Practice Test for Test 2, Math 292, April 25, 2013

Eric A. Carlen[1]
Rutgers University

April 24, 2014

1. The differential equation

$$t^2x''(t) - 3tx'(t) + 4x(t) = 0$$

has polynomial coefficients.

(a) Find one polynomial solution to this equation.

(b) Find the general solution of this equation.

(c) Find the general solution of

$$t^2x''(t) - 3tx'(t) + 4x(t) = t^2 \ln t\ .$$

2. Consider the equation

$$y''(x) - xy'(x) + \frac{x^2}{2}y(x) = 0\ . \tag{0.1}$$

(a) Find a function $q(x)$ so that whenever $y(x)$ is a solution of (0.2), there is a solution $z(x)$ of

$$z''(x) + q(x)z(x) = 0 \tag{0.2}$$

that has the same set of zeros as $y(x)$.

(b) Find a number $L > 0$ so that if $y(x)$ solves (0.1) and satisfies $y(0) = 0$ and $y'(0) = 1$, then for some x_1 with $0 < x_1 < L$, $y(x_1) = 0$. Justify your answer.

3. Find the continuously differentiable curve $y(x)$ such that $y(0) = 1$ and $y(1) = 0$ that minimizes the functional

$$I[y] = \int_0^1 [|y'(x)|^2 + |y(x)|^2]\mathrm{d}x\ .$$

Justify your answer.

4. Let

$$\mathcal{L}u = (xu')' - \frac{1}{x}u\ .$$

(a) Show that for all $0 < a < b < \infty$, there is no solution of

$$\mathcal{L}u = 0 \qquad \text{with} \qquad u(a) = u(b) = 0\ .$$

(b) Let $a = 1$ and $b = 2$. Find the Green's function $G(x, y)$ such that the solution of $\mathcal{L}u = f$ with $u(1) = u(2) = 0$ is given by

$$u(x) = \int_a^b G(x, y) f(y) \mathrm{d}y$$

for all continuous f on $[a, b]$.

(c) Find the function $u(x)$ on $[1, 2]$ such that

$$\mathcal{L}u = x^2$$

with $u(1) = 1$ and $u(2) = 2$.