Practice Final Exam, Math 292, 2014

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1. Find the general solution of

$$t^3x'(t) + t^2x(t) - x^2(t) = 2t^4 .$$

2. Cinside the two equations

$$\textbf{I} \quad (y')^2 + y^2 = 1 \qquad \text{and} \qquad \textbf{II} \quad (y')^2 - y^2 = 1 .$$

One has a unique solution with $y(t) = y_0$, and the other has infinitely many such solutions. Which one is which? Justify your answer.

3. Let $\mathbf{v}(x, y) = (xy + 12, x^2 + y^2 - 25)$. Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are Lyapunov stable, assymptoticaly stable, or unstable whenver this can be deremined by linearization. Justify your answer.

4. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ Let $\mathbf{f}(t) = (1, t)$. Find the solutions of

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

with $\mathbf{x}(0) = (1, 1)$.

5. The differential equation

$$t2x''(t) + (2 + 2t)x'(t) + 2x(t) = 0$$

has polynomial coefficients.

(a) Find one polynomial solution to this equation.

(b) Find the general solution of this equation.

(c) Find the general solution of

$$t2x''(t) + (2 + 2t)x'(t) + 2x(t) = 8e^{2t} .$$

6. Consider the equation

$$y''(x) + 3xy'(x) + x^2y(x) = 0 . \qquad (0.1)$$

(a) Find a function $V(x)$ so that whenever $y(x)$ is a solution of (0.2), there is a solution $z(x)$ of

$$z''(x) + V(x)z(x) = 0 \tag{0.2}$$

that has the same set of zeros as $y(x)$.

(b) Are there any non-trivial solutions of $y''(x) + 3xy'(x) + x^2y(x) = 0$ that have more than one zero? Justify your answer.

7. Find the continuously differentiable curve $y(x)$ such that $y(0) = 1$ and $y(1) = 0$ that minimizes the functional

$$I[y] = \int_0^1 (y')^2 \mathrm{d}x$$

subject to the constraint $\int_0^1 y^2 = 1$. Justify your answer.

8. Let

$$\mathcal{L}u = (x^{-3}u')' + 4x^{-5}u \ .$$

(a) Show that for all $0 < a < b < \infty$, there is no solution of

$$\mathcal{L}u = 0 \qquad \text{with} \qquad u(a) = u(b) = 0 \ .$$

(b) Let $a = 1$ and $b = 2$. Find the Green's function $G(x, y)$ such that the solution of $\mathcal{L}u = f$ with $u(1) = u(2) = 0$ is given by

$$u(x) = \int_a^b G(x, y)f(y)\mathrm{d}y$$

for all continuous f on $[a, b]$.

(c) Find the function $u(x)$ on $[1, 2]$ such that

$$\mathcal{L}u = x^2$$

with $u(1) = 1$ and $u(2) = 2$.