# Notes on the Inclusion-Exclusion Formula

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**Abstract**

These are some notes on the Inclusion-Exclusion Formula and its applications

## 0.1 The Inclusion-Exclusion Formula and Counting

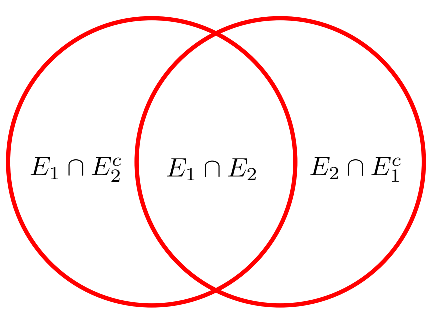
Let $S$ be any finite set, and let $E_1, E_2$ be any subsets of $S$. How are the cardinalities of $E_1$, $E_2$, $E_1 \cup E_2$ and $E_1 \cap E_2$ related? The answer is:

**0.1 PROPOSITION.** *Let $S$ be any finite set, and let $E_1, E_2$ be any subsets of $S$. Then*

$$\#(E_1 \cup E_2) = \#(E_1) + \#(E_2) - \#(E_1 \cap E_2) \ . \tag{0.1}$$

The proof we give is not the shortest, but it has the merit of being readily adaptable to the more complicated case of arbitrarily many subsets of $S$, and it is still quite clear. The *Inclusion-Exclusion Formula* is the generalization of (0.3) to arbitrarily many sets.

*Proof of Proposition 0.1.* The union of the two sets $E_1$ and $E_2$ may always be written as the union of *three* non-intersecting sets $E_1 \cap E_2^c$, $E_1 \cap E_2$ and $E_1^c \cap E_2$. This is illustrated in the Venn diagram below: $E_1$ is represented by the circle on the left, and $E_2$ is represented by the circle on the right. Then $E_1 \cap E_2$ is the overlap, while $E_1 \cap E_2^c$ and $E_1^c \cap E_2$ are the parts in one circle and not the other.



Evidently, $E_1 \cap E_2^c$, $E_1 \cap E_2$ and $E_1^c \cap E_2$ are mutually disjoint. To make the formulas that follow more easily read, define

$$F_1 := E_1 \cap E_2^c \ , \quad F_2 := E_1 \cap E_2 \quad \text{and} \quad F_3 = E_1^c \cap E_2 \ .$$

Then,

$$E_1 = F_1 \cup F_2, \qquad E_2 = F_2 \cup F_3 \quad \text{and} \quad E_1 \cup E_2 = F_1 \cup F_2 \cup F_3 \ . \tag{0.2}$$

Since the sets $F_1$, $F_2$ and $F_3$ are mutually disjoint, this means

$$\#(E_1) = \#(F_1)+\#(F_2), \qquad \#(E_2) = \#(F_2)+\#(F_3) \quad \text{and} \quad \#(E_1 \cup E_2) = \#(F_1)+\#(F_2)+\#(F_3) \ .$$

Therefore,

$$\begin{aligned} \#(E_1) + \#(E_2) = \#(F_1) + \#(F_2) + 2\#(F_3) &= \#(E_1 \cup E_2) + \#(F_3) \\ &= \#(E_1 \cup E_2) + \#(E_1 \cap E_2) \ . \end{aligned}$$

Rearranging terms we obtain (0.3). □

## 0.2 The Inclusion-Exclusion Formula and Probability

In the proof of Proposition 0.1, the only property of the set function $E \mapsto \#(E)$ that was used is is that this function is additive over *disjoint* unions. That is, if $\{F_1, \ldots, F_n\}$ is any collection of mutually disjoint subsets of some finite set $S$, then

$$\#\left(\cup_{j=1}^n F_j\right) = \sum_{j=1}^n \#(F_j) \ .$$

If $S$ is any finite set equipped with any probability measure $P$, and $\{F_1, \ldots, F_n\}$ is any collection of mutually exclusive events in $S$, then

$$P\left(\cup_{j=1}^n F_j\right) = \sum_{j=1}^n P(F_j) \ .$$

Therefore, the proof of Proposition 0.1 also yields the probabilistic proposition:

**0.2 PROPOSITION.** *Let $S$ be any finite set equipped with a probability measure $P$, and let let $E_1, E_2$ be any events in $S$. Then*

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \ . \tag{0.3}$$

**0.3 EXAMPLE.** *Janet goes on vacation and takes two books. The probability that she will like the first book is $\frac{2}{3}$ The probability that she will like the second book is $\frac{1}{2}$. The probability that she will like both books is $\frac{1}{3}$. What is the probability that she likes neither book?*

*Let $E_1$ be the event that she likes the first book. Let $E_2$ be the event that she likes the second book. Then $E_1 \cap E_2$ is the event that she likes both books, and $E_1 \cup E_2$ is the event that she likes at least one of the books. The probability we seek is $P((E_1 \cup E_2)^c)$, and since*

$$P((E_1 \cup E_2)^c) = 1 - P(E_1 \cup E_2) \ ,$$

*we can answer the question if we can compute $P(E_1 \cup E_2)$. By Proposition 0.2 and the information given above,*

$$P(E_1 \cup E_2) = \tfrac{2}{3} + \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{5}{6} \ ,$$

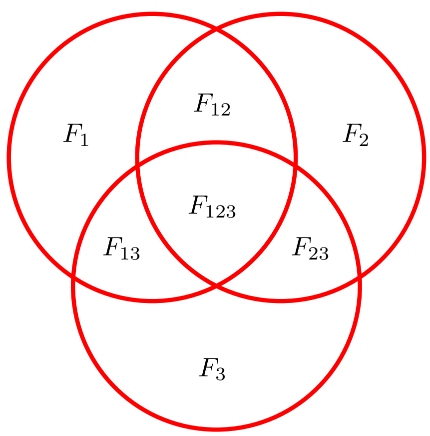
*and so the probability that Janet likes neither book is $\frac{1}{6}$.*

## 0.3 The Inclusion-Exclusion Formula for Three Subsets

Let $S$ be any finite set, and let $E_1, E_2, E_3$ be any subsets of $S$. Associated to the set $\{E_1, E_2, E_3\}$ is a set of 7 disjoint sets $\{F_1, F_2, F_3, F_{12}.F_{13}, F_{23}, F_{1,2,3}\}$ such that

$$E_1 \cup E_2 \cup E_3 = F_1 \cup F_2 \cup F_3 \cup F_{12}.F_{13} \cup F_{23} \cup F_{123} \ , \tag{0.4}$$

and these sets are displayed in the following Venn diagram:




In this diagram, $E_1$ is represented by the upper-left circle, $E_2$ is represented by the upper-right circle, and $E_3$ by the lower circle. Then, for $j = 1, 2, 3$, $F_j$ is the set of elements of $S$ that belong to $E_j$, but *not* to the other two subsets. For $1 \leq i < j \leq 3$, $F_{i,j}$ is the set of elements of $S$ that belong to $E_i$ and $E_j$, but *not* to the third subset. Finally, $F_{123} = E_1 \cap E_2 \cap E_3$, the sets of elements of $S$ that belong to all three subsets.

Evidently, the sets in $\{F_1, F_2, F_3, F_{12}.F_{13}, F_{23}, F_{1,2,3}\}$ are mutually disjoint, and $E_1 \cup E_2 \cup E_3$ is the union of all 7 of them, as in (0.4). Moreover, by the disjointness, adding up the cardinalities of the components in each circle, we have that

$$\begin{aligned}
\#(E_1) &= \#(F_1) + \#(F_{12}) + \#(F_{13}) + \#(F_{123}) \\
\#(E_2) &= \#(F_2) + \#(F_{12}) + \#(F_{23}) + \#(F_{123}) \\
\#(E_3) &= \#(F_3) + \#(F_{13}) + \#(F_{23}) + \#(F_{123})
\end{aligned}$$

and likewise

$$\begin{aligned}
\#(E_1 \cap E_2) &= \#(F_{12}) + \#(F_{123}) \\
\#(E_1 \cap E_3) &= \#(F_{13}) + \#(F_{123}) \\
\#(E_2 \cap E_3) &= \#(F_{23}) + \#(F_{123}) \ .
\end{aligned}$$

Summing, we find

$$\sum_{j=1}^{3} \#(E_j) = \sum_{j=1}^{3} \#(F_j) + 2 \sum_{1 \le i < j \le 3} \#(F_{i,j}) + 3\#(F_{123}) \tag{0.5}$$

and

$$\sum_{1 \le i < j \le 3} \#(E_i \cap E_j) = \sum_{1 \le i < j \le 3} \#(F_{i,j}) + 3\#(F_{123}) \ . \tag{0.6}$$

Subtracting each side of (0.6) from the corresponding side of (0.6), and then adding $\#(F_{123}) = \#(E_1 \cap E_2 \cap E_3)$ to both sides, we obtain

$$\begin{aligned}
&\sum_{j=1}^{3} \#(E_j) - \sum_{1 \le i < j \le 3} \#(E_i \cap E_j) + \#(E_1 \cap E_2 \cap E_3) = \\
&\qquad \#(F_1) + \#(F_2) + \#(F_3) + \#(F_{12}) + \#(F_{13}) + \#(F_{23}) + \#(F_{123})
\end{aligned} \tag{0.7}$$

But by (0.4) and the disjointness of the $F$'s,

$$\#(E_1 \cup E_2 \cup E_3) = \#(F_1) + \#(F_2) + \#(F_3) + \#(F_{12}) + \#(F_{13}) + \#(F_{23}) + \#(F_{123}) \ ,$$

and this proves:

**0.4 PROPOSITION.** *Let $S$ be any finite set, and let let $E_1, E_2, E_3$ be any subsets of $S$. Then*

$$\#(E_1 \cup E_2 \cup E_3) = \sum_{j=1,2,3} \#(E_j) - \sum_{1 \le i < j \le 3} \#(E_i \cap E_j) + \#(E_1 \cap E_2 \cap E_3) \ . \tag{0.8}$$

That is, the cardinality of $E_1 \cup E_2 \cup E_3$ is obtained by adding up the cardinalities of the individual sets, subtracting out the cardinalities of the intersections of distinct pairs, and then adding back in the cardinality of the triple intersection.

**0.5 EXAMPLE.** *In a sports club, there are 36 members who play tennis, 28 who play squash, and 18 that play badminton. There are 22 members that play both tennis and squash, 12 that play both tennis and badminton, and 9 that play both squash and badminton. Finally, there are 4 members that play all three games. How many members play at least one of the games? By Proposition 0.4, this number is*

$$(36 + 28 + 18) - (22 + 12 + 9) + 4 = 82 - 43 + 4 = 43 \ .$$

Again, in proving Proposition 0.4, all we used about the cardinality set function is that it is additive over disjoint unions. Any probability measure is a set function with this same property, and thus the same result applies when cardinality is replaced by any probability measure, but we shall go into this only when we have proved the formula for $N$ subsets, and $N$ is a arbitrary natural number.

## 0.4 Inclusion-Exclusion for $N$ subsets

Let $S$ be any finite set, and for any natural number $N$, let $E_!, \dots, E_N$ be subsets of $S$. Let $J$ denote the index set

$$J = \{1, \dots, N\} .$$

For each subset $X \subset J$, define subsets $E_X$ and $F_X$ of $S$ as follows,

$$E_X := \bigcap_{j\in X} E_j \tag{0.9}$$

and

$$F_X = \left(\bigcap_{j\in X} E_j\right) \cap \left(\bigcap_{k\notin X} E_j^c\right) . \tag{0.10}$$

Notice that

$$F_X = E_X \cap \left(\bigcap_{k\notin X} E_j^c\right) \subset E_X .$$

Let $X$ and $Y$ be two distinct subsets of $J$. Then either there is some element $j$ of $X$ that does not belong to $Y$, or *vice-versa*, or both. Suppose that there is some $j \in X$ such that $j \notin Y$. If $\omega \in F_X$, then $\omega \in E_j$, and hence $\omega \notin F_Y$. On the other hand, if $\omega \in F_Y$, then $\omega \notin E_j$, and hence $\omega \notin F_X$. Swapping the roles of $Y$ and $X$ if need be, if follows that

$$Y \neq X \Rightarrow F_X \cap F_Y = \emptyset . \tag{0.11}$$

That is, the $F's$ are disjoint.

Also every $\omega \in S$ that belongs to at least one of the sets $E_j$, $j \in J$, belongs to $F_X$ for some non-empty subset $X \subset J$: In fact, $X$ is simply the set of all $j \in J$ such that $\omega$ belongs to $E_j$. In symbols, $X = \{j \ : \ \omega \in E_j\}$.

Therefore,

$$\bigcup_{j\in J} E_j = \bigcup_{X\subset J} F_X , \tag{0.12}$$

where on the right, we need only take the union over the $2^N - 1$ *non-empty* subsets of $J$. Since the sets on the right in (0.12) are mutually disjoint,

$$\#\left(\bigcup_{j\in J} E_j\right) = \sum_{X\subset J} \#(F_X) , \tag{0.13}$$

So far, so good, but the problem with trying to apply this as it stands is that the sets $F_X$, $X \subset J$ are quite complicated compared to the sets $E_X$, $X \subset J$, as one see by comparing (0.9) and (0.10). Therefore, we get a more useful formula if we can express the right hand side of (0.13) in terms of the cardinalities of the sets $E_X$, $X \subset J$. This can be done, and the result is the general *Inclusion-Exclusion Formula*:

**0.6 THEOREM.** *Let $S$ be any finite set, and for any natural number $N$, let let $E_1, \ldots, E_N$ be any $N$ subsets of $S$. Let $J = \{1, \ldots, N\}$. For $X \subset J$, let $E_X \subset S$ be given by* (0.9). *Then*

$$\#\left(\bigcup_{j\in J} E_j\right) = \sum_{k=1}^{N} (-1)^{k-1} \left(\sum_{X\subset J, \#(X)=k} \#(E_X)\right) . \tag{0.14}$$

There is a virtually identical version for probabilities:

**0.7 THEOREM.** *Let $S$ be any set equipped with a probability measure $P$, and for any natural number $N$, let let $E_1, \ldots, E_N$ be any $N$ events in $S$. Let $J = \{1, \ldots, N\}$. For $X \subset J$, let $E_X \subset S$ be given by* (0.9). *Then*

$$P\left(\bigcup_{j\in J} E_j\right) = \sum_{k=1}^{N} (-1)^{k-1} \left(\sum_{X\subset J, \#(X)=k} P(E_X)\right) . \tag{0.15}$$

Before proving the theorems, we give some examples illustrating its use.

**0.8 EXAMPLE.** *Consider a deck of $N$ distinct cards arranged in some prescribed order.* What is the probability that after a random shuffle of the deck, not a single card is in it original place? *A rearrangement, or permutation, of a set, that does not leave any element in its place is called a* derangement. *The question therefore, is:* What is the probability that a randomly chosen permutation is a derangement?. *By "random" we mean, as usual, that all permutations are taken to be equality likely. Since there are $N!$ permutations, an equivalent question is:* How many of the permutations are derangements? *Counting the derangements, and then dividing by $N!$, we get the probability that a random shuffle leaves no card in its place.*

*For each $j \in J = \{1, \ldots, N\}$, define $E_j$ to be the set of shuffles that leave the $j$th card fixed. The* only *requirement for membership in $E_j$ is that the $j$th card is kept in its place. Other cards may or may not be. The thing that makes this event "simple" is that it only depends on the position of a single card. Evidently then, the event that* at least one *card is kept fixed is $\bigcup_{j\in J} E_j$ and then the event that* no *card is kept fixed is the complementary event $\left(\bigcup_{j\in J} E_j\right)^c$. Since the probabilities of complementary sets sum to 1, the probability we seek is*

$$1 - P\left(\bigcup_{j\in J} E_j\right) \tag{0.16}$$

*To apply Theorem 0.15, we need to compute $P(E_X)$ for all non-empty $X \subset J$. This is easy: Suppose $\#(X) = k$. Then the $k$ cards with indices in $X$ must be kept in their places. The general shuffle that fixes these $k$ cards is obtained by shuffling the $N-k$ remaining cards, and then inserting the $k$ chosen cards into the correct places. Therefore:*

$$\#(X) = k \quad \Rightarrow \quad P(E_X) = \frac{(N-k)!}{N!} \ ,$$

*no mater which $k$ cards are to be kept in place. Clearly there are $\binom{N}{k}$ sets $X \subset J$ with $\#(X) = k$. Therefore*

$$\sum_{X \subset J, \#(X)=k} P(E_X) = \binom{N}{k} \frac{(N-k)!}{N!} = \frac{1}{k!} \ .$$

*Now* (0.15) *yields* $P\left(\sum_{j\in J} E_j\right) = \sum_{k=1}^{N} (-1)^{k-1}\frac{1}{k!}$, *and then by what we have noted above, the probability that a random shuffle leaves no card fixed is*

$$1 - \left(\sum_{k=1}^{N} (-1)^{k-1}\frac{1}{k!}\right) = \sum_{k=0}^{N} (-1)^{k}\frac{1}{k!} \ .$$

*The sum on the right converges very rapidly to $1/e$. Since the series for $e^{-1}$ is an alternating decreasing series, the result of truncating at a negative term is less than $1/e$, and the result of truncating at a positive term is greater than $1/e$.*

*Let $p$ denote the probability of a deranged shuffle for a standard deck of 52 cards. Since 52 is even, the last term in the sum for $p$ is positive, and so $p > 1/e$, but $p - 1/52! < 1/e$. That is,*

$$\frac{1}{e} < p < \frac{1}{e} + \frac{1}{52!} \ .$$

*Since $52! \approx 8.068 \times 10^{67}$, for all practical purposes, the probability of a deranged shuffle of a standard deck of cards is $1/e$.*

**0.9 EXAMPLE.** *Consider a dinner party at which 10 married couples will be present, and everyone will be seated in a random order at a round table. What is the probability no two spouses are seated next to each other?*

*To answer this, we assign an index $j \in J := \{1,\ldots,10\}$ to each couple, and for each $j \in J$, define $E_j$ to be the event that the $j$th couple gets seated next to each other. The event that* at least *one couple is seated next to each other is $\bigcup_{j\in J} E_j$, and then the event that* no *couple is seated next to each other is the complementary event $\left(\bigcup_{j\in J} E_j\right)^c$. Since the probabilities of complementary sets sum to 1, the probability we seek is*

$$1 - P\left(\bigcup_{j\in J} E_j\right) \tag{0.17}$$

*First, let's fix the sample space: If everyone shifts their place at the table by $\ell$ seats in a clockwise order, this simply "rotates" the seating arrangement, and does no affect who sits by whom. We call two seating arrangements* equivalent *is they are related by such a rotation. Let $S$ be the set of equivalence classes of such arrangements. Let's designate one of the participants as "Guest of Honor". Let the Guest of Honor have their choice of any seat. There are 19 people remaining. Chose one of them, and place them in the next seat clockwise form the Guest of Honor. keep filling in the seats in the clockwise order. Since it does not matter where the Guest of Honor chose to sit, there are 19! arrangements in $S$. Having finished counting the outcomes in $S$, forget about the Guest of Honor, which was introduced into the story only to help with this counting task.*

*Now, suppose $X \subset J$ is such that $\#(X) = k$. Leave the $k$ husbands aside for the moment, and start seating the $20 - k$ remaining guests. There would be (20- k-1)! ways to seat them around a round table, but as soon as a wife from the $k$ designated couples is chosen, make one of the two choices for seating husband next or wife next, seat them both, and keep going. Thus, there are $(19 - k)!2^k$ ways to do the seating, no matter which of the $\binom{10}{k}$ sets of $k$ couples $X$ may be. It follows that*

$$\sum_{X\subset J, \#(X)=k} P(E_X) = \binom{10}{k}\frac{(19-k)!2^k}{19!} \ .$$

*Now* (0.15) *yields*

$$P\left(\bigcup_{j\in J} E_j\right) = \sum_{k=1}^{10}\binom{10}{k}\frac{(19-k)!2^k}{19!} \approx 0.6605 \ ,$$

*and hence the probability than nobody is seated next to their spouse is approximately 0.3395.*

## 0.5 Proof of the Inclusion-Exclusion Formula and More

We use the notation of the previous subsection. We have already observed that for each $X \subset J$, $F_X \subset E_X$, and we have that for $X, Y \subset J$, $X \neq Y$, $F_X$ and $F_Y$ are disjoint. We can say more:

**0.10 LEMMA.** *For $X, Y \subset J$, if $X \subset Y$, then $F_Y \subset E_X$, but if $X \not\subset Y$, $F_Y$ and $E_X$ are disjoint.*

*Proof.* Suppose $X \subset Y$. It is clear from the definition $E_X = \cap_{j\in X} E_j$ that $E_Y \subset E_X$, and we have already seen that $F_Y \subset E_Y$, so $F_Y \subset E_X$.

Next, suppose that $X \not\subset Y$. then there is some $j \in X$ such that $j \notin Y$. By definition, if $\omega \in E_X$, $\omega \in E_j$. Also by definition, if $\omega \in F_Y$, $\omega \notin E_j$. Hence $E_X$ and $F_Y$ are mutually exclusive. □

*Proof of Theorem 0.6.* By the previous lemma, for each $X \subset J$, $E_X = \displaystyle\bigcup_{Y \ : \ X\subset Y} F_Y$, and since the union is disjoint, $\#(E_X) = \displaystyle\sum_{Y \ : \ X\subset Y} \#(F_Y)$. Therefore,

$$\sum_{X \ : \ \#(X)=k} \#(E_X) = \sum_{X \ : \ \#(X)=k}\left(\sum_{Y \ : \ X\subset Y} \#(F_Y)\right) \ .$$

For $\ell = k, \dots, N$, if $Y \subset J$ has cardinality $\ell$, there are $\binom{\ell}{k}$ subsets $X$ of cardinality $k$ that are contained in $Y$. Thus, doing the double sum over $X$ and $Y$, each set $Y$ with cardinality $\ell$ comes up $\binom{\ell}{k}$ times. Therefore,

$$\sum_{X\ :\ \#(X)=k} \#(E_X) = \sum_{\ell=k}^{N} \binom{\ell}{k} \left( \sum_{Y\ :\ \#(Y)=\ell} \#(F_Y) \right) = \sum_{\ell=0}^{N} \binom{\ell}{k} \left( \sum_{Y\ :\ \#(Y)=\ell} \#(F_Y) \right) ,$$

where the last equality is valid because $\binom{\ell}{k} = 0$ for $\ell < k$.

Multiplying by $(-1)^{k-1}$ and summing on $k$, yields

$$\begin{aligned} \sum_{k=1}^{N}(-1)^{k-1} \left( \sum_{X\ :\ \#(X)=k} \#(E_X) \right) &= \sum_{k=1}^{N}(-1)^{k-1} \sum_{\ell=0}^{N} \binom{\ell}{k} \left( \sum_{Y\ :\ \#(Y)=\ell} \#(F_Y) \right) \\ &= \sum_{\ell=0}^{N} \left( \sum_{Y\ :\ \#(Y)=\ell} \#(F_Y) \right) \left( \sum_{k=1}^{N} \binom{\ell}{k} (-1)^{k-1} \right) \end{aligned} \tag{0.18}$$

By the Binomial Theorem,

$$\sum_{k=1}^{N} \binom{\ell}{k} (-1)^{k-1} = 1 - \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k 1^{\ell-k} = 1 - (1 + (-1))^{\ell} = 1 .$$

Therefore, we have the further simplification that

$$\sum_{k=1}^{N}(-1)^{k-1} \left( \sum_{X\ :\ \#(X)=k} \#(E_X) \right) = \sum_{\ell=0}^{N} \left( \sum_{Y\ :\ \#(Y)=\ell} \#(F_Y) \right) = \sum_{Y \subset J} \#(F_Y) .$$

By (0.13), the right side is the same as $\#\left(\cup_{j=1}^{N} E_j\right)$. □

*Proof of Theorem 0.7.* The only property of the cardinality set function $E \mapsto \#(E)$ that was used in the proof of Theorem 0.6 was that it is additive over disjoint unions. Probability measures are also additive over disjoint unions, and hence the same proof applies when $\#(E)$ is replaced by $P(E)$. □

We easily see something more from the proof of Theorem 0.7, or equivalently of Theorem 0.7. We shall use the following lemma.

**0.11 LEMMA.** *For $M < \ell$, $\sum_{k=0}^{M} \binom{\ell}{k}(-1)^k$ has the same sign as $(-1)^M$.*

*Proof.* For $1 \leq j \leq \ell$, define $b_j := \begin{pmatrix} \ell \\ j \end{pmatrix} - \begin{pmatrix} \ell \\ j-1 \end{pmatrix}$. Since $\binom{\ell}{j} \geq \binom{\ell}{j-1}$ for $j \leq \frac{\ell+1}{2}$, $b_j$ is non-negative for $j \leq \frac{\ell+1}{2}$. For $j > \frac{\ell+1}{2}$, $b_j$ is non-positive. Suppose that $M$ is odd and write $M = 2n+1$. Then

$$\sum_{k=0}^{M} \binom{\ell}{k} (-1)^k = -\sum_{k=0}^{n} b_{2k+1} =: -f(M) \tag{0.19}$$

Observe that $f(M)$ is increasing in $M$ as long as $2n+1 \leq \frac{\ell+1}{2}$, because all summands are positive. As $M$ increases beyond this point, $f(M)$ decreases because negative terms are being added in. However, since $f(\ell) = (1+(-1))^\ell = 0$, $f(M)$ does not decrease to 0 until $M$ reaches $\ell$. Thus, it stays positive as long as $M < \ell$, and then by (0.19), $\sum_{k=0}^{M} \binom{\ell}{k}(-1)^k < 0$ for all odd $M < \ell$.

The proof for $M = 2n$ is similar. Then

$$g(M) := \sum_{k=0}^{M} \binom{\ell}{k}(-1)^k = 1 + \sum_{k=1}^{n} b_{2k} \ .$$

As above, $g(M)$ is increasing in $M$ as long as $2n \leq \frac{\ell+1}{2}$, and then it starts to decrease. However, $1 + g(\ell) = 0$, so $1 + g(M)$ is positive for all even $M < \ell$ □

**0.12 THEOREM.** *Let $S$ be any finite set, and for any natural number $N$, let let $E_1, \dots, E_N$ be any $N$ subsets of $S$. Let $J = \{1, \dots, N\}$. For $X \subset J$, let $E_X \subset S$ be given by (0.9). Then for any $M < N$, if $M$ is odd*

$$\#\left(\bigcup_{j\in J} E_j\right) \leq \sum_{k=1}^{M}(-1)^{k-1}\left(\sum_{X\subset J, \#(X)=k} \#(E_X)\right) \ , \tag{0.20}$$

*and if $M$ is even,*

$$\#\left(\bigcup_{j\in J} E_j\right) \geq \sum_{k=1}^{M}(-1)^{k-1}\left(\sum_{X\subset J, \#(X)=k} \#(E_X)\right) \ , \tag{0.21}$$

*Proof.* Suppose, for $M < N$, in (0.18) we only sum from $k = 1$ to $k = M$, instead of the $k = N$. Then the same reasoning that yields (0.18) yields

$$\sum_{k=1}^{M}(-1)^{k-1}\left(\sum_{X\ :\ \#(X)=k} \#(E_X)\right) = \sum_{\ell=0}^{N}\left(\sum_{Y\ :\ \#(Y)=\ell} \#(F_Y)\right)\left(\sum_{k=1}^{M}\binom{\ell}{k}(-1)^{k+1}\right) \ .$$

Note that

$$\sum_{k=1}^{M}\binom{\ell}{k}(-1)^{k+1} = 1 - \sum_{k=0}^{M}\binom{\ell}{k}(-1)^k$$

By Lemma 0.11, this quantity is at most 1 if $M$ is even, and is at least 1 if $M$ is odd. □

Of course, the analogous theorem for probabilities is valid as well. The utility of this theorem, in the probabilistic setting, is that the finite sequence of numbers

$$q_k := \sum_{X\subset J, \#(X)=k} P(E_X)$$

decrease quite rapidly as $k$ increases. Theorem 0.12 (in its probabilistic version) says that for all $M < N$,

$$\left|\sum_{k=1}^{M-1} q_k - P\left(\bigcup_{j\in J} E_j\right)\right| \le q_M \ .$$

Suppose we want to know $P\left(\bigcup_{j\in J} E_j\right)$ to within plus or minus 0.01. If we can find some $M$ for which $q_M < 0.01$, then

$$\left|\sum_{k=1}^{M-1} q_k - P\left(\bigcup_{j\in J} E_j\right)\right| \le 0.01 \ .$$

When $N$ is large, $M$ may be much smaller than $N$, and it might be much easier to compute $\sum_{k=1}^{M-1} q_k$ than to compute $\sum_{k=1}^{N} q_k$.

**0.13 EXAMPLE.** *Consider again the dinner party from Example 0.9. Using the notation introduced just above, we have found that*

$$q_k = \binom{10}{k}\frac{(19-k)!2^k}{19!} \ .$$

*Evaluating the $q_k$, one finds, with 3 significant digits,*

$$q_1 \approx 1.0526 \ , \quad q_2 \approx 0.5263, \quad q_3 \approx 0.1651, \quad q_4 \approx 0.0361, \quad q_5 \approx 0.0058 \ .$$

*Hence the probability that at least one couple is seated next to each other is*

$$q_1 + q_2 + q_3 + q_4 \pm 0.01 = 0.0653 \pm 0.01 \ .$$