Notes on uniform integrability and Vitali's Theorem for Math 501

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1 Introduction

Integrable functions cannot be too heavily concentrated on small sets, nor can they be too spread out over sets of infinite measure. As we know, they can be well-approximated n the L^1 metric by simple functions that are bounded and supported on sets of finite measure. The following results is one way to quantify this that turns out to be very useful:

1.1 THEOREM (Concentration Properties of Integrable Functions)**.** *Let f be an integrable function on $(\mathcal{O}, \mathcal{S}, \mu)$. Then the integral of $|f|$ over a measurable sets E with small measure $\mu(E)$ is uniformly small in the sense that*

$$\lim_{\delta \to 0} \left(\sup \left\{ \int_E |f| \mathrm{d}\mu \;\middle|\; \mu(E) \leq \delta \right\} \right) = 0 \ . \tag{1.1}$$

Moreover, for any $\epsilon > 0$, there is a measurable set A_ϵ such that

$$\mu(A_\epsilon) < \infty \qquad \text{and} \qquad \int_{A_\epsilon^c} |f| \mathrm{d}\mu < \epsilon \ . \tag{1.2}$$

Proof: Define $f_N := f 1_{\{1/N \leq |f| \leq N\}}$ and define $B_N := \{x \mid N \geq |f(x)| \geq 1/N\}$. Clearly,

$$\int_\Omega |f| \mathrm{d}\mu \geq \int_{B_N} |f| \mathrm{d}\mu \geq \int_{B_N} \frac{1}{N} \mathrm{d}\mu = \frac{\mu(B_N)}{N} \ .$$

Thus,

$$\mu(B_N) \leq N \int_\Omega |f| \mathrm{d}\mu < \infty \ .$$

Next,

$$\lim_{N \to \infty} |f_N(x)| = |f(x)|$$

for all x, and morover, $\{|f_N(x)|\}$ is an increasing sequence. Thus, by the Montone Convergence Theorem,

$$\lim_{N \to \infty} \int_\Omega |f_N| \mathrm{d}\mu = \int_\Omega |f| \mathrm{d}\mu \ .$$

Now pick N so large that

$$\int_\Omega |f_N|\mathrm{d}\mu \geq \int_\Omega |f|\mathrm{d}\mu - \epsilon \ .$$

Then, since $|f| = |f_N|$ on B_N,

$$\int_{B_N^c} |f|\mathrm{d}\mu = \int_\Omega (|f| - |f|_N)\mathrm{d}\mu \leq \epsilon \ .$$

Therefore, we can take $A_\epsilon = B_N$ for this choice of N.

Next observe that $|f(x)| \leq N_\epsilon$ on A_ϵ, where we have used N_ϵ to denote the particular value of N chosen above, which depends on ϵ. Therefore, for any measurable set E,

$$\begin{aligned}\int_E |f|\mathrm{d}\mu &\leq \int_{E\cap A_\epsilon} N_\epsilon \mathrm{d}\mu + \int_{A_\epsilon^c} |f|\mathrm{d}\mu \\ &\leq \int_E N_\epsilon \mathrm{d}\mu + \epsilon \\ &\leq N_\epsilon \mu(E) + \epsilon \ .\end{aligned}$$

In particular, define $\delta_\epsilon = \epsilon/N_\epsilon$. Then

$$\mu(E) \leq \delta_\epsilon \Rightarrow \int_E |f|\mathrm{d}\mu < 2\epsilon \tag{1.3}$$

Note that (1.1) is equivalent to the existence, for each $\epsilon > 0$, of a $\delta_\epsilon > 0$ such that (1.3) is true. □

When a set of $\mathcal{F}$ of integrable functions have their concentration quantitatively controlled in a *uniform way*, then it is possible to deduce a convergence theorem for integrals of sequences of functions $\{f_n\}$ chosen from $\mathcal{F}$ using Egoroff's Theorem. The next definition explains what we mean by uniform control on concentration.

1.2 DEFINITION (Uniform Integrability). Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, and $\mathcal{F}$ a set of measurable functions on Ω. Then $\mathcal{F}$ is *uniformly integrable* in case

(i)

$$\lim_{\delta\to 0}\left(\sup\left\{\int_E |f|\mathrm{d}\mu \ \middle|\ \mu(E) \leq \delta\ ,\ f \in \mathcal{F}\right\}\right) = 0\ , \tag{1.4}$$

and for all $\epsilon > 0$ there exists a measurable set A_ϵ with $\mu(A_\epsilon) < \infty$ such that for all f in $\mathcal{F}$,

(ii)

$$\int_{A_\epsilon^c} |f|\mathrm{d}\mu < \epsilon\ . \tag{1.5}$$

(iii) For some finite C, $\int_\Omega |f|\mathrm{d}\mu \leq C$ for all $f \in \mathcal{F}$.

Notice that (1.4) is equivalent to the existence, for each $\epsilon > 0$, of a $\delta_\epsilon > 0$ such that

$$\mu(E) \leq \delta_\epsilon \Rightarrow \int_E |f|\mathrm{d}\mu < \epsilon \qquad \text{for all} \quad f \in \mathcal{F}\ . \tag{1.6}$$

1.3 EXAMPLE. *Let h be a non negative integrable function, and let $\mathcal{F}$ be the set of complex valued measurable functions defined by*

$$\mathcal{F} = \{ f \mid |f| \le h \}$$

Then $\mathcal{F}$ is uniformly integrable.

Indeed, given $\epsilon > 0$ let A_ϵ and δ_ϵ be such that $\mu(A_\epsilon) < \infty$, $\int_{A_\epsilon^c} |h| \mathrm{d}\mu < \epsilon$, and $\mu(E) < \delta_\epsilon \Rightarrow \int_E |h| \mathrm{d}\mu < \epsilon$. Since $|f| \le |h|$, the same A_ϵ and δ_ϵ work for each f in $\mathcal{F}$.

1.4 THEOREM (Vitali's Theorem)**.** *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, and let $\mathcal{F}$ be a uniformly integrable set of functions on $(\Omega, \mathcal{S}, \mu)$. Suppose that $\{f_n\}$ is a sequence of functions in $\mathcal{F}$ with $\lim_{n\to\infty} f_n = f$ pointwise or in measure. Then*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \mathrm{d}\mu = 0 \ . \tag{1.7}$$

Conversely, suppose that $\{f_n\}$ is any sequence of integrable functions and that (1.7) holds. Then the set $\mathcal{F}$ consisting of the functions f_n in the sequence, together with the limit f, is uniformly integrable.

Proof: Fix $\epsilon > 0$, and let A_ϵ and δ_ϵ be such that (1.4) and (1.5) hold for all g in $\mathcal{F}$, and each f_n in particular. Then when $\lim_{n\to\infty} f_n = f$ pointwise, by Fatou's Lemma,

$$\begin{aligned}
\int_\Omega |f| \mathrm{d}\mu &\le \liminf_{n\to\infty} \int_\Omega |f_n| \mathrm{d}\mu \le C \\
\int_{A_\epsilon^c} |f| \mathrm{d}\mu &\le \liminf_{n\to\infty} \int_{A_\epsilon^c} |f_n| \mathrm{d}\mu < \epsilon \\
\int_E |f| \mathrm{d}\mu &\le \liminf_{n\to\infty} \int_E |f_n| \mathrm{d}\mu
\end{aligned}$$

so that $\int_\Omega |f| \mathrm{d}\mu \le C$, and $\mu(E) \le \delta_\epsilon \Rightarrow \int_E |f| \mathrm{d}\mu \le \epsilon$. Thus, f has the same uniform integrability properties as the members of $\mathcal{F}$.

When $\lim_{n\to\infty} f_n = f$ in measure, then a subsequence converges pointwise, and arguing as above, we draw the same conclusion concerning f in this case.

Now let us suppose $\lim_{n\to\infty} f_n = f$ pointwise. We will use condition (i) in the definition of uniform integrability to reduce the proof to that of the special case in which $\mu(\Omega) < \infty$:

$$\begin{aligned}
\int_\Omega |f_n - f| \mathrm{d}\mu &= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} |f_n - f| \mathrm{d}\mu \\
&= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} (|f_n| + |f|) \mathrm{d}\mu\epsilon \\
&\le \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + 2\epsilon \ .
\end{aligned}$$

It therefore suffices to show that

$$\lim_{n\to\infty} \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu = 0 \ .$$

This is the same as (1.7), except A_ϵ has finite measure. Therefore, when $\lim_{n\to\infty} f_n = f$ pointwise, we can apply Egoroff's Theorem to obtain a subset B_ϵ of A_ϵ with $\mu(B_\epsilon) < \delta_\epsilon$ such that f_n converges uniformly to f on $G_\epsilon := A_\epsilon \cap B_\epsilon^c$. Then

$$\begin{aligned}\int_{A_\epsilon} |f_n - f|\mathrm{d}\mu &= \int_{G_\epsilon} |f_n - f|\mathrm{d}\mu + \int_{B_\epsilon} |f_n - f|\mathrm{d}\mu\\ &\le \int_{G_\epsilon} |f_n - f|\mathrm{d}\mu + \int_{B_\epsilon} (|f_n| + |f|)\mathrm{d}\mu\\ &\le \int_{G_\epsilon} |f_n - f|\mathrm{d}\mu + 2\epsilon\ .\end{aligned}$$

Since the convergence is uniform on G_ϵ, and since $\mu(G_\epsilon) < \infty$,

$$\lim_{n\to\infty}\int_{G_\epsilon} |f_n - f|\mathrm{d}\mu = 0\ .$$

Then

$$\limsup_{n\to\infty}\int_{\Omega} |f_n - f|\mathrm{d}\mu \le \limsup_{n\to\infty}\int_{G_\epsilon} |f_n - f|\mathrm{d}\mu + 4\epsilon = 4\epsilon\ .$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_{n\to\infty}\int_{\Omega} |f_n - f|\mathrm{d}\mu = 0\ .$$

In case $\lim_{n\to\infty} f_n = f$ in measure, things are even easier: By definition, there is an N so that for all $n \ge N$,

$$\mu(\{x\ :\ |f_n(x) - f(x)| > \epsilon\}) \le \delta_\epsilon\ .$$

In this case,

$$\int_{\mathcal{O}} |f_n - f|\mathrm{d}\mu \le 2\epsilon\ .$$

Thus, the first part of Theorem 1.4 is proved.

Now we prove the converse. For any set B,

$$\int_B |f_n|\mathrm{d}\mu \le \int_B |f|\mathrm{d}\mu + \int_B |f_n - f|\mathrm{d}\mu \le \int_B |f|\mathrm{d}\mu + \int_\Omega |f_n - f|\mathrm{d}\mu\ .$$

Now, for any fixed $\epsilon > 0$, choose N_ϵ so that

$$n > N_\epsilon \Rightarrow \int_\Omega |f_n - f|\mathrm{d}\mu < \epsilon/2\ .$$

We then have that for all $n > N_\epsilon$,

$$\int_B |f_n|\mathrm{d}\mu \le \int_B |f|\mathrm{d}\mu + \epsilon/2\ .$$

Since $\{f\}$ itself is uniformly integrable, there is a number $\tilde{\delta}_\epsilon > 0$ so that

$$\mu(B) \le \tilde{\delta}_\epsilon \Rightarrow \int_B |f|\mathrm{d}\mu \le \epsilon/2\ .$$

Hence, for all $n > N_\epsilon$,

$$\mu(B) \le \tilde{\delta}_\epsilon \Rightarrow \int_B |f_n|\mathrm{d}\mu \le \epsilon\ .$$

Finally, using the fact that for each $n \leq N_\epsilon$, $\{f_n\}$ is uniformly integrable, there is a $\delta_\epsilon^{(n)} > 0$ so that

$$\mu(B) \leq \delta_\epsilon^{(n)} \Rightarrow \int_B |f_n| \mathrm{d}\mu \leq \epsilon \ .$$

Define

$$\delta_\epsilon = \min\{\delta_\epsilon^{(1)}, \delta_\epsilon^{(2)}, \ldots, \delta_\epsilon^{(N_\epsilon)}, \tilde{\delta}_\epsilon\} \ .$$

Since the minimum of a *finite* set of strictly positive numbers is strictly positive, we have that $\delta_\epsilon > 0$ Also,

$$\mu(B) \leq \delta_\epsilon \Rightarrow \int_B |f_n| \mathrm{d}\mu \leq \epsilon$$

for all n and for f as well. Thus, condition (i) is satisfied. Condition (ii) may be proved in the exact same way, and condition (iii) is even simpler. □

1.5 COROLLARY (Generalized Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on $(\Omega, \mathcal{S}, \mu)$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x. Let $\{g_n\}$ be a seqeunce of non-negaitve integrable functions on $(\Omega, \mathcal{S}, \mu)$ such that for some $g \in L^1(\Omega, \mathcal{S}, \mu)$, $\lim_{n\to\infty} \int_\Omega |g_n - g| \mathrm{d}\mu = 0$.*

Then, if $|f_n| \leq g_n$ for all n, $\lim_{n\to\infty} |f_n - f| \mathrm{d}\mu = 0$.

Proof: since by the converse of Vitali's Theorem, the set $\mathcal{G} := \{g_n\}$ is uniformly integrable, the set $\mathcal{F} = \{f \ : \ |f| \leq g_n \quad \text{for some } n \ \}$ is uniformly integrable: As in the first Example, one can use the same sets A_ϵ and numbers δ_ϵ that work for $\mathcal{G}$. Therefore, the second part of the theorem also follows form Vitali's Theorem. □

The next theorem provides a useful way to establish uniform integrability without using a dominating function or sequence – which may not exist.

1.6 THEOREM (Integral Limits on Concentration)**.** *Let ϕ be a non–negative increasing function on $\mathbb{R}_+$ with*

$$\lim_{t\to\infty} \phi(t) = \infty \ .$$

Then for any measure space $(\Omega, \mathcal{S}, \mu)$ and any $C > 0$, let $\mathcal{F}_C$ be the set of functions satisfying

$$\int_\Omega |f| \phi(|f|) \mathrm{d}\mu \leq C \ . \tag{1.8}$$

Then

$$\lim_{\delta\to 0} \left(\sup \left\{ \int_E |f| \mathrm{d}\mu \ : \ \mu(E) \leq \delta \ , \ f \in \mathcal{F}_C \ \right\} \right) = 0 \ .$$

In particular, if $\mu(\Omega) < \infty$, $\mathcal{F}_C$ is uniformly integrable.

Proof: Let E be any measurable sets, and f any memeber of $\mathcal{F}_C$. Then for any $a > 0$, let

$$B_a = \{x \mid |f(x)| > a \ \} \ .$$

$$\begin{aligned}
\int_E |f|\mathrm{d}\mu &= \int_{E\cap B_a} |f|\mathrm{d}\mu + \int_{E\cap B_a^c} |f|\mathrm{d}\mu \\
&\le \int_{E\cap B_a} |f|\frac{\phi(|f|)}{\phi(a)}\mathrm{d}\mu + \int_{E\cap B_a^c} a\mathrm{d}\mu \\
&\le \int_\Omega |f|\frac{\phi(|f|)}{\phi(a)}\mathrm{d}\mu + \int_E a\mathrm{d}\mu \\
&\le \frac{C}{\phi(a)} + a\mu(E)\ .
\end{aligned}$$

Now given $\epsilon > 0$, choose a so that $C/\phi(a) < \epsilon/2$, and then choose $\delta_\epsilon = \epsilon/(2a)$. It then follows that

$$\mu(E) < \delta_\epsilon \Rightarrow \int_E |f|\mathrm{d}\mu < \epsilon$$

and f was an arbirary member of $\mathcal{F}_C$. Finally, in a finite measure space, we can just take $A_\epsilon = \Omega$; the second requirement in the definition of uniform integrability is vacuous in this case, and it is easy to see that (1.8) provides a uniform bound on $\int_\Omega |f|\mathrm{d}\mu$. □

Here is a typical example where Vitali's Theorem may be applied, because of Theorem 1.6, and where neither the Dominated Convergence Theorem nor the the Monotone Convergence Theorem is applicable.

1.7 EXAMPLE. *Suppose* $(\Omega, \mathcal{S}, \mu)$ *is a finite measure space. Suppose* $\{f_n\}$ *is a sequence of measurable functions with* $\lim_{n\to\infty} f_n = f$ *pointwise. Suppose also that*

$$\sup_n \int_\Omega |f_n| \ln(1 + |f_n|)\mathrm{d}\mu = C < \infty$$

Then

$$\lim_{n\to\infty} \int_\Omega |f_n - f|\mathrm{d}\mu = 0\ .$$

This follows directly from the prevoius theorem and then Vitali's Theorem. There is no dominating function in this case. In later applications to evolution equations, one often has that some integral of the form $\int_\Omega |f|\phi(|f|)\mathrm{d}\mu$ *is monotone decreasing under the evolution. The kind of argument in this example can be used to prove convergence of integrals, provided of course, that one has pointwise convergence.*

We will see another example of the application of uniform integrability when we examine the error term in Fatou's Lemma.