Solutions for Test I, Math 292 Spring 2014

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1: (a) Find the general solution of

$$x'(t) = -\frac{1}{t^2} - \frac{x(t)}{t} + x^2(t) \qquad (0.1) \quad \boxed{\texttt{p1}}$$

for $t > 0$.

(b) Find the solution with $x(1) = x_0$ for arbitrary $x_) \in \mathbb{R}$. For which values of x_0 does this solution exist for all $t > 0$?

SOLUTION Looking for a solution of the form $x_1 = Ct^\alpha$, we find

$$\alpha C t^{\alpha-1} + t^{-2} + \frac{C^{\alpha-1}}{t} - C^2 t^{2\alpha} = 0 \ .$$

We must have $\alpha - 1 = 2\alpha$, so $\alpha = -1$. We then have $-C + 1 + C - C^2 = 0$, or $C^2 = 1$. We have two solutions $C = \pm 1$.

If we choose $x_1 = -t^{-1}$, i.e., $C = -1$, and let $x = u + x_1$, then

$$u' = -\frac{3}{t}u + u^2 \ .$$

Defining $z = 1/u$, we find

$$z' - \frac{3}{t}z = -1 \ ,$$

which leads to

$$t^{-3}z = \frac{1}{2}t^{-2} + C$$

so finally,

$$x(t) = \left(\frac{1}{2}t + Ct^3\right)^{-1} - \frac{1}{t} \ .$$

Alternately, if we use $x_1 = t^{-1}$; i.e., $C = 1$, and again $x = x_1 + u$, we find

$$u' = \frac{1}{t}u + u^2 \ ,$$

and then with $z = 1/u$,

$$z' + \frac{1}{t}z = -1 ,$$

which leads to

$$tz = -\frac{1}{2}t^2 + C$$

so finally,

$$x(t) = \left(-\frac{1}{2}t + Ct^{-1}\right)^{-1} + \frac{1}{t} .$$

A bit of algebra shows the two solutions agree, as they must.

For **(b)** , we have

$$x_0 = \left(-\frac{1}{2} + C\right)^{-1} + 1 ,$$

so that

$$C = \frac{1}{2}\frac{x_0 + 1}{x_0 - 1} .$$

The solution then is

$$x(t) = \left(-\frac{1}{2}t + \frac{1}{2}\frac{x_0 + 1}{x_0 - 1}t^{-1}\right)^{-1} + \frac{1}{t} .$$

The solution exists for all $t > 0$ if and only if

$$\frac{1}{2}\frac{x_0 + 1}{x_0 - 1} < 0$$

since only in this case is there no division by zero. Therefore, the solution exists for all $t > 0$ if and only if

$$-1 \le x(1) \le 1 .$$

Notice that the endpoints of this interval correspond to our two particular solutions.

2: Consider the differential equation $x'(t) = v(x(t)$ where

$$v(x) = x - x^3 .$$

Note that $(0, 1)$ is a maximal interval.

(a) Is $v(x)$ Lipschitz on $(0, 1)$? Is it Lipschitz on all of $\mathbb{R}$? Justify your answers.

(b) Find the general solution $x(t)$ for $x(0) = x_0 \in (0, 1)$, and find the corresponding flow transformation.

(c) Let $x(0) = 1/2$. Find T such that $x(T) = 3/4$.

SOLUTION: (a) Differentiating, we find $v'(x) = 1 - 3x^2$. This is continuous on all of $\mathbb{R}$, and therefore bounded on any bounded interval, and so v is Lipschitz on $(0, 1)$. More specifically,

$$|v'(x)| \le 2$$

for all $x \in (0,1)$, and so $|v(y) - v(x)| \leq 2|y-x|$ for all $x, y \in (0,1)$. However, $v(x)$ is not Lipschitz on all of $\mathbb{R}$ since $v'(x)$ is not bounded on all of $\mathbb{R}$.

For **(b)**, by Barrow's formula

$$\begin{aligned} t(x) - t(x_0) &= \int_{x_0}^{x} \frac{1}{z - z^3} \mathrm{d}z = \int_{x_0}^{x} \left[\frac{1}{z} + \frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1+z} \right] \mathrm{d}z \\ &= \left. \frac{z}{\sqrt{1-z^2}} \right|_{x_0}^{x} . \end{aligned}$$

Solving for $x(t)$ we find

$$x(t) = \frac{e^{t-t_0} x_0}{\sqrt{1 + x_0^2 (e^{2(t-t_0)} - 1)}} .$$

For **(c)**, by Barrow's formula,

$$t(3/4) - t(1/2) = \int_{1/2}^{3/4} \frac{1}{z - z^3} \mathrm{d}z = \frac{1}{2} \ln(27/7) \approx 0.67496 .$$

3: Consider the vector field

$$\mathbf{v}(x,y) = ((y-1)(x-y) \ , \ (x+y)(x+1)) .$$

Alternatively, this equation is a Bernoulli equation, and can be solved by the standard Bernouli substitution. But Barrow's formula is still the easiest way to answer the final part.

(a) Find all equilibrium points, and for each one, determine whether it is asymptotically stable, Lyapunov stable or unstable, or if this cannot be decided by linearization. Explain your reasoning and justify your answer with appropriate calculations.

(b) Sketch the solution curves in the vicinity of each equilibrium point for which the linearization determines the stability.

SOLUTION: At an equilibrium point, we must have either $y = 1$ or $y = x$, and we must also have $(x+y)(x+1) = 0$. If $y = 1$, the latter condition is $(x+1)^2 = 0$, so $x = -1$. Thus $(-1,1)$ is the only equilibrium point with $y = 1$. If $y = x$, the second condition becomes $2x(x+1) = 0$ which has the solutions $x = 0$ and $x = -1$. Hence the only other equilibrium points we get are $(0,0)$ and $(-1,-1)$. Thus,

$$\mathbf{x}_1 = (-1,-1) \qquad \text{and} \qquad \mathbf{x}_2 = (0,0) \qquad \text{and} \qquad \mathbf{x}_3 = (-1,1) .$$

are the only equilibrium points.

Computing the Jacobian matrix of $\mathbf{v}$ at $\mathbf{x}_1$, we find

$$[D_{\mathbf{v}}(x,y)] = \left[\begin{array}{cc} y-1 & x-2y+1 \\ 2x+y-1 & x+1 \end{array} \right] .$$

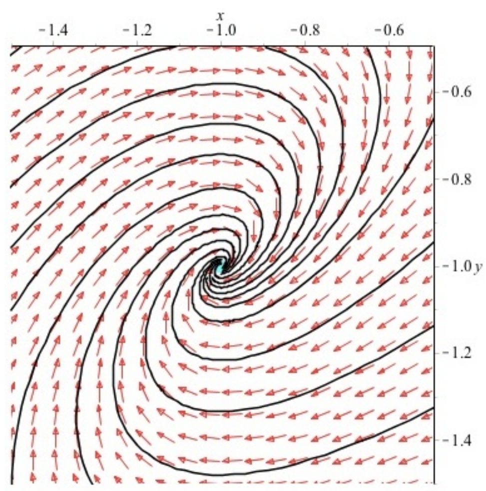
Therefore,

$$[D_{\mathbf{v}}(\mathbf{x}_1)] = \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} .$$

The characteristic polynomial is $t^2 + 2t + 4 = (t+1)^2 + 3$, so that eigenvalues are

$$\mu_{\pm} = -1 \pm i\sqrt{3} .$$

The imaginary parts are both strictly negative so $\mathbf{x}_1$ is an asymptotically stable equilibrium point. Here is the phase portrait in the region $-1/2 \le x, y \le 1/2$:



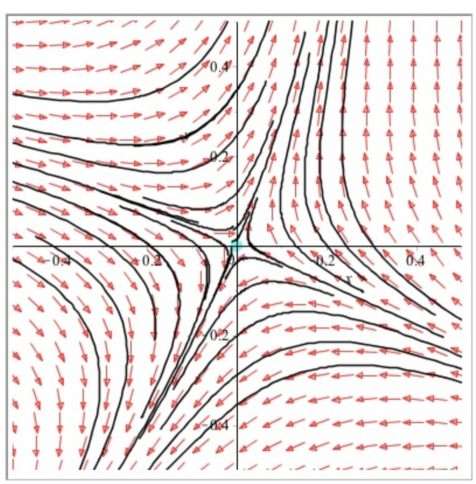
Next,

$$[D_{\mathbf{v}}(\mathbf{x}_2)] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} .$$

The characteristic polynomial is $t^2 - 2$, so that eigenvalues are

$$\mu_{\pm} = \pm\sqrt{2} .$$

One eigenvalue is strictly positive, so $\mathbf{x}_2$ is unstable. Here is the phase portrait in $-3/2 \le x, y \le 1/2$:



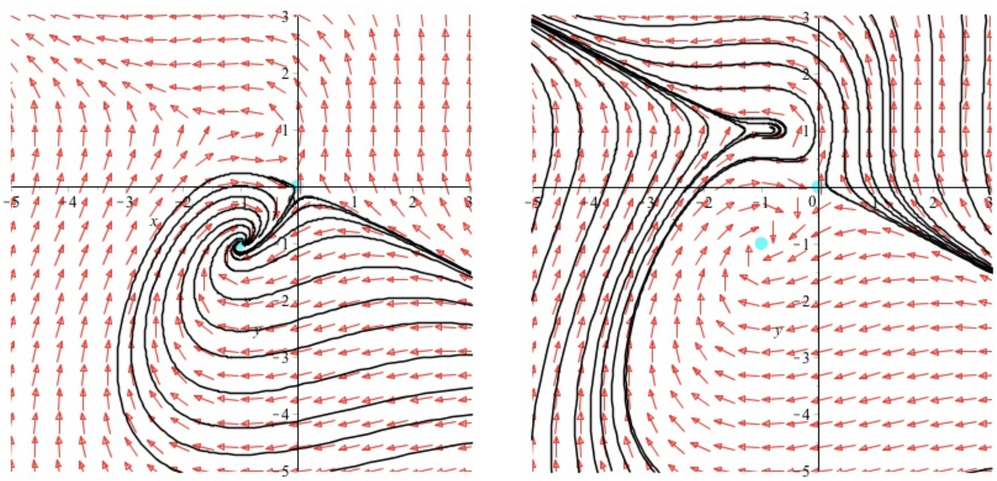
Finally,

$$[D_{\mathbf{v}}(\mathbf{x}_2)] = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} .$$

The only eigenvalue is 0. The stability (or not) of this equilibrium point cannot be decided by linearization.

The following was not part of the problem, but here is a more global picture showing the vicinity of both equilibrium points. Both figures show the region $-5 \le x, y \le 3$. The figure on the left shows phase curves passing through points that get 'swept into' the stable equilibrium point. This is its *basin of attraction*. The figure on the right shows phase curves passing through points that get swept away.

Looking at the plots, one can see that the equilibrium point $\mathbf{x}_3$ is unstable.



4: Consider the system

$$\begin{aligned} x' &= 2x - 5y \\ y' &= x - 2y \end{aligned} \tag{0.2}$$

(a) Find a matrix A so that this system can be written as $\mathbf{x}' = A\mathbf{x}$, and compute e^{tA}.

(b) Use Duhamel's formula to find the solution of

$$\mathbf{x}'(t) = A\mathbf{x}(t) + (\cos t, \sin t)$$

with $\mathbf{x}(0) = (1, 1)$ where A is the matrix form part **(a)**.

Extra Credit: Show that the solution curves of the system in part **(a)** are ellipses, and find the equations of these ellipses.

SOLUTION: The Matrix is

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} .$$

The characteristic polynomial is $t^2 + 1 = (t + i)(t - i)$, so the eigenvalues are $\mu_\pm = \pm i$. The eigenvectors are

$$\mathbf{v}_\pm = (2 \pm i, 1) .$$

Hence one complex solution is

$$\begin{aligned}\mathbf{z}(t) &= e^{t\mu_+}\mathbf{v}_+ = e^{it}(2+i,1) = (\cos t + i\sin t)(2+i,1)\\ &= (2\cos t - \sin t, \cos t) + i(\cos t + 2\sin t, \sin t) = \mathbf{x}(t) + i\mathbf{y}(t)\ .\end{aligned}$$

This gives us two real solution, $\mathbf{x}(t)$ and $\mathbf{y}(t)$, and

$$\begin{aligned}e^{tA} &= [\mathbf{x}(t),\mathbf{y}(t)][\mathbf{x}(0),\mathbf{y}(0)]^{-1}\\ &= \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t\end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0\end{bmatrix}^{-1}\\ &= \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2\end{bmatrix}\\ &= \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t\end{bmatrix}\ .\end{aligned}$$

(b). The solution is

$$\begin{aligned}\mathbf{x}(t) &= e^{tA}\mathbf{x}_0 + \int_0^t e^{(t-s)A}(\cos s, \sin s)\mathrm{d}s\\ &= e^{tA}\left(\mathbf{x}_0 + \int_0^t e^{-sA}(\cos s, \sin s)\mathrm{d}s\right)\ .\end{aligned}$$

Next,

$$\begin{bmatrix} \cos s + 2\sin s & -5\sin s \\ \sin s & \cos s - 2\sin s\end{bmatrix}(\cos s, \sin s) = (5 - 4\cos^2 s - 2\sin s\cos s\ ,\ 2\sin^2 s)\ ,$$

and then we compute

$$\int_0^t (5 - 4\cos^2 s - 2\sin s\cos s\ ,\ 2\sin^2 s)\mathrm{d}s = (\cos^2 t - 2\cos t\sin t + 3t - 1\ ,\ -\sin t\cos t + t)\ ,$$

so that

$$\mathbf{x}_0 + \int_0^t e^{-sA}(\cos s, \sin s)\mathrm{d}s = (\cos^2 t - 2\cos t\sin t + 3t\ ,\ -\sin t\cos t + t + 1)\ ,$$

Finally then,

$$\mathbf{x}(t) = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t\end{bmatrix}(\cos^2 t - 2\cos t\sin t + 3t\ ,\ -\sin t\cos t + t + 1)\ ,$$

and you can leave the answer in this form. However, just to complete the analysis, here is the result of multiplying it all out, and simplifying:

$$\mathbf{x}(t) = (t(3\cos t - 2\sin t) - 2\sin t\ ,\ t(\cos t + \sin t) - \sin t)\ .$$

For the extra credit, let $f(x,y) = Ax^2 + Bxy + Cy^2$. Let $\mathbf{x}(t)$ be a solution of the system. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)), y(t)) = 2A(2x - 5y) + Bx(2 - 2y) + By(2x - 5y) + 2Cy(x - 2y) \ .$$

Setting this equal to zero, we get

$$(4A + B)x^2 + (2C - 10A)xy + (-4C - 5B)y^2 = 0$$

So the ellipses are given by

$$x^2 - 4xy + 5y^2 = \text{constant} \ .$$