Notes on $L^2(\Omega, \mathcal{M}, \mu)$ and the Riesz–Fischer Theorem

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Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Then for any two complex valued measurable functions f and g, we have the Cauchy-Schwarz inequality:

0.1 THEOREM (Cauchy-Schwarz inequality)**.** *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Then for any two complex valued measurable functions f and g such that $\int_\Omega |f|^2 \mathrm{d}\mu$ and $\int_\Omega |g|^2 \mathrm{d}\mu$ are both finite, f^*g is integrable, and*

$$\left| \int_\Omega f^* g \mathrm{d}\mu \right| \le \left(\int_\Omega |f|^2 \mathrm{d}\mu \right)^{1/2} \left(\int_\Omega |g|^2 \mathrm{d}\mu \right)^{1/2} , \tag{0.1}$$

and in the case that neither f nor g is zero almost everywhere, there is equality if and only if for some numbers a and b, not both zero, $af + bg = 0$.

Proof: To see this, we assume without loss of generality that neither f nor g vanishes almost everywhere, so that both $\int_O |f|^2 \mathrm{d}\mu$ and $\int_\Omega |g|^2 \mathrm{d}\mu$ are strictly positive. We may then define

$$u := \left(\int_\Omega |f|^2 \mathrm{d}\mu \right)^{-1/2} f \qquad \text{and} \qquad v := \left(\int_\Omega |g|^2 \mathrm{d}\mu \right)^{-1/2} g .$$

One readily checks that

$$\int_\Omega |u|^2 \mathrm{d}\mu = \int_\Omega |v|^2 \mathrm{d}\mu = 1 .$$

We note that for every x, $0 \le (|u(x)| - |v(x)|)^2 = |u(x)|^2 + |v(x)|^2 - 2|u||v|$, and thus $2|u||v| \le |u|^2 + |v|^2$. Since the right hand side is integrable, so is the left hand side.

Now let α be a complex number of unit modulus such that

$$\alpha \int_\Omega u^* v \mathrm{d}\mu \ge 0 .$$

Then

$$0 \le \int_\Omega |u - \alpha v|^2 \mathrm{d}mu = \int_\Omega |u|^2 \mathrm{d}\mu + \int_\Omega |v|^2 \mathrm{d}\mu - 2 \int_\Omega |u||v| \mathrm{d}\mu = 2 \left(1 - \alpha \int_\Omega u^* v \mathrm{d}\mu \right) .$$

We conclude

$$\left| \int_\Omega u^* v \mathrm{d}\mu \right| \le 1 ,$$

which by the definition of u and v is equivalent to (0.1), and evidenlty there is equality if and only if $u - \alpha v$ almost everywhere. □

0.2 DEFINITION. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Then $L^2(\Omega, \mathcal{M}, \mu)$ is the set of all equivalence classes of complex valued measurable functions on f Ω, identified under the relation of almost everywhere equality, such that

$$\int_\Omega |f|^2 \mathrm{d}\mu < \infty \ .$$

equipped with the *means square metric* d_2, also called the L^2 metric, defined by

$$d_2(f,g) = \left(\int_\Omega |f-g|^2 \mathrm{d}\mu \right)^{1/2} \ .$$

To work effectively with the L^2 metric, it is helpful to define the L^2 inner product and norm:

0.3 DEFINITION (L^2 inner product and norm)**.** Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Then for $f, g \in L^2(\Omega, \mathcal{M}, \mu)$ the *inner product* of f and g, $\langle f, g \rangle$, is defined by

$$\langle f, g \rangle := \int_\Omega f^* g \mathrm{d}\mu \ ,$$

and the L^2 *norm* of f, $\|f\|_2$ is defined by

$$\|f\|_2 = \left(\int_\Omega |f|^2 \mathrm{d}\mu \right)^{1/2} = (\langle f, f \rangle)^{1/2} \ .$$

Then we have $d_2(f,g) = \|f-g\|_2$. To see that the triangle inequality is satisfied, note that for f, g and h in $L^2(\Omega, \mathcal{M}, \mu)$,

$$\begin{aligned} \|f-h\|_2^2 = \|(f-g)-(g-h)\|_2^2 &= \langle (f-g)-(g-h) \ , \ (f-g)-(g-h) \rangle \\ &= \|f-g\|_2^2 + \|g-h\|_2^2 + \langle (g-h), (f-g) \rangle - \langle (f-g), (g-h) \rangle \\ &\le \|f-g\|_2^2 + \|g-h\|_2^2 + 2\|(g-h)\|_2 \|(f-g)\|_2 \rangle \\ &= (\|f-g\|_2 + \|g-h\|_2)^2 \ . \end{aligned}$$

where the inequality is the Cauchy-Schwarz inequality. Taking the square root, we have the triangle inequality. Thus, $L^2(\Omega, \mathcal{S}, \mu)$ is a metric space.

The Riesz–Fischer Theorem asserts that $L^2(\Omega, \mathcal{S}, \mu)$ is complete:

0.4 THEOREM (Riesz-Fischer)**.** *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Then $L^2(\Omega, \mathcal{S}, \mu)$ is a complete metric space, and if $\{f_n\}$ is any Cauchy sequence in $L^2(\Omega, \mathcal{S}, \mu)$, there is a subsequence $\{f_{n_k}\}$ that converges almost everywhere to a function $f \in L^2(\Omega, \mathcal{S}, \mu)$.*

Proof: Let $\{f_n\}$ be any Cauchy sequence in $L^2(\Omega, \mathcal{S}, \mu)$. Choose n_1 so that for all $j, k \ge n_1$, $\|f_j - f_k\|_2 < 2^{-1}$. Then with n_{k-1} defined, choose $n_k > n_{k-1}$ and such that for all $j, k \ge n_k$, $\|f_j - f_k\|_2 < 2^{-k}$. It follows that for all k,

$$\|f_{n_k} - f_{n_{k-1}}\|_2 \le 2^{-(k-1)} \ .$$

Now note that

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k} (f_{n_k} - f_{n_{k-1}}) \ .$$

Define

$$F_k(x) := |f_{n_1}(x)| + \sum_{j=1}^{k} |f_{n_k}(x) - f_{n_{k-1}}(x)| \ .$$

By the triangle inequality,

$$\|F_k\|_2 \leq \|f_{n_1}\|_2 + \sum_{j=1}^{k} \|f_{n_k} - f_{n_{k-1}}\|_2 \leq \|f_{n_1}\|_2 + 1 \ .$$

Define $F(x) := \lim_{k\to\infty} F_k(x)$ which exists by monotonicity. By the Monotone Convergence Theorem,

$$\int_\Omega F^2 \mathrm{d}\mu = \lim_{k\to\infty} \int_\Omega F_k^2 \mathrm{d}\mu = \lim_{k\to\infty} \|F_k\|_2^2 \leq (\|f_{n_1}\|_2 + 1)^2 \ .$$

It follows that F is finite almost everywhere, and then since absolute convergence implies convergence,

$$\lim_{k\to\infty} f_{n_k}(x) =: f(x)$$

exists for each x with $F(x) < \infty$. Define $f(x)$ to be zero elsewhere. Then $\lim_{k\to\infty} f_{n_k} = f$ almost everywhere. By Fatou's Lemma,

$$\int_O |f|^2 \mathrm{d}\mu \leq \liminf_{k\to\infty} \int_\Omega |f_{n_k}|^2 \mathrm{d}\mu \leq \|F\|_2^2 \ ,$$

and so $f \in L^2(\Omega, \mathcal{M}, \mu)$

Then, since $|f(x) - f_{n_k}(x)|^2 \leq (|f(x)| + |f_{n_k}(x)|)^2 \leq 4F^2(x)$, which is integrable, the Dominated Convergence Theorem then tells us that

$$\lim_{k\to\infty} \|f - f_{n_k}\|_2^2 = \lim_{k\to\infty} \int_\Omega |f - f_{n_k}|^2 \mathrm{d}\mu = \int_\Omega \lim_{k\to\infty} |f - f_{n_k}|^2 \mathrm{d}\mu = 0 \ .$$

Thus, the subsequence converges to f. The rest follows from standard properties of metric spaces. □

The first application of this result was to the convergence of Fourier series. In fact, it was proved in exactly that context. Let us now explain this application.

Consider the case that Ω is the unit circle, S^1 paramterized by angles t with $-\pi < t \leq \pi$, $\mathcal{B}$ is the Borel sigma algebra, and μ is Lebesgue measure on S^1. For each $k \in \mathbb{Z}$, define

$$u_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt} \ .$$

By an easy computation, one sees that

$$\int_{-\pi}^{\pi} u_k^*(t) u_\ell(t) \mathrm{d}t = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases} \tag{0.2}$$

Hence the set of functions $\{u_k\}_{k\in\mathbb{Z}}$ is orthonormal in $L^2(S^1, \mathcal{B}, \mu)$. We now ask:

- *Which functions f in $L^2(S^1, \mathcal{B}, \mu)$ can be written in the form*

$$f(t) = \sum_{k\in\mathbb{Z}} a_k u_k(t) \tag{2}$$

for some sequence of numbers $\{a_k\}_{k\in\mathbb{Z}}$*?*

An infinite sum is always defined as a limit, so

$$\sum_{k\in\mathbb{Z}} a_k u_k = \lim_{N\to\infty} \sum_{|k|\leq N} a_k u_k \ ,$$

where we reuire the limit to exist in the $L^2(S^1, \mathcal{B}, \mu)$ metric.

Let us try to answer the question, first going as far as we can without without using the Riesz–Fischer Theorem. Given a function $f \in L^2(S^1, \mathcal{B}, \mu)$, suppose that there is a sequence of numbers $\{a_k\}_{k\in\mathbb{Z}}$ such that (2) is true. Define the function sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ by

$$g_n(t) = \sum_{k\ \leq n} a_k u_k(t) \ .$$

Then, by (1), for all $n \geq |k|$,

$$\langle u_k, g_n\rangle = \sum_{|\ell|\leq n} a_\ell \langle u_k, u_\ell\rangle = a_k \ ,$$

since only the $\ell = k$ term is not zero. But then if (2) is true,

$$\langle u_k, f\rangle = \lim_{n\to\infty} \langle u_k, g_n\rangle = a_k \ ,$$

the sequence $\{a_k\}_{k\in\mathbb{Z}}$ for which it is true is uniquely determined: It must be the case that

$$a_k = \langle u_k, f\rangle \tag{3}$$

for all k.

Now let a_k be defined by (3) for al k. Let us compute $\|f - g_n\|_2^2$. This is

$$\begin{aligned} \|f - g_n\|_2^2 &= \langle f - g_n, f - g_n\rangle > \\ &= \langle f, f\rangle - \langle f, g_n\rangle - \langle g_n, f\rangle + \langle g_n, g_n\rangle \end{aligned}$$

Then by (3),

$$\langle g_n, f\rangle = \sum_{|k|\leq n} a_k^* \langle u_k, f\rangle = \sum_{|k|\leq n} |a_k|^2 \ .$$

Since this is real, and since $\langle f, g_n\rangle$ is the complex conjugate of $\langle g_n, f\rangle$, we have the same result for $\langle f, g_n\rangle$. Finally, by (1) once more

$$\langle g_n, g_n\rangle = \sum_{|k|\leq n, |\ell|\leq n} a_k^* a_\ell \langle u_k, u_\ell\rangle = \sum_{|k|\leq n} |a_k|^2 \ .$$

Thus, going back to (4) we have:

$$\|f - g_n\|_2^2 = \|f\|_2^2 - \sum_{|k|\leq n} |a_k|^2 \ . \tag{5}$$

This is known as *Bessel's equality*, and it immediately yields *Bessel's inequality*:

$$\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \|f\|_2^2 \ . \tag{6}$$

This has the following important consequence:

• *The seuqence $\{g_n\}$ is a Cauchy sequence.*

To see this, we compute $\|g_n - g_m\|^2$ for $n > m$. Since $g_n - g_m = \sum_{m<|k|\leq n} a_k u_k$, this is

$$\sum_{m<|k|\leq n, m<|\ell|\leq n} a_k^* a_\ell \langle u_k, u_\ell\rangle = \sum_{m<|k|\leq n} |a_k|^2 \ ,$$

where we have used (1) once more. Thus, for all $m, n \geq N$,

$$\|g_n - g_m\|^2 \leq \sum_{|k|>N} |a_k|^2 \ ,$$

and since $\sum_{\in\mathbb{Z}} |a_k|^2 < \infty$ by Bessel's inequality, the right hand side tends to zero as N increases. This means that $\{g_n\}$ is a Cauchy sequence.

Now we invoke the Riesz–Fischer Theorem: *Since $L^2(S^1, \mathcal{B}, \mu)$ is complete, the limit*

$$g = \lim_{n\to\infty} g_n$$

exists in $L^2(S^1, \mathcal{B}, \mu)$.

Our only remaining question is this:

• *Does $g = f$?*

The key to this is to observe that the function $g - f$ is orthognal to u_k for every k. Indeed:

$$\langle u_k, g - f\rangle = \langle u_k, g\rangle - \langle u_k, f\rangle = \langle u_k, g\rangle - a_k \ .$$

But

$$\langle u_k, g\rangle = \lim_{n\to\infty} \sum_{|\ell|\leq n} a_\ell \langle u_k, u_\ell\rangle = a_k$$

by (1) once more. Thus, $\langle u_k, g - f\rangle = a_k - a_k = 0$. We conclude:

• *Either $f = g$, or else $f - g$ is a non zero function in $L^2(S^1, \mathcal{B}, \mu)$ that is orthogonal to u_k for each k. Thus, if we show that no such function exists, we have proved that $g = f$.*

Up to this point in the analysis, we have not used any of the specific structure of $L^2(S^1, \mathcal{B}, \mu)$ or the specific definition of the u_k, apart from their orthonormality. The same reasoning would apply to *any* orthonormal sequence $\{v_k\}_{k\in\mathbb{N}}$ in *any* L^2 space $L^2(\Omega, \mathcal{S}, \nu)$. (We switch back to the usual indexing set $\mathbb{N}$ in place of $\mathbb{Z}$ for the general case.) The conclusion to this point are very much worth summarizing in a theorem:

Theorem 1 (Orthonormal bases in L^2) *Let $\{v_k\}_{k\in\mathbb{N}}$ be any orthonormal sequence of vectors in an L^2 space $L^2(\Omega, \mathcal{S}, \nu)$. Then the following statements are equivalent:*

(i) For every $f \in L^2(\Omega, \mathcal{S}, \nu)$, $f = \sum_{k=1}^\infty \langle v_k, f\rangle v_k$, and $\|f\|_2^2 = \sum_{k=1}^N |\langle v_k, f\rangle|^2$.

(ii) The only function f such that $\langle v_k, f\rangle = 0$ for all k is $f = 0$.

Of course, when we write "function" , we mean the equivalence class of the function.

A an orthonormal sequence $\{v_k\}_{k\in\mathbb{N}}$ in an L^2 space $L^2(\Omega, \mathcal{S}, \nu)$ is called an *orthonormal basis* for $L^2(\Omega, \mathcal{S}, \nu)$ if and only if statement *(i)* in Theorem 1 is true for $\{v_k\}_{k\in\mathbb{N}}$. Any set of S of functions in $L^2(\Omega, \mathcal{S}, \nu)$ is called *total* in case

$$\langle v, f\rangle = 0 \text{ for all } v \in S \qquad \Rightarrow \qquad f = 0 \ .$$

Thus, the theorem can be expressed as saying that an orthonormal sequence $\{v_k\}_{k\in\mathbb{N}}$ is an orthonormal basis if and only if it is total.

To show that any particular orthonormal sequence is total, one must make use of the definition of that orthonormal sequence: Some orthonormal sequences are total, and some are not. (If one drops any terms from a total orthonormal sequence, then what is left is still orthonormal, but not total.)

There are several ways to show that the functions $u_k(t) = (2\pi)^{-1/2}e^{ikt}$, $k \in \mathbb{Z}$, are total in $L^2(S^1, \mathcal{B}, \mu)$. Here is one, using the Stone-Wierstrass Theorem. We first prove a density lemma:

0.5 LEMMA. *The set of continuous functions is dense in $L^2(S^1, \mathcal{B}, \mu)$.*

Proof: Let $f \in L^2(S^1, \mathcal{B}, \mu)$, and let $\epsilon > 0$ be given. By the montone convergence Theorem, there is an N so that with $f_N(x) := 1_{\{x \ \ |f(x)|\le N\}}f(x)$, $\|f - f_N\|_2 \le \epsilon/2$.

By the Cauchy-Scwarz inequality, and the fact that $|f_N| \le |f|$,

$$\int_{S^1} |f_N|\mathrm{d}\mu = \langle 1, |f_N|\rangle \le \|1\|_2\|f_N\|_2 \le \sqrt{2\pi}\|f\|_2 \ .$$

Thus $|f_N|$ is in $f \in L^1(S^1, \mathcal{B}, \mu)$. We have already proved that continuous functions are dense in $L^1(S^1, \mathcal{B}, \mu)$. Hence for any $\tilde{\epsilon} > 0$, we can find a continuous function g on S^1 such that $\|f_N - g\|_1 < \tilde{\epsilon}$. A simple argument shows we can arrange the $|g| < 2N$ everywhere, by truncating its real and imaginary parts when they exceed N in magnitide. Then $\|\|\|f_N - g\|_2 \le \sqrt{2N}\sqrt{\|f_N - g\|_1} \le \sqrt{2N}\tilde{\epsilon}$. We now choose $\tilde{\epsilon}$ so that the right hand side is no more that $\epsilon/2$. Then

$$\|f - g\|_2 \le \|f - f_N\|_2 + \|f_N - g\|_2 < \epsilon \ ,$$

which is what we needed to show.

Now, was we have seen, the Complex Stone-Wierstrass Theorem says that the finite linear combinations of the functions u_k, $k \in \mathbb{Z}$ are uniformly dense in the continuous function on S^1.

0.6 THEOREM. *The functions $u_k(t) = (2\pi)^{-1/2}e^{ikt}$, $k \in \mathbb{Z}$, are total in $L^2(S^1, \mathcal{B}, \mu)$, and consequently, for every $f \in L^2(S^1, \mathcal{B}, \mu)$, with $a :=< u_k, f >$,*

$$\lim_{n\to\infty}\left\|\sum_{k=-n}^{n} a_k u_k - f\right\|_2 = 0 \ .$$

Proof. Suppose that $f \in L^2(S^1, \mathcal{B}, \mu)$ has the property that $\langle f, u_k\rangle = 0$ for each $k \in \mathbb{Z}$. We must show that $f = 0$. Suppose not. Then $\|f\|_2 > 0$, and we can find a continuous function g with

$$\|f - g\|_2 \le \frac{\|f\|_2}{3} \ .$$

Then by the Stone-Wierstrass Theorem, we can find a trigonometric polynomial $h = \sum_{k=-n}^{n} b_k u_k$ for some n and some coefficients b_k, $|k| \leq n$, such that

$$\|g - h\|_2 \leq \frac{\|f\|_2}{3} \ .$$

But then

$$\|f - h\|_2 \leq \frac{2\|f\|_2}{3} \ .$$

But since $< f, h >= 0$,

$$\|f - h\|_2^2 = \|f\|_2^2 + \|h\|_2^2 \geq \|f\|_2^2$$

We conclude

$$\|f\|_2^2 \leq \|f - h\|_2^2 \leq \frac{2\|f\|_2}{3} \ .$$

This is a contradiction. □

At the end of this section, we will give another proof of this result due to Fejer. It is more involved, but gives additional useful information.

Theorem (Fejer) *With $u_k(t)$ defined by*

$$u_k(t) = \frac{1}{\sqrt{2\pi}} e^{-kt} \ ,$$

the sequence $\{u_k\}_{k\in\mathbb{Z}}$ is total in $L^2(S^1, \mathcal{B}, \mu)$ and hence it is an orthonormal basis.

The basis $\{u_k\}_{k\in\mathbb{Z}}$ is called the *Fourier basis* for $L^2(S^1, \mathcal{B}, \mu)$. Fejer's Theorem resolved a question that had been open for many decades about what sorts of functions could be represented as infinite series of multipled of the u_k.

Before starting into the proof, recall that if $\mathbf{v}$ is any vector in $\mathbb{C}^n$, and if $\{_{\mathbf{1}}, \ldots, _{\mathbf{n}}\}$ is any set of orthonormal vecotors in $\mathbb{C}^n$, then

$$\mathbf{v} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \langle_{\mathbf{k}}, \mathbf{v}\rangle_{\mathbf{k}} \ .$$

The analog of this in $L^2(S^1, \mathcal{B}, \mu)$ would be

$$f(t) = \sum_{k\in\mathbb{Z}} \left(\int_{-\pi}^{\pi} u_k^*(s) f(s) \mathrm{d}s \right) u_k(t) \ ,$$

where we are now explicitly writing out the inner product and an integral. By this we mean that in the $L^2(S^1, \mathcal{B}, \mu)$ metric,

$$f(t) = \lim_{N\to\infty} \sum_{k\leq N} \left(\int_{-\pi}^{\pi} u_k^*(s) f(s) \mathrm{d}s \right) u_k(t) \ . \tag{7}$$

Defining the *Dirichlet kernel* $D_N(t - s)$ by

$$D_N(t - s) = \sum_{k\leq N} u_k(t) u_k^*(s) = \frac{1}{2\pi} \sum_{k\leq N} e^{ik(t-s)} \ ,$$

we can rewrite (7) as

$$f(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} D_N(t,s) f(s) \mathrm{d}s \ . \tag{8}$$

The original question with which we began this section is whether this is true, or not, for all f in $L^2(S^1, \mathcal{B}, \mu)$.

Let us investigate this question directly, before turning to the issue of totality. An explicit formula for $D_N(t)$ should help, and with the realization that $D_N(t)$ is a geometric sum, this is easy to obtain.

Recall that for any number r,

$$(r-1)\sum_{k=-N}^{N} r^k = r^{N+1} - r^{-N}$$

so that

$$\sum_{k=-N}^{N} r^k = \frac{r^{N+1} - r^{-N}}{r-1} = \frac{r^{N+1/2} - r^{-N-1/2}}{r^{1/2} - r^{-1/2}} \ .$$

Now taking $r = e^{i(t-s)}$ we get

$$D_N(t) = \frac{\sin((N+1)t/2)}{\sin(t/2)} \ .$$

We now ask:

- *For which functions f in $L^2(S^1, \mathcal{B}, \mu)$ does*

$$f(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} D_N(t-s) f(s) \mathrm{d}s \tag{9}$$

where the limit is taken in the $L^2(S^1, \mathcal{B}, \mu)$ metric?

It is not so easy to analyze this limit because $\sin((N+1)t/2)$ oscillates rapidly for large N, and cancelation plays an important role in determining the value of the integral. This is usually delicate, and certainly the cancelation is difficult to deal with in the case at hand.

Fejer made the following brilliant observation: If $\{b_n\}$ is any convergent sequence sequence of numbers with

$$\lim_{n\to\infty} b_n = b \ ,$$

and if we define

$$c_n = \frac{1}{n}\sum_{k=0}^{n-1} b_k \ ,$$

then $\lim n \to \infty c_n = b$ also. *Therefore*, if (6) is true, we should have

$$f(t) = \lim_{N\to\infty} \left(\frac{1}{N}\sum_{n=0}^{N-1} \int_{-\pi}^{\pi} D_n(t-s) \right) f(s) \mathrm{d}s \tag{10}$$

The *Fejer kernel* $F_N(t)$ is defined by

$$F_N(t) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(t) \ .$$

Using this definition, we can rewrite (10) as

$$f(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} F_N(t-s) f(s) \mathrm{d}s \qquad (11)\ .$$

We now ask:

- *For which functions f in $L^2(S^1, \mathcal{B}, \mu)$ does*

$$f(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} F_N(t-s) f(s) \mathrm{d}s \qquad (12)$$

where the limit is taken in the $L^2(S^1, \mathcal{B}, \mu)$ metric?

We are building up a large store of questions and as of yet no answers. But now this shall change. Once we compute a formula for $F_N(t)$, it will be easy to show that (12) is true whenever f is continuous. From that result, we shall easily obtain the proof Fejer's Theorem.

To obtain the explicit formula for $F_N(t)$, notice that

$$\begin{aligned}
\sin(t/2)F_N(t) &= \frac{1}{N}\sum_{n=0}^{N-1} \sin((n+1/2)t) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{I}\left(e^{i(n+1/2)t}\right) \\
&= \frac{1}{N}\mathcal{I}\left(e^{it/2}\sum_{n=0}^{N-1}\left(e^{it}\right)^n\right)
\end{aligned}$$

Now by the formula for geometric sums,

$$\sum_{n=0}^{N-1}\left(e^{it}\right)^n = \frac{e^{iNt}-1}{e^{it}-1} = \frac{e^{iNt/2}}{e^{it/2}}\frac{e^{iNt/2}-e^{-iNt/2}}{e^{it/2}-e^{-it/2}} = \frac{e^{iNt/2}}{e^{it/2}}\frac{\sin(Nt/2)}{\sin(t/2)}\ .$$

Thus,

$$\sin(t/2)F_N(t) = \frac{1}{N}\mathcal{I}(e^{iNt/2})\frac{\sin(Nt/2)}{\sin(t/2)} = \frac{1}{N}\frac{\sin^2(Nt/2)}{\sin(t/2)}\ .$$

Therefore,

$$F_N(t) = \frac{1}{N}\frac{\sin^2(Nt/2)}{\sin^2(t/2)}\ . \qquad (9)$$

The big advantage of the Fejer kernel over the Dirichlet Kernel is that it is everywhere positive. The following Theorem summarizes its key properties.

Theorem 3 (Properties of the Fejer kernel) *The Fejer kernel has the following properties:*

(1) $F_N(t) \geq 0$ for all t and N.

(2) $\int_{-\pi}^{\pi} F_N(t)\mathrm{d}t = 1$ for all N.

(3) For all $\delta > 0$,

$$\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} F_N(t)\mathrm{d}t \leq \frac{1}{N}\frac{1}{\sin^2(\delta/2)}$$

for all N.

Proof: First (1) is obvious from (9). Next, for each n,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\mathrm{dt} = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(\int_{-\pi}^{\pi} e^{ikt}\mathrm{dt}\right) = 1 \ .$$

Only the $k = 0$ term is non zero, and evaluating it easily leads to this conclusion.

Now,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t)\mathrm{d}t &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{N}\sum_{n=0}^{N-1} D_n(t)\mathrm{d}t\right)\\ &= \frac{1}{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} D_n(t)\mathrm{d}t\right)\\ &= \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1 \ . \end{aligned} \tag{0.3}$$

Finally, for all $\delta > 0$, notice that on $0 \le t \le \pi$, $\sin(t/2)$ is monotonically increasing so that for all $|t| \ge \delta$,

$$\frac{\sin^2(Nt/2)}{\sin^2(t/2)} \le \frac{1}{\sin^2(\delta/2)} \ ,$$

and this proves (3). □

This theorem has the following consequence:

Theorem 4 (The Fejer kernel is an approximate identity) *Let f be any continuous function on S^1. Then*

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} F_N(t-s)f(s)\mathrm{d}s = f(t) \ ,$$

and the convergence is uniform in t.

Proof: We first note that

$$\int_{-\pi}^{\pi} F_N(t-s)f(s)\mathrm{d}s = \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s$$

by the rotation invariance of Lebesgue measure on S^1. Then, by (2) in Theorem 1,

$$f(t) - \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s = \int_{-\pi}^{\pi} F_N(s)(f(t)-f(t-s))\mathrm{d}s \ .$$

Therefore,

$$\left|f(t) - \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s\right| \le \int_{-\pi}^{\pi} F_N(s)|f(t)-f(t-s)|\mathrm{d}s \ . \tag{10}$$

Now, since f is continuous and S^1 is compact, f is uniformly continuous, meaning that for each $\epsilon > 0$, there is a $delta_\epsilon > 0$ such that

$$|s-t| < \delta_\epsilon \quad \Rightarrow \quad |f(s)-f(t)| < \epsilon \ .$$

and also, there is a number B so that

$$f(t) \le B$$

for all t. Therefore, picking up with (10),

$$\begin{aligned}\int_{-\pi}^{\pi} F_N(s)|f(t) - f(t-s)|\mathrm{d}s &= \int_{|s|\le\delta_\epsilon} F_N(s)|f(t) - f(t-s)|\mathrm{d}s \\ &+ \int_{\delta_\epsilon<|s|\le\pi} F_N(s)|f(t) - f(t-s)|\mathrm{d}s \\ &\le \epsilon\int_{-\pi}^{\pi} F_N(s)\mathrm{d}s + \frac{2B}{N\sin^2(\delta_\epsilon)} \\ &= \epsilon + \frac{2B}{N\sin^2(\delta_\epsilon)}\end{aligned}$$

Combining this with (10), we see that

$$\limsup_{N\to\infty}\left|f(t) - \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s\right| \le \epsilon\ .$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{N\to\infty}\left|f(t) - \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s\right| = 0\ .$$

Since the bound in (11) is independent of t, the convergence is even uniform. □

We are finally ready to prove Theorem 2, Fejer's Theorem:

Proof of Theorem 2: Let f be any function that is orthogonal to each u_k. We know that all functions in $L^2S^1, \mathcal{B}, \mu)$, can be approximated arbitrarily closely by continuous functions, and so for any $\epsilon > 0$, there is a continuous functions h with $\|f - h\|_2 \le \epsilon$.

Now,

$$\int_{-\pi}^{\pi} F_N(s)h(t-s)\mathrm{d}s = \int_{-\pi}^{\pi} F_N(s)(h(t-s) - f(t-s))\mathrm{d}s + \int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s\ .$$

Consider the second term on the right.

$$\int_{-\pi}^{\pi} F_N(s)f(t-s)\mathrm{d}s = \int_{-\pi}^{\pi} F_N(t-s)f(s)\mathrm{d}s\ .$$

Since for each t, $D_N(t-s)$ and hence $F_N(t-s)$ is a finite linear combination of the functions $u_k^*(s)$, and since f is orthogonal to each u_k,

$$\int_{-\pi}^{\pi} F_N(t-s)f(s)\mathrm{d}s = 0\ .$$

Therefore

$$\int_{-\pi}^{\pi} F_N(s)h(t-s)\mathrm{d}s = \int_{-\pi}^{\pi} F_N(s)(h(t-s) - f(t-s))\mathrm{d}s\ .$$

By the Minkowski inequality, and the fact that $\int_{-\pi}^{\pi} F_N(s)\mathrm{d}s = 1$,

$$\left\|\int_{-\pi}^{\pi} F_N(s)(h(t-s) - f(t-s))\mathrm{d}s\right\|_2 \le \|h - f\|_2 \le \epsilon\ .$$

Since by Theorem 4, $h(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} F_N(s)h(t-s)\mathrm{d}s$ with uniform convergence, and hence convergence in $L^2(S^1, \mathcal{B}, \mu)$, it follows that $\|h\|_2 \le \epsilon$. But then by Minkowski once more,

$$\|f\|_2 \le \|f - h\|_2 + \|h\|_2 = 2\epsilon \ .$$

Since ϵ is arbitrary, it follows that $f = 0$. $\square$