Final Exam Solutions, Math 477, Dec 21, 2018

December 27, 2018

1. If 14 identical blackboards are to be divided among 4 schools, how many divisions are possible? What if each school is to receive at least 2 blackboards?

SOLUTION By the "Stars and Bars" Theorems, the answer to fiirst part is

$$\binom{14+4-1}{4-1} = \binom{17}{3} = 680 \ .$$

For the second part, we distrubute one to each school. That leaves us with 10 to distribute among 4 schools with each school recieving at least one of the 10. So this time the answer is

$$\binom{10-1}{4-1} = \binom{9}{3} = 84 \ .$$

2. Three airlines serve a city. Airline A carries 50% of all passengers, airline B carries 30% of all passengers, and airline C carries 20% of all passengers. The flights of airline A are cancelled at a rate affecting 1% of their passengers, the flights of airline B are cancelled at a rate affecting 3% of their passengers, and the flights of airline C are cancelled at a rate affecting 5% of their passengers. If a randomly selected passenger had their flight cancelled, what is the probability that they were (attempting to) fly with airline A?

SOLUTION For $X = A, B, C$, let E_X be the event that the passenger has a ticket with airline X. Let F be the event that their flight is canceled. We are asked to cpmpute $P(E_A|F)$. By Bate's Formula,

$$P(E_A|F) = \frac{P(E_A)}{P(F)} P(F|E_A) \ ,$$

and we are given that $P(F|E_A) = 0.01$ and $P(E_A) = 0.5$. If remains to determine $P(F)$:

$$\begin{aligned} P(F) &= P(F \cap E_A) + P(F \cap E_B) + P(F \cap E_C) \\ &= P(F|E_A)P(E_A) + P(F|E_B)P(E_B) + P(F|E_C)P(E_C) \\ &= (0.01)(0.5) + (0.03)(0.3) + (0.05)(0.2) = 0.024 \ . \end{aligned}$$

Hence

$$P(E_A|F) = \frac{0.5}{0.024} 0.01 = 0.2083333333...$$

3. A fair tetrahedral (4 sided) die is rolled 10 times? What is the probability all four sides land "face down" at least once? (When a tetrahedral die is at rest on a table top after a toss, one face is on the table and three are "up", so it is the "down" side we focus upon.)

SOLUTION For $j = 1, 2, 3, 4$, let E_j be the event that the jth side does not land face down on any of the 10 tosses. Then $\cup_{j=1}^4 E_j$ is the event that at least one side does not land face down on any of the 10 tosses. We are asked to compute

$$1 - P\left(\cup_{j=1}^4 E_j\right) \ .$$

By the Inclusion-Exclusion formula and symmetry, together with the fact that $E_1 \cap E_2 \cap E_3 \cap E_4$ is impossible,

$$\begin{aligned} P\left(\cup_{j=1}^4 E_j\right) &= \binom{4}{1} P(E_1) - \binom{4}{2} P(E_1 \cap E_2) + \binom{4}{3} P(E_1 \cap E_2 \cap E_3)) \\ &= 4\left(\frac{3}{4}\right)^{10} - 6\left(\frac{2}{4}\right)^{10} + 4\left(\frac{1}{4}\right)^{10} = \frac{28757}{131072} = 0.2193984985... \end{aligned}$$

Thus, the probability all four sides land "face down" at least once is 0.7806015015....

4. Among 18 students in a room, 7 study mathematics, 10 study music, and 10 study computer programming. Also, 3 study mathematics and music, 4 study mathematics and computer programming, and 5 study music and computer programming. We know that 1 student studies all three subjects. If I pick one student at random whats the probability that this student doesn't study any of those 3 subjects?

SOLUTION Let E_1 denote the set of students that study mathematics. Let E_2 denote the set of students that study music. Let E_3 denote the set of students that study computer science. By the Inclusion-Eclusion formula for counting, and the given information,

$$\begin{aligned} \#(E_1 \cup E_2 \cup E_3) &= [\#(E_1) + \#(E_2) + \#(E_3)] \\ &- [\#(E_1 \cap E_2) + \#(E_1 \cap E_3) + \#(E_2 \cap E_3)] + \#(E_1 \cap E_2 \cap E_3) \\ &= [7 + 10 + 10] - [3 + 4 + 5] + 1 = 16 \ . \end{aligned}$$

Hence, 2 of the 18 students do not study any of the three subjects; required probability is $\frac{1}{9}$.

5. Suppose that 1 in $5,000$ light bulbs are defective. Let X denote the number of defective lightbulbs in a sample of size $10,000$. Estimate $P(X \geq 3)$.

SOLUTION The distribution of X is well appoximated by the Posson distribution with parameter

$$\lambda = 10,000 \times \frac{1}{5,000} = 2 \ .$$

Hence

$$P(X < 3) \approx e^{-2}\left(1 + 2 + \frac{2^2}{2}\right) = e^{-2}5 \approx 0.6767 \ .$$

Hence the required probability is 0.3233.

6. Along a road that is 1 mile long, 3 people are distributed uniformly at random and independently. What is the probability that at least a half mile separate the second and third person?

SOLUTION Let X_j^* denote the position of the jth person along the road, orderef left to right. By the theory of order statsitics, the joint distribution of there randome variables is

$$f(x_1, x_2, x_3) = \begin{cases} 6 & x_1 < x_2 < x_3 \\ 0 & \text{else} \end{cases} .$$

The event in question is $\{X_3^* > X_2^* + \frac{1}{2}\}$, and when this event occurs, $X_1^* < X_2^* \leq \frac{1}{2}$. Hence

$$\begin{aligned} P\left(X_3^* > X_2^* + \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \left(\int_{x_1}^{\frac{1}{2}} \left(\int_{x_2+\frac{1}{2}}^{1} f(x_1, x_2, x_3) \mathrm{d}x_3 \right) \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &= 6 \int_0^{\frac{1}{2}} \left(\int_{x_1}^{\frac{1}{2}} \left(\frac{1}{2} - x_2 \right) \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &= 6 \int_0^{\frac{1}{2}} \left(\frac{1}{8} - \frac{1}{2}x_1 + \frac{1}{2}x_1^2 \right) \mathrm{d}x_1 = \frac{1}{8} \end{aligned}$$

7. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. There are N floors above the ground floor, and each person is equally likely to get off at any of the N floors, independent of what the others do. Let X be the first floor among $\{1, \ldots, N\}$ after which no passengers remain. Compute the distribution function of X.

SOLUTION Let M be the number of people who enter on the ground floor. Let $\{Y_1, \ldots, Y_M\}$ be the floors these people select. The random variable X is given by

$$X = \begin{cases} \max\{Y_1, \ldots, Y_M\} & M > 0 \\ 1 & M = 0 \end{cases} .$$

We condition on $M = m$. If m people enter on the ground floor, For any $r \in \{1, \ldots, N\}$,

$$P(\max\{Y_1, \ldots, Y_m\} \leq r) = P\left(\cap_{j=1}^m \{Y_j \leq r\}\right) = \prod_{j=1}^m P(Y_j \leq r) = \left(\frac{r}{N}\right)^m ,$$

noting that this is also valid for $m = 0$ as well as $m > 0$. That is, for all $m \geq 0$ and all $r \in \{1, \ldots, N\}$,

$$P(X \leq r | M = m) = \left(\frac{r}{N}\right)^m .$$

Next, for $r \leq N$,

$$P(X \leq r) = \sum_{m=0}^{\infty} e^{-10} \frac{10^m}{m!} \left(\frac{r}{N}\right)^m = e^{-10(N-r)/N} .$$

Evidently $P(X \leq r) = 1$ for $r \geq N$.

8. Let X and Y be independent and uniformly distributed on $[0, 1]$. Define $U = X + Y$ and $V = X/Y$.

(a) Compute the joint probability density of U and V. Are U and V independent?
(b) Compute $P(V > v)$ for all $v > 0$.

SOLUTION Define $f(x,y) = 1$ for $0 \le x, y \le 1$, and 0 otherwise. Define

$$u(x,y) = x + y \qquad \text{and} \qquad v(x,y) = x/y \ .$$

Solving for x and y, we find

$$x = \frac{uv}{1+v} \qquad \text{and} \qquad y = \frac{u}{1+v} \ .$$

We compute the Jacobian:

$$\begin{bmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial x}{\partial v}(u,v) \\ \frac{\partial y}{\partial u}(u,v) & \frac{\partial y}{\partial v}(u,v) \end{bmatrix} = \begin{bmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{1}{1+v} & \frac{-u}{(1+v)^2} \end{bmatrix}$$

and therefore

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = u(1+v)^{-2} \ .$$

The condition $0 \le x, y \le 1$ gives us

$$0 \le \frac{uv}{1+v} \le 1 \qquad \text{and} \qquad 0 \le= \frac{u}{1+v} \le 1 \ ,$$

so that, since u and v are non-negative when x and y non-negative,

$$0 \le u \le 1 + \frac{1}{v} \qquad \text{and} \qquad 0 \le u \le 1 + v \ .$$

Hence the joint probability density function for U and V, $g(u,v)$, is given by

$$g(u,v) = u(1+v)^{-2} \quad \text{for} \quad 0 \le v \le \infty \quad \text{and} \quad 0 \le u \le 1 + \min\{v, 1/v\} \ .$$

This is not a product of functions of u and v, so U and V are not indepndent.

for the second part, we compute, for $0 \le v \le 1$,

$$g_V(v) = \int_0^{1+v} g(u,v)\mathrm{d}u = \frac{1}{2} \ ,$$

and for $v > 1$,

$$g_V(v) = \int_0^{1+\frac{1}{v}} g(u,v)\mathrm{d}u = \frac{1}{2v^2} \ .$$

That is,

$$g_V(v) = \begin{cases} \frac{1}{2} & 0 \le v \ leq 1 \\ \frac{1}{2v^2} & 1 \le v \end{cases} \ .$$

Then $P(V > v) = \int_v^\infty g_V(w)\mathrm{d}w$, so

$$P(V > v) = \begin{cases} 1 - \frac{1}{2}v & 0 \le v \le 1 \\ \frac{1}{2v} & 1 \le v \end{cases} \ .$$

9. For $N \in \mathbb{N}$, let $\{X_1, \ldots, X_N\}$ be independent random variables that are uniformly distributed on $[0,1]$. Compute $\mathrm{E}(\max\{X_1, \ldots, X_N\})$ and $\mathrm{E}(\min\{X_1, \ldots, X_N\})$

SOLUTION Let $Y := \max\{X_1, \ldots, X_N\}$. Then $\{Y \le y\} = \cap_{j=1}^N \{X_j \le y\}$, and so $P(Y \le y) = y^N$. Differentiating, the probability density function of Y is Ny^{N-1}. Hence

$$\mathrm{E}(Y) = N \int_0^1 y^N \mathrm{d}y = \frac{N}{N+1} \ .$$

Let $Z := \min\{X_1, \ldots, X_N\}$. Then $\{Z > z\} = \cap_{j=1}^N \{X_j > z\}$, and so $P(Z > z) = (1-z)^N$. Differentiating, the probability density function of Z is $N(1-z)^{N-1}$. Hence

$$\mathrm{E}(Z) = N \int_0^1 z(1-z)^{N-1} \mathrm{d}z = \int_0^1 (1-z)^N \mathrm{d}z = \frac{1}{N+1} \ .$$

10. Consider two Markov chains with the state space $\{1,2,3\}$ and the transitions matrices

$$P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad P_2 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} .$$

The one with transition matrix P_1 is the *first Markov chain*, and the one with transition matrix P_2 is the *second Markov chain*.

(a) One of these two transition matrices satisfies the condition that $P^n_{i,j} > 0$ for all i,j, and the other does not. Which one is it and why? Find the smallest value of n for which $P^n_{i,j} > 0$ for all i,j for the one that has this property.

(b) For one of these Markov chains,

$$\lim_{n\to\infty} P(X_n = j | X_0 = i)$$

is independent of i. Compute this limit for $j = 1,2,3$ for the chain with this property.

SOLUTION (a) The first one satisfies this property since it is possible to get from any state to state 1 in one step, and then from state 1 to any other in one more step. Sp the minimal n is 2. For the second one, if $X_0 = 1$, then $X_{2n} = 1$, while $X_{2n-1} \in \{2,3\}$, so this does not satisfy the property.

(b) By part **(a)**, it is the first one. To compute the limit, we find the eigenvactor of P^T with eigenvalue 1. We find

$$\boldsymbol{\pi}_\infty = \frac{1}{17}(9,3,5) \ .$$

Hence

$$\begin{aligned} \lim_{n\to\infty} P(X_n = 1 | X_0 = i) &= \frac{9}{17} \\ \lim_{n\to\infty} P(X_n = 2 | X_0 = i) &= \frac{3}{17} \\ \lim_{n\to\infty} P(X_n = 3 | X_0 = i) &= \frac{5}{17} \end{aligned}$$

11. A person has 100 light bulbs whose lifetimes are independent exponentials with a mean of 5 hours. If the bulbs are used one at a time, with a new bulb immediately replacing a failed bulb, what is the probability that there is still a working bulb after 525 hours?

SOLUTION Let X_j be the lifetime of the jth bulb. For exponential variables, the variance is the square of the mean, so we also know that the variance is 25. We are asked to estimate the probability that

$$\sum_{j=1}^{100} X_j > 525 \ .$$

This event is the same as

$$\sum_{j=1}^{100} \frac{(X_j - 5)}{50} > \frac{525 - 500}{50} = \frac{1}{2} \ .$$

By the Central Limit Theorem, the random variable on the left is apprximately satandard normal, so the probability we seek is

$$P\left(Z > \frac{1}{2}\right) = 1 - P\left(Z \le \frac{1}{2}\right) = 1 - \Phi(0.5) \approx 1 - 0.6915 = 0.3085 \ .$$

12. Customers arrive at a bank at a Poisson rate λ. Given that 2 customers arrived during the first hour, what is the probability that at least one of them arrived in the first 20 minutes?

SOLUTION Interpret the rate λ is the expected number of customers to arrive in one hour. (You will get the same answer if you takes is as the expected number that arrive in one minute, as you will see below.) Let A be the event that at least one customer arrives in first 20 minutes. Let F be the event that exactly two customes arrive in the first hour.

We must compute

$$P(A|F) = \frac{P(A \cap F)}{P(F)} \ .$$

The number of customers arriving in the first two hours is Possoin distributed with parameter 2λ, and hence and $P(F) = e^{-2\lambda}\frac{(2\lambda)^2}{2}$.

Next, we compute $P(A \cap F)$. There are exactly two ways the event $A \cap F$ can occur: *(1)* Exactly one customer arrives in the first 20 minutes *and* exactly one cutomer arrives in the next 100 minutes. *(2)* Exactly two customers arrive in the first 20 minutes *and* no cutomer arrives in the next 100 minutes.

We will use the fact that the numbers of custumers arriving in the first 20 minutes is independent of the number of custometrs arriving in the next 20 minutes. Let X denote hte number of customers arriving in the first 20 minutes, and let Y denote the number of customers arriving in the next 100 minutes. By what has been notes above

$$P(A \cap F) = P(X = 1, Y = 1) + P(X = 2, Y = 0) = P(X = 1)P(Y = 1) + P(X = 2)P(Y = 0) \ .$$

Finally, X is Poisson with parameter $\lambda/3$, and Y is Poison with parameter $5\lambda/3$. Therefore,

$$
\begin{aligned}
P(A \cap F) &= P(X=1)P(Y=1) + P(X=2)P(Y=0) \\
&= \left(e^{-\lambda/3}\frac{\lambda}{3}\right)\left(e^{-5\lambda/3}\frac{5\lambda}{3}\right) + \left(e^{-\lambda/3}\frac{1}{2}\left(\frac{\lambda}{3}\right)^2\right)\left(e^{-5\lambda/3}\right) \\
&= e^{-2\lambda}\left[\frac{\lambda}{3}\frac{5\lambda}{3} + \frac{1}{2}\left(\frac{\lambda}{3}\right)^2\right] = e^{-2\lambda}\frac{11}{18}\lambda^2 \; .
\end{aligned}
$$

Altogether,

$$
P(A|F) = \frac{11}{36} \; ,
$$

independent of λ.