such that*

$$F \subset E \subset G \qquad \text{and} \qquad m_n(G \cap F^c) = 0 \ .$$

In fact, we can take F to be a a countable union of closed sets, and G to be a countable intersection of open sets.

Proof. Write $\mathbb{R}^n$ as the disjoint union of a family of bounded half open rectangles $\{C_j\}$, which we may as well take to be cubes of unit side length. Given $k \in \mathbb{N}$, for each j there exists an open set $U_{j,k}$ such that $E \cap C_j \subset U_{j,k}$ and

$$m_n(U_{j,k}) \le m_n(E \cap C_j) + \frac{1}{k2^j} \ .$$

Let $U_k = \cup_{j=1}^\infty U_{j,k}$ which is open. Then $E \subset U_k$ and

$$\mu(U_k \cap E^c) \le \sum_{j=1}^\infty m_n(U_{j,k} \cap E^c) \le \sum_{j=1}^\infty m_n(U_{j,k} \cap (E \cap C_j)^c) \le \frac{1}{k} \ .$$

Define $G = \cap_{k=1}^\infty U_k$. Then G is a G_δ set, hence Borel, $E \subset G$, and $m_n(G \cap E^c) = 0$.

Likewise, choose $F_{j,k}$ to be a compact set contained in $E \cap C_j$ so that

$$m_n(F_{j,k}) \ge m_n(E \cap C_j) - \frac{1}{k2^j} \ .$$

Let $F = \cup_{j,k=1}^\infty F_{j,k}$. Then $F \subset E$ and F is an F_σ set. Finally,

$$m_n(E \cap F^c) = \sum_{j=1}^\infty m_n((E \cap C_j) \cap F^c) \le \sum_{j=1}^\infty m_n((E \cap C_j) \cap F_{j,k}^c) \le \frac{1}{k} \ .$$

Since this is true for all k, $m_n(E \cap F^c) = 0$. □

1.6 THEOREM (Approximation of measurable sets by finite unions of rectangles)**.** *Let $E \in \mathcal{L}_n$ be such that $m_n(E) < \infty$. Then for all $\epsilon > 0$, there is a set $A \in \mathcal{A}_n$ such that*

$$m_n(A \Delta E) \le \epsilon \ . \tag{1.5}$$

Proof. We have already seen, using the Monotone Class Theorem, that for any σ-finite measure μ on the σ algebra $\sigma(\mathcal{A})$ generated by some algebra $\mathcal{A}$, the following is true: For any measurable set E with $\mu(E) < \infty$, and any $\epsilon > 0$, there is an $A \in \mathcal{A}$ so that $\mu(A \Delta E) \le \epsilon$. Since the Borel σ-algebra of $\mathbb{R}^n$ is generated by $\mathcal{A}_n$ and since m_n is σ-finite, it follows that (1.5) is true whenever E is a Borel set of finite Lebesgue measure. But since we have shown above that every Lebesgue measurable set differs form a Borel set by a set of measure zero, the theorem is true in general. □

1.7 THEOREM. *$L^1(\mathbb{R}^n, \mathcal{L}_n, m_n)$ is separable and $C_c^\infty(\mathbb{R}^n)$ is dense in it.*

Proof. Let $f \in L^1(\mathbb{R}^n, \mathcal{L}_n, m_n)$, and let $\epsilon > 0$ be given. We know from the general theory of Lebesgue integration that there is a simple function ϕ such that $\|f - \phi\|_1 \le \epsilon$. Without loss of generality, we may assume each w_j is rational. Let $\phi = \sum_{j=1}^M z_j 1_{E_j}$. Approximating each E_j by a disjoint union of half-open rectangles, that we may assume to have rational boundaries, we find a function ψ of the form $\psi = \sum_{k=1}^N w_k 1_{R_k}$ where each w_k is rational, and each R_k is a half-open rectangle with rational boundary, and $\|\phi - \psi\|_1 \le \epsilon$. By Minkowski's inequality, $\|f - \psi\|_1 \le \|f - \phi\|_1 + \|\phi - \psi\|_2 \le 2\epsilon$. since there are only countably many functions of the form of ψ, $L^1(\mathbb{R}^n, \mathcal{L}_n, m_n)$, is separable.

To see that $C_c^\infty(\mathbb{R}^n)$ is dense, consider the function

$$\varphi(t) = \begin{cases} \exp(-(1-t^2)^{-1}) & -1 < t < 1 \\ 0 & |t| \ge 1 \ . \end{cases}$$

Note that $\varphi^{1/n}(t)$ increases monotonically with n to $1_{(-1,1)}(t)$. It follows that for any $a < b$ in $\mathbb{R}$, $\varphi^{1/k}((2t - (a+b))/(b-a))$ increases monotonically to $1_{(a,b)}(t)$ as $k \to \infty$, and hence that

$$\prod_{j=1}^n \varphi^{1/k}((2\langle \mathbf{e}_j, x\rangle - (a_j + b_j))/(b_j - a_j)) \quad \uparrow \quad \prod_{j=1}^n 1_{(a_j,b_j)}(\langle \mathbf{e}_j, x\rangle) \ .$$

Hence, by the Lebesgue Monotone Convergence Theorem, the characteristic function of any open rectangle, and hence any half-open rectangle, can be arbitrarily closely approximated in $L^1(\mathbb{R}^n, \mathcal{L}_n, m_n)$ by $C_c^\infty(\mathbb{R}^n)$ functions. Combining this with our approximation of f by a finite linear combination of characteristic functions of open rectangles, we obtain the result. □

2 Transformation of measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(Y, \mathcal{N})$ be a measurable space. Let $T : X \to Y$ be measurable. Define a function $T\#\mu$ on $\mathcal{N}$ with values in $[0, \infty]$ by

$$T\#\mu(F) = \mu(T^{-1}(F)) \ .$$

Notice that if $\{F_j\}_{j\in\mathbb{N}}$ is disjoint in Y, then $\{T^{-1}(F_j)\}_{j\in\mathbb{N}}$ is disjoint in X and $T^{-1}(\cup_{j=1}^\infty F_j) = \cup_{j=1}^\infty T{-}1(F_j)$,

$$\begin{aligned} T\#(\cup_{j=1}^\infty (F_j)) &= \mu(T^{-1}(\cup_{j=1}^\infty F_j) = \mu(\cup_{j=1}^\infty T{-}1(F_j) \\ &= \sum_{j=1}^\infty \mu(T^{-1}(F_j)) = \sum_{j=1}^\infty T\#\mu(F_j) \ . \end{aligned}$$

Thus, $T\#\mu$ is a countably additive measure on $\mathcal{N}$. it is called the *push-forward* of μ by T. The following identity is the root of a number of *change of variables* formulae.

2.1 THEOREM. *Let $f \in L^+(Y, \mathcal{N})$, and let $T : X \to Y$ be measurable with respect to $\mathcal{N}$ on Y and $\mathcal{M}$ on X. Let μ be any measure on X. Then $f \circ T \in L^+(X, \mathcal{M})$, and*

$$\int_X f \circ T \mathrm{d}\mu = \int_Y f T\#\mu \ .$$

Proof. Clearly $f \circ T$ is non-negative and is measurable, so $f \circ T \in L^+(X, \mathcal{M})$, and both integrals are defined. Next suppose that f is a simple functions, say $f = \sum_{j=1}^M z_j 1_{F_j}$ where each F_j belongs to $\mathcal{N}$. Notice that

$$\mu(T^{-1}(F_j)) = \int_X 1_{F_j} \circ T(x) \mathrm{d}\mu \ .$$

for all $x \in X$, Therefore.

$$\int_Y f T\#\mu = \sum_{j=1}^M z_j \mu(T^{-1}(F_j)) = \int_X \sum_{j=1}^M z_j 1_{F_j}(T(x)) \mathrm{d}\mu = \int_X f \circ T \mathrm{d}\mu \ .$$

This proves the result for simple functions, and now the general result follows from approximation by simple functions. □

This theorem becomes useful if it can be combined with a concrete description of $\mathbb{R}\#\mu$. That is, we would like to identify $T\#\mu$ with some explicit measure μ on $(Y, \mathcal{N})$. In case $\mathcal{N}$ is generated by an algebra $\mathcal{A}$ and is strongly σ-finite under either $T\#mu$ or ν, the following theorem is useful in this regard:

2.2 THEOREM (Uniqueness of measures). *Let $(Y,\mathcal{N})$ be a measure space, and suppose that $\mathcal{N} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is an algebra of set in Y. Let ν_1 and ν_2 be two strongly σ-finite measures on $(Y,\mathcal{N})$ such that*

$$\nu_1(A) = \nu_2(A)$$

for all $A \in \mathcal{A}$. Then $\nu_1 = \nu_2$ everywhere on $\mathcal{N}$.

Proof. Suppose first that $\nu_1(Y) < \infty$, and hence that $\nu_2(Y) < \infty$, since $Y \in \mathcal{A}$. Let $\mathcal{C}$ be given by

$$\mathcal{C} = \{\ E \in \mathbb{N}\ :\ \nu_1(E) = \nu_2(E)\ \}\ .$$

Since $\mathcal{A} \subset \mathcal{C}$, it suffices to show that $\mathcal{C}$ is a monotone class. Let $\{A_j\}$ be a descending sequence of sets in $\mathcal{C}$, and let $A = \cap_{j=1}^{\infty} A_j$. We must show that $\nu_1(A) = \nu_2(A)$. But, by continuity from above, using finiteness of the measures,

$$\nu_1(A) = \lim_{j\to\infty} \nu_1(A_j) = \lim_{j\to\infty} \nu_2(A_j) = \nu_2(A)\ .$$

A similar argument using continuity form below handles the case of an ascending sequence in $\mathcal{C}$. Thus, $\mathcal{C}$ is a monotone class.

In general, write Y as a countable disjoint union of sets in A, and apply the above to each piece to conclude that the restrictions of ν_1 and ν_2 agree on each piece. By countable additivity, they agree everywhere. □

One simple consequence of this is the invariance of Lebesgue measure under translation.

2.3 THEOREM (Translation invariance of Lebesgue measure). *For all $a \in \mathbb{R}^n$, define $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau(x) = x + a$. Then*

$$\tau_a \# m_n = m_n\ .$$

Proof. Let $A \in \mathcal{A}_n$. Note that τ_a is invertible and $\tau_a^{-1} = \tau_{-a}$. Notice that $\tau_a^{-1}(A) \in \mathcal{A}$. By definition,

$$\tau_a \# m_n(A) = m_n(\tau_a^{-1}(A)) = \rho_n(\tau_a^{-1}(A))\ ,$$

since on $\mathcal{A}_n$, m_n agrees with ρ_n. But ρ_n is invariant under translations as an obvious consequence of the formula that defines it. Hence

$$\rho_n(\tau_a^{-1}(A)) = \rho_n(A) = m_n(A)$$

for all $A \in \mathcal{A}_n$. Thus, $\tau_a \# m_n$ and m_n agree on $\mathcal{A}_n$, and so by the uniqueness theorem, they agree on all Borel sets. Finally, since every Lebesgue measurable contains and is contained in a Borel set of the same measure, the result is true for all $A \in \mathcal{L}_n$ as well. □

3 The push-forward of Lebesgue measure under an invertible linear transformation

3.1 THEOREM. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then*

$$T \# m_n = |\det(T)|^{-1} m_n\ . \tag{3.1}$$

Consequently, for $f \in L^+(\mathbb{R}^n, \mathcal{L}_n, m_n)$,

$$|\det T| \int_{\mathbb{R}^n} f \circ T \mathrm{d}m_n = \int_{\mathbb{R}^n} f \mathrm{d}m_n \ . \tag{3.2}$$

Proof. The second part follows from the first part and Theorem 2.1. Hence it suffices to prove (3.1). The proof of this rests on the following elementary fact from linear algebra: Let E be any parallelepiped in $\mathbb{R}^n$. That is, E is some translate of the image of the unit cube in $\mathbb{R}^n$ under some invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then for every $\epsilon > 0$, there are to sets F and G in $\mathcal{A}_n$ such that $F \subset E \subset G$ and

$$\rho_n(G) - \epsilon \le |\det(T)| \le \rho_n(F) + \epsilon \ .$$

Since

$$\rho_n(F) \le m_n(E) \le m_n(G) \ ,$$

it follows that

$$m_n(E) = |\det(T)| \ .$$

Now let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Let S denote the inverse of T. Then S is also linear and has a matrix representation. Let $\mathbf{v}_j$ denote the jth row of this matrix so that

$$S(x) = (\langle \mathbf{v}_1, x\rangle, \dots, \langle \mathbf{v}_n, x\rangle) \ . \tag{3.3}$$

Since S is invertible, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, and hence a basis for $\mathbb{R}^n$. Conversely, any basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ generates an invertible linear transformation S on $\mathbb{R}^n$ through (3.3).

Now let

$$E := \{x \in \mathbb{R}^n \ : \ \langle \mathbf{e}_j, x\rangle \in (a_j, b_j] \quad j = 1, \dots, n\ \} \tag{3.4}$$

be a generic element of $\mathcal{E}_n$. Then

$$T(E) = S^{-1}(E) = \{x \in \mathbb{R}^n \ : \ \langle \mathbf{v}_j, x\rangle \in (a_j, b_j] \quad j = 1, \dots, n\ \} \ ,$$

is a rectangle in coordinates based on the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. It is evident that the sets of this form are an elementary family, and so the set of all finite disjoint unions of such sets forms an algebra that we denote $\mathcal{A}_T$. It is clear that T sets up a one-to-one correspondence between $\mathcal{A}_T$ and $\mathcal{A}_n$. That is, $E \in \mathcal{A}_n$ if and only if $T(A) \in \mathcal{A}_T$.

We define a premeasure ρ_T on $\mathcal{A}_T$ by

$$\rho_T(A) = \rho_n(T^{-1}(A)) \tag{3.5}$$

for all $A \in \mathcal{A}_T$, This is well defined since $T(A) \in \mathcal{A}_n$ for all $A \in \mathcal{A}_T$. Since ρ_n is strongly σ-finite, and hence strongly semifinite, so is ρ_T.

Moreover, if $\{A_j\}$ is a decreasing sequence with $\cap_{j=1}^\infty A_j = \emptyset$, then $\cap_{j=1}^\infty T^{-1}(A_j) = \emptyset$. It follows from the fact that ρ_n is continuous at the empty set that ρ_T has this property too.

It is clear that just as with $\mathcal{A}_n$, $\sigma(\mathcal{A}_T) = \mathcal{B}_n$. Let m_T denote the Borel measure on $\mathbb{R}^n$ obtained by restricting the outer measure generated by ρ_T to $\mathcal{B}_n$. Since, as noted at the beginning of the proof, the Lebesgue measure of every parallelepiped in $\mathbb{R}^n$ may be computed by evaluating an appropriate determinant, and since m_n extends ρ_n, it follows from (3.5) that

$$\rho_n(T^{-1}(A)) = |\det(T)|^{-1} m_n(A) \ ,$$

and hence that for all $A \in \mathcal{A}_T$,

$$m_T(A) = \rho_T(A) = |\det(T)|^{-1} m_n(A) \ , \tag{3.6}$$

Since m_T and $|\det(T)|m_n$ agree on the algebra $\mathcal{A}_T$, and since this algebra generates $\mathcal{B}_n$, it follows from Theorem 2.2 that $m_T = |\det(T)|^{-1} m_n$ on all of $\mathcal{B}_n$.

Furthermore, since m_T agrees with ρ_T on $\mathcal{A}_T$ and m_n agrees with ρ_n on $\mathcal{A}_n$, (3.5) tells us that

$$m_T(A) = m_n(T^{-1}A) \tag{3.7}$$

for all $A \in \mathcal{A}_T$. By Theorem 2.2 once more, this holds for all Borel sets A. That is, for all $A \in \mathcal{B}_n$, $m_T(A) = (T^{-1})\# m_n(A)$. Combining this with (3.6) we obtain

$$T\# m_n(A) = |\det(T)|^{-1} m_n(A) \ , \tag{3.8}$$

□

3.2 THEOREM (Rotation invariance of Lebesgue measure)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any orthogonal transformation. Then*

$$T\# m_n = m_n \ .$$

Proof. If T is orthogonal, $|\det(T)| = 1$, and the result follows from the previous theorem. □

3.3 THEOREM (Rotation invariance of Lebesgue measure)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any orthogonal transformation. Then*

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Proof. If T is orthogonal, $|\det(T)| = 1$, and the result follows from the previous theorem. □

3.4 THEOREM (Transformation of Lebesgue measure under dilation)**.** *For $t \in \mathbb{R}$ define $\varsigma_t : \mathbb{R}^n \to \mathbb{R}^n$ by $\varsigma_t(x) = e^{-t}x$. Then*

$$\varsigma_t \# m_n = e^{nt} m_n \ .$$

Proof. The transformation ς_t is linear and the corresponding matrix is e^{-t} times the identity, whose determinant is e^{-tn}. Hence this result is a direct consequence of (3.1). □

3.5 Remark. The set of transformations ς_t, $t \in \mathbb{R}$, form an abelian group of transformation acting on $\mathbb{R}^n$ since for all $s, t \in \mathbb{R}$ $\varsigma_s \circ \varsigma_t = \varsigma_{s+t}$, and ς_0 is the identity transformation. The group of translation is another abelian group action on $\mathbb{R}^n$, while the rotation are a non-abelian group of transformation on $\mathbb{R}^n$.

4 Lebesgue measure on the sphere S^{n-1}

We let S^{n-1} denote the unit sphere in $\mathbb{R}^n$; i.e., $S^{n-1} = \{x \in \mathbb{R}^n \ : \ \|x\| = 1 \ \}$. Define the map $\Omega : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ by

$$\Omega(x) = \frac{x}{\|x\|} \ .$$

Notice the Ω is continuous, and hence a Borel transformation.

4.1 DEFINITION (Lebesgue measure on S^{n-1})**.** Let X denote the punctured closed unit ball in $\mathbb{R}^n$; i.e, $X = \{x \in \mathbb{R}^n \ : \ 0 < \|x\| \le 1 \ \}$, and let μ denote the restriction of Lebesgue measure m_n to X. That is, for E a Borel set in $\mathbb{R}^n$,

$$\mu(E) = m_n(E \cap X) \ .$$

We define the *Lebesgue measure on* S^{n-1} to be the Borel measure σ_n on S^{n-1} given by

$$\sigma_n(A) = n\Omega\#\mu(A) = n\mu(\Omega^{-1}(A)) \tag{4.1}$$

for all Borel sets $A \subset S^n$.

4.2 Remark. The factor of n in the definition of σ_n is to make $\sigma_n(S^{n-1})$ equal to n times the Lebesgue measure of the unit ball which will be seen to be the "right" normalization below.

Notice that the map Ω commutes with rotations. That is, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is any rotation

$$T\Omega(x) = \Omega T(x)$$

for all $x \in \mathbb{R}^n$ since $\|T(x)\| = \|x\|$ for all x, and T is linear. This has the following consequence:

4.3 THEOREM (Invariance of σ_n under rotations)**.** *Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be any rotation, which then acts on S^{n-1} be restriction.*

$$T\#\sigma_n = \sigma_n \ . \tag{4.2}$$

Proof. Let A be any Borel set in S^{n-1}, and T any rotation. Then, since T^{-1} is also a rotation, $\Omega^{-1} \circ T^{-1} = T^{-1} \circ \Omega^{-1}$. Therefore,

$$\sigma_n(T^{-1}A) = n\mu(\Omega^{-1}T^{-1}(A)) = n\mu(T^{-1}\Omega^{-1}(A)) = nT\#\mu(\Omega^{-1}(A)) = n\mu(\Omega^{-1}(A)) = \sigma_n(A) \ .$$

□

We now define a homeomorphism $\Phi : (0,\infty) \times S^n$ onto $\mathbb{R}^n\backslash\{0\}$ by

$$\Phi(r,\omega) = r\omega \ .$$

Since both Φ and Φ^{-1} are continuous, they are both Borel. Notice that

$$\Omega^{-1}(A) \cap X = \Phi((0,1] \times A) \ ,$$

and therefore

$$\frac{\sigma_n(A)}{n} = m_n(\Phi((0,1] \times A)) \ . \tag{4.3}$$

4.4 LEMMA. *For any $a < b \in (0,\infty)$, and any Borel set $A \subset S^{n-1}$,*

$$m_n(\Phi((a,b] \times A)) = \frac{b^n - a^n}{n}\sigma_n(A) \ . \tag{4.4}$$

Proof. Notice that (4.3) is the special case of (4.4) corresponding to $a = 0$ and $b = 1$. To get the general case, we use the dilation propeties of Lebsgue measure. Let t be such that $e^{-t} = b$. Then

$$\Phi((0, b] \times A) = \varsigma_t(\Phi((0, 1] \times A)) = \varsigma_{-t}^{-1}(\Phi((0, 1] \times A)) ,$$

and therefore,

$$m_n(\Phi((0, b] \times A)) = \varsigma_{-t}\#m_n(\Phi((0, 1] \times A)) = e^{-tn}m_n((0, 1] \times A) ,$$

where in the last equality we have used (4.2). Combining this with (4.3) and recalling that $e^{-t} = b$, we obtain

$$m_n(\Phi((0, b] \times A)) = \frac{b^n}{n}\sigma_n(A) .$$

But then

$$m_n(\Phi((a, b] \times A)) = m_n(\Phi((0, b] \times A)) - m_n(\Phi((0, a] \times A))$$

and we obtain (4.4). □

4.5 Remark. Taking $a = 0$, $b = 1$ and $A = S^{n-1}$, we see that with B_n denoting the unit ball on $\mathbb{R}^n$,

$$m_n(B_n) = \frac{1}{n}\sigma_n(S^{n-1}) ,$$

and that with this normalization of σ_n,

$$\sigma_n(S^{n-1}) = \lim_{r\uparrow 1} \frac{m_n(\Phi((r, 1] \times S^{n-1})}{1 - r} ,$$

where $\Phi((r, 1] \times S^{n-1})$ is a spherical shell of thickness $1-r$. Thus, the factor of n in (4.1) is natural.

4.6 DEFINITION. Let $F_n(t) = t^n/n$ for $t \in (0, \infty)$. Since F_n is right continuous, There is a unique Lebesgue-Stiltjes measure ϱ_n on $(0, \infty)$ such that for all $a < b \in (0, \infty)$,

$$\varrho_n((a, b]) = F_n(b) - F_n(a) .$$

We write ϱ_n to denote this measure in what follows.

4.7 Remark. Note that for $a < b \in (0, \infty)$,

$$\varrho_n((a, b]) = \int_a^b r^{n-1}\mathrm{d}r .$$

The point of this definition is that we may now rewrite (4.4) as

$$m_n(\Phi((a, b] \times A)) = \varrho_n \otimes \sigma_n((a, b] \times A) ,$$

and thus,

$$\Phi^{-1}\#m_n((a, b] \times A) = \varrho_n \otimes \sigma_n((a, b] \times A) , \tag{4.5}$$

Let $\mathcal{S}_n$ be the algebra consisting of all disjoint unions of sets of the form $(a, b] \times A$, $a < b \in (0, 1]$ and $A \in \mathcal{B}_{S^{n-1}}$. Then $\sigma(\mathcal{S}_n)$ is easily seen to be the product σ-algebra $\mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{S^{n-1}}$. By Theorem 2.2, we conclude that

$$\Phi^{-1}\#m_n = \varrho_n \otimes \sigma_n \qquad \text{and consequently} \qquad m_n = \Phi\#(\varrho_n \otimes \sigma_n) .$$

This identification of m_n as the push-forward under Φ of $\varrho_n \otimes \sigma_n$ leads to the following theorem for integration in polar coordinates:

4.8 THEOREM (Integration in polar coordinates)**.** *Let* $f \in L^+(\mathbb{R}^n)$*. Then*

$$\int_{\mathbb{R}^n} f \mathrm{d}m_n = \int_{(0,\infty)} \left[\int_{S^{n-1}} f(r\omega)\mathrm{d}\sigma_n(\omega) \right] \mathrm{d}\varrho_n(r) = \int_{S^{n-1}} \left[\int_{(0,\infty)} f(r\omega)\mathrm{d}\varrho_n(r) \right] \mathrm{d}\sigma_n(\omega) \ . \tag{4.6}$$

Proof. For $r \in (0,\infty)$ and $\omega \in S^{n-1}$, $\Phi(r,\omega) = r\omega$, and hence

$$\int_{(0,\infty)\times S^{n-1}} f(r\omega)\mathrm{d}\varrho_n \otimes \sigma_n = \int_{(0,\infty)\times S^{n-1}} f \circ \Phi(r,\omega)\mathrm{d}\varrho_n \otimes \sigma_n = \int_{\mathbb{R}^n\backslash\{0\}} f \mathrm{d}m_n = \int_{\mathbb{R}^n} f \mathrm{d}m_n \ .$$

Finally, by Tonelli's Theorem we have (4.8). □

In particular, since

$$\int_{\mathbb{R}} e^{-t^2/2}\mathrm{d}t = \sqrt{2\pi} \ ,$$

by Tonelli's Theorem once more,

$$\begin{aligned} (2\pi)^{n/2} &= \int_{\mathbb{R}^n} e^{-}\|x\|^2/2\mathrm{d}m_n \\ &= \sigma_n(S^{n-1}) \int_{(0,\infty)} e^{-r^2/2} r^{n-1}\mathrm{d}r \\ &= \sigma_n(S^{n-1}) 2^{n/2-1} \int_{(0,\infty)} e^{-u} u^{n/2-1}\mathrm{d}u \\ &= \sigma_n(S^{n-1}) 2^{n/2-1}\Gamma\left(\frac{n}{2}\right) \ . \end{aligned}$$

Therefore,

$$\sigma_n(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad m_n(B_n) = n\frac{2\pi^{n/2}}{\Gamma(n/2)} \ .$$

5 Properties of Lebesgue measure on S^{n-1}

There is an important identification of S^{n-1} with the one-point compactification of $\mathbb{R}^{n-1}$ through the *stereographic projection.*

5.1 DEFINITION (Stereographic projection)**.** Let $\omega_0 = -\mathbf{e}_n$ be the "South Pole" in S^{n-1}. For any other $\omega \in S^{n-1}$, define $T(\omega)$ to be the intersection of the line through ω_0 and ω with the hyperplane $\{x \in \mathbb{R}^n \ : \ \langle \mathbf{e}_n, x\rangle = 0\}$, which we may idetify with $\mathbb{R}^{n-1}$ in the natural way. Then $T : S^{n-1}\backslash\omega_0 \to \mathbb{R}^{n-1}$ is the stereographic of $S^{n-1}\backslash\omega_0$ onto $\mathbb{R}^{n-1}$. Let $\mathbb{R}^{n-1} \cup \infty$ be the one point compactification of $\mathbb{R}^{n-1}$, so that the neighborhoods of ∞ are the complements of compact sets in $\mathbb{R}^{n-1}$. Then we define $T(\omega_0) = \infty$, which yields the stereographc projection of S^{n-1} onto the one-point compactification of $\mathbb{R}^{n-1}$.

It is easy to work out a formula for T. Let us write vectors in $\mathbb{R}^n$ in the form (v,z) where $\mathbf{v} \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$. Then for any $\omega = (v,z)$ other than ω_0, the line through ω_0 and ω is parameterized by

$$(1-t)\omega_0 + t\omega = (tv, t(1+z)-1) \ .$$

Then $\langle (tv, t(1+z)-1), \mathbf{e}_n\rangle = 0$ reduces to $t = (1+z)^{-1}$, and hence

$$T((v,z)) = \frac{1}{1+z}v \ . \tag{5.1}$$

It is also useful to have a formula for the inverse. Let $x := T((v,z)$, so that $(1+z)x = v$. Since $(v,z)\ inS^{n-1}$, $\|v\|^2 + z^2 = 1$, and so $(1+z)^2\|x\|^2 = 1 - z^2$ which is readily solved for z in terms of $\|x\|^2$, and then $(1+z)x = v$ gives us v:

$$T^{-1}(x) = \frac{1}{1+\|x\|^2}(2x, 1-\|x\|^2) \ . \tag{5.2}$$

The map T is evidently a homeomorphism of $S^{n-1}\backslash\omega_0$ onto $\mathbb{R}^{n-1}$. We may use it to transfer the half open reactangle algebra of sets $\mathbb{R}^{n-1}$ to an algebra of sets in $S^{n-1}\backslash\omega_0$, since any bijective image of an algebra is an algebra. Call this algebra $\mathcal{A}_{S^{n-1}}$. We know that every open set in $\mathbb{R}^{n-1}$ can be written as a countable union of sets in the half-open rectagle algebra on $\mathbb{R}^{n-1}$, and then, since T is a homeomorhpism, every open set in $S^{n-1}\backslash\omega_0$ is a countable union of sets in $\mathcal{A}_{S^{n-1}}$. It follows that

$$\mathcal{B}_{S^{n-1}} = \sigma(\mathcal{A}_{S^{n-1}}) \ .$$

Now let μ be *any* Borel measure μ on $S^{n-1}\backslash\omega_0$ such that $\mu(S^{n-1}) < \infty$. By our general results concerning measures on σ-algebras generated by algebras, it follows that every set $E \in mathcalB_{S^{n-1}\backslash\omega_0}$ has the property that for every $\epsilon > 0$, there is a set $A \in \mathcal{A}_{S^{n-1}}$ such that

$$\mu(E\Delta A) \leq \epsilon \ . \tag{5.3}$$

(The condition that $\mu(S^{n-1}) < \infty$ ensures that $\mu(E) < \infty$, a requirement of the general theorem.)

It then follows, in the usual way, that every $f \in L^1(S^{n-1}\backslash, \omega_0\mathcal{B}_{S^{n-1}}, \mu)$ may be approximated by a really simple function, and then rrounding the corners", by a continuous function.

Finally, if $\mu(\{\omega_0\}) = 0$, there is no difference between $L^1(S^{n-1}\backslash\omega_0, \mathcal{B}_{S^{n-1}}, \mu)$ and $L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \mu)$. We have proved:

5.2 THEOREM. *Let μ be any Borel measure on S^{n-1} such that $\mu(S^{n-1}) < \infty$ and such that $\mu(\omega) = 0$ for all single points ω. Then $L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \sigma_n)$ is seperable, and the continuous functions $C(S^{n-1})$ are dense in it.*

We next prove a uniqueness theorem for Lebesgue measure on S^{n-1}.

5.3 THEOREM. *Let μ be any Borel measure on S^{n-1} such that $\mu(S^{n-1}) < \infty$, and such that for all rotations R,*

$$R\#\mu = \mu \ .$$

Then

$$\frac{\mu(S^{n-1})}{\sigma_n(S^{n-1})}\mu = \sigma_n \ .$$

The proof of this theorem is based on some lemmas of independent interest that we now present. In what follows, L^2 denote $L^2(S^{n-1}, \mathcal{B}_{S^{n-1}}, \sigma_n)$ and $\|f\|_2^2 = \int_{S^{n-1}} |f|^2 \mathrm{d}\sigma_n$.

Given any $f \in L^2$, we define Rf to be the function $f\circ R$. Then $f \to Rf$ is linear, and since σ_n is rotation invariant,

$$\|f\|_2 = \|Rf\|_2 \ .$$

Next, for any rotation R define

$$A_R f = \frac{1}{2}(f + R) \ .$$

This operation is also linear, and it is a contraction in the L^2 norm by the Minkowski inequality and the fact that $\|Rf\|_2 = \|f\|_2$. Moreover, if f is continuous, Rf has the same modulus of continuity as f, and the average of f and Rf, namely $A_R f$, therefore has a modulus of continuity no greater than that of f.

5.4 LEMMA. *For any real valued $f \in L^2$, and any rotation R,*

$$\|A_R f\|_2 \leq \|f\|_2 \ ,$$

and there is equality if and only if $Rf = f$ in L^2. More generally, for any finite set $\{R_1, \ldots, R_m\}$ of rotations,

$$\|A_{R_m} \cdots A_{R_1} f\|_2 \leq \|f\|_2 \ ,$$

and there is equality if and only if $R_j f = f$ for each $j = 1, \ldots, m$.

Proof. We compute

$$\left\|\frac{1}{2}(f + Rf)\right\|_2^2 = \frac{1}{4}(\|f\|_2^2 + \|Rf\|_2^2 + 2\int_{S^{n-1}} fRf \mathrm{d}\sigma_n \ .$$

However, by the rotation invariance of σ_n, $\|Rf\|_2 = \|f\|_2$, and by the Cauchy-Schwarz inequality,

$$\int_{S^{n-1}} fRf \mathrm{d}\sigma_n \leq \|f\|_2 \|Rf\|_2 = \|f\|_2^2 \ .$$

This proves the inequality. Note that there is equality in the Cauchy-Scwarz inequality if and only if $\|Rf\|_2 f = \|f\|_2 Rf$, whihc reduces to $f = Rf$.

For the second part, consider first the case $m = 2$, and suppose $\|A_{R_2} A_{R_1} f\|_2 = \|f_2\|$. By the first part,

$$\|A_{R_2} A_{R_1} f\|_2 \leq \|A_{R_1} f\|_2 \qquad \text{and} \qquad \|A_{R_1} f\|_2 \leq \|f\|_2 \ .$$

We must have euqality in both inequalities. By what we have proved above, equality on the right implies that $R_1 f = f$, and then of course $A_{R_1} f = f$. Then the inequaity on the left reduces to $\|A_{R_2} f\|_2 \leq \|f\|_2$, and be what we have proved above, equality here implies that $R_2 f = f$. The general case follows in the same way. □

5.5 LEMMA. *Let f be any continuous functions on S^{n-1}. Let $\{R_1, \ldots, R_m\}$ be any finite set of rotations. Define a sequence $\{f_j\}$ by*

$$f_0 = f \qquad \text{and} \qquad f_{j+1} = A_{R_m} \cdots A_{R_1} f_j \ .$$

Then f_j converges uniformly to a continuous function h such that

$$R_j h = h$$

for $j = 1, \ldots, m$. Moreover, there is a choice of $m = n - 1$ rotations $\{R_1, \ldots, R_{n-1}\}$ for which the limiting function h is constant.

Proof. By what we have noted above, the sequence $\{f_j\}$ is uniformly equicontinuous and equibounded, by Arzela-Ascoli Theorem, there is a subsequence $\{f_{j_k}\}$ and $h \in C(S^{n-1})$ such that $f_{j_k} \to h$ uniformly.

Next note that by the lemma, $\|f_j\|_2$ is monotone decreasing. Define

$$c = \lim_{\mathrm{J}\to\infty} \|f_j\|_2 \ .$$

Since uniform convergence implies L^2 convergence for finite measure spaces,

$$\|h\|_2 = \lim_{k\to\infty} \|f_{j_k}\|_2 = c \ .$$

Since the linear operator $A_{R_m} \cdots A_{R_1}$ is continuous (it is even a contraction),

$$[A_{R_m} \cdots A_{R_1}]f_{j_k} \to [A_{R_m} \cdots A_{R_1}]h$$

uniformly. But the left hand side is f_{j_k+1}, and so

$$\|A_{R_m} \cdots A_{R_1} h\|_2 = \lim_{k\to\infty} \|f_{j_k+1}\|_2 = c \ .$$

That is,

$$\|A_{R_m} \cdots A_{R_1} h\|_2 = \|h\|_2 \ .$$

The lemma now implies that h is invariant under each of $R_1, \dots, B_m$.

Let $\|\cdot\|_\infty$ denote the supremum norm, whch gives the uniform topology. Since for any continuous g,

$$\|A_{R_m} \cdots A_{R_1} g - h\|_\infty = \|A_{R_m} \cdots A_{R_1}(g-h)\|_\infty \leq \|g-h\|_\infty \ ,$$

The fact that $\|f_{j_k} - h\|_\infty \to 0$ implies that $\|f_j - h\|_\infty \to 0$ along the whole seuqence. The final part is left as an exercise for the reader. □

Proof of Theorem 5.3. We may suppose $\mu(S^{n-1}) \neq 0$, or else the claim is trivial. Then normalizing, we may suppose without loss of generality that $\mu(S^{n-1}) = \sigma_n(S^{n-1})$.

Next, if $\mu(\{\omega\}) = c > 0$ for some ω, $\mu(\{R\omega\}) = c$ for every rotation R. We can choose an infinite sequence of rotations to obtain an infinite sequences of distinct points in this way. This would force $\mu(S^{n-1}) = \infty$, and so μ does not charge single points. Thus, Theorem 5.2 applies to μ, σ_n and to $\mu + \sigma_n$.

It suffices to show that

$$\int_{S^{n-1}} f \mathrm{d}\mu = \int_{S^{n-1}} f \mathrm{d}\sigma_n \tag{5.4}$$

for all $f \in C(S^{n-1})$. This is beacuse if E is any Borel set, 1_E may be approximated by continuous functions in $L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \mu + \sigma_n)$ by Theorem 5.2. But if

$$\lim_{n\to\infty} \|f_n - 1_E\|_{L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \mu+\sigma_n)} = 0 \ ,$$

then $f_n \to 1_E$ in both $L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \mu)$ and $L^1(S^{n-1}, \mathcal{B}_{S^{n-1}}, \sigma_n)$.

Therefore, if (5.4) is true for all continuous functions f,

$$\begin{aligned}\mu(E) = \int_{S^{n-1}} 1_E \mathrm{d}\mu &= \lim_{n\to\infty} \int_{S^{n-1}} f_n \mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_{S^{n-1}} f_n \mathrm{d}\sigma_n = \int_{S^{n-1}} 1_E \mathrm{d}\sigma_n = \sigma_n(E) \ .\end{aligned}$$

Next, given any continuous function f, in the lemmas above we have constructed a seuqence $\{f_j\}$ of continuous function that converges uniformyl to constant functions h. Morover since each f_j is an average over rotations of f and since both μ and σ_n are rotation inavariant,

$$\int_{S^{n-1}} f_j \mathrm{d}\sigma_n = \int_{S^{n-1}} f \mathrm{d}\sigma_n \quad \text{and} \qquad \int_{S^{n-1}} f_j \mathrm{d}\mu = \int_{S^{n-1}} f \mathrm{d}\mu$$

for all j .

Since uniform convergence implies convergence of integrals on a finite measure space,

$$\int_{S^{n-1}} f_j \mathrm{d}\sigma_n \to \int_{S^{n-1}} h \mathrm{d}\sigma_n \quad \text{and} \qquad \int_{S^{n-1}} f_j \mathrm{d}\mu \to \int_{S^{n-1}} h \mathrm{d}\mu$$

as $j \to \infty$. But since h is constant and μ and σ_n have the same total mass,

$$\int_{S^{n-1}} h \mathrm{d}\sigma_n = \int_{S^{n-1}} h \mathrm{d}\mu \ .$$

Combining the last three identities, we obtain (5.4) □