Solutions for Homework Assignment 7, Math 292, Spring 2014

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1. Let

$$I[y] = \int_1^2 \frac{\sqrt{1+(y')^2}}{y} \mathrm{d}x$$

Consider the problem of minimizing $I[y]$ subject to $y(1) = a$ and $y(2) = b$. Find the corresponding solution, or solutions, of the Euler-Lagrange equation.

SOLUTION This is a functional in which x is not present, the Euler-Lagrange equation integrates to

$$f - y'\left(\frac{\partial f}{\partial y'}\right) = c \ .$$

In the present case, this works out to be

$$\frac{\sqrt{1+(y')^2}}{y} - \frac{(y')^2}{y\sqrt{1+(y')^2}} = c \ ,$$

which simplifies to

$$y\sqrt{1+(y')^2} = C \tag{0.1}$$

where $C = 1/c$. Solving for y', we find

$$y' = \pm\frac{\sqrt{C^2 - y^2}}{y} \ .$$

Introducing $z = C^2 - y^2$, $z' = -2yy'$, so our equation is $z' = \pm\frac{1}{2}\sqrt{z}$, or equivalently,

$$(\sqrt{z})' = \pm 1 \ ,$$

so that $z(x) = (B \pm x)^2$. Since B is an arbitrary constant, we can write this as

$$z(x) = (x - B)^2 \ .$$

Recalling that $z = C^2 - y^2$, we have

$$(x - B)^2 + y^2 = C^2 \ .$$

That is, the graph of $y(x)$ is an arc of a some circle centered somewhere on the x-axis.

2. Let

$$I[y] = \int_0^4 [xy' - (y')^2]\mathrm{d}x$$

Consider the problem of maximizing $I[y]$ subject to $y(0) = 0$ and $y(4) = 3$. Find the corresponding solution, or solutions, of the Euler-Lagrange equation. Also, explain why no minimum exists.

SOLUTION Since y is not present, the the Euler-Lagrange equation integrates to

$$\frac{\partial f}{\partial y'} = c \ ,$$

which is

$$y' = \frac{1}{2}x + C$$

for an arbitrary constant C. Integrating, we find

$$y(x) = \frac{1}{4}x^2 + Cx + B \ ,$$

where B is another arbitrary constant. To match our boundary conditions, we must solve $y(0) = B = 0$ and $y(4) = 4 + 4C = 3$, so that $C = -\frac{1}{4}$. Thus, the unique solution is

$$y(x) = \frac{1}{4}(x^2 - x) \ .$$

Since we found a unique solution, it cannot be the case that the functional has only a maximum and a minimum: The functional is not constant, so they would have to be different, and so if both existed, there would have to be two solutions of the Euler-Lagrange equation, since when minima or maxima exist, they must satisfy the Euler-Lagrange equation. So at least one is missing.

Here is one way to see that a minimum cannot exist, which answers the question. Using a zig-zag curve you can create a piecewise linear function satisfying the boundary conditions wholes slope is $\pm s$ at every point where the derivative is defined; i.e., on all of the pieces. Then $\int_0^4 (y')^2\mathrm{d}x = 4s^2$. On the other hand, integrating by parts,

$$\int_0^4 xy'\mathrm{d}x = 4y(4) - \int_0^4 y\mathrm{d}x = 12 - \int_0^4 y\mathrm{d}x \ .$$

since we may arrange that $y(x) > 0$ for all x, we have

$$I[y] \leq 12 - 4s^2 \ .$$

Since s is arbitrary, there is no minimum.

Here is a second way. This is better, since we shall show that the solution we found above *is a maximum*, and so there is no minimum, by what we have said above.

To see this,complete the square

$$xy' - (y')^2 = \frac{1}{4}x^2 - \left(\frac{1}{2}x - y'\right)^2 \ .$$

Therefore, for any admissible y,

$$I[y] = \int_0^4 \frac{1}{4}x^2 \mathrm{d}x - \int_0^4 \left(\frac{1}{2}x - y'\right)^2 \mathrm{d}x = \frac{16}{3} - \int_0^4 \left(\frac{1}{2}x - y'\right)^2 \mathrm{d}x \ .$$

The second term on the right is zero if and only if $y' = 1/2$, and otherwise it is strictly negative. Since we can match the boundary conditions with such a function, it is our unique maximum. In fact, we see that the maximum value of this functional is independent of the boundary conditions; it is always 16/3.

3. Consider the problem of finding a curve $y(x)$ with $y(0) = 1, y(1) = 0$, and such that

$$\int_0^1 \sqrt{1 + (y')^2} \mathrm{d}x = L$$

with L given, and such that the area under the curve and above the x-axis is minimal. For which values of L does such a curve exist, and what is it?

SOLUTION By the method of Lagrange multipliers, e must seek the solutions of the Euler-Lagrange equation for

$$i[y] = \int_0^1 [y + \lambda\sqrt{1 + (y')^2}] \mathrm{d}x \ .$$

Since x is not present, the the Euler-Lagrange equation integrates to

$$y + \lambda\sqrt{1 + (y')^2} - \lambda \frac{(y')^2}{\sqrt{1 + (y')^2}} = c \ ,$$

which becomes

$$(c - y)\sqrt{1 + (y')^2} = \lambda \ .$$

This is easily reduced to the equation we studied in Exercise 1: Replace y by $y - c$, which does not change y', and C by λ. We see that the solution is

$$(x - B)^2 + (y - C)^2 = \lambda^2 \ .$$

The solution is a circle. since the circle is to pass through the points $(0, 1)$ and $(1, 0)$, the center must be of the form (s, s) for some $s \geq 1$. For $s = 1$, the graph is a quarter-circle of radius 1, whose arc-length is $\pi/2$. As $s \to \infty$, the arc tends to the straight line segment joining $(0, 1)$ and $(1, 0)$ which has length $\sqrt{2}$. So we have solutions of the Euler-Largange equation whenever

$$\sqrt{2} \leq L \leq \frac{\pi}{4} \ .$$

For $2 > L > \pi/4$. there is no solutions of the Euler-Lagrange equation. What the optimal curve wants to do is to stick to the line segments between $(0, 1)$ and $(0, 0)$, and between $(0, 0)$ and $(0, 1)$, and then using an arc of a circle in the corner. For $L > 0$, one can make he area arbitrarily small, but there is no minimizing curve.

4. Find the Green's function for

$$\mathcal{L}u = ((1 + x)^2 u')' - u$$

subject to $u(0) = u(1) = 0$, and solve

$$((1+x)^2 u')' - u = e^x$$

SOLUTION We look for solutions of the form $u = (1+x)^\alpha$. For this u, we find

$$\mathcal{L} = [\alpha(\alpha+1) - 1](1+x)^\alpha = [(\alpha + 1/2) - 5/4](1+x)^\alpha \ .$$

Thus we have the solutions

$$u(x) = (1+x)^{1/2 \pm \sqrt{5}/2} \ .$$

We take u_1 and u_2 to be linear combinations that are zero at $x = 0$ and $x = 1$ respectively. These are

$$u_1(x) = (1+x)^{1/2+\sqrt{5}/2} - (1+x)^{1/2-\sqrt{5}/2}$$

and

$$u_2(x) = (1+x)^{1/2+\sqrt{5}/2} - 2^{\sqrt{5}}(1+x)^{1/2-\sqrt{5}/2} \ .$$

Our next task is to compute the constant

$$C = [u_1 u_2' - u_2 u_1']p \ .$$

we can do this at any convenient $x \in [0,1]$ since we know it is constant. Since $p(0) = 1$ and $u_1(0) = 0$, we choose $x = 0$. We get

$$C = -u_2(0)u_1'(0) = -(1 - 2^{\sqrt{5}})\sqrt{5} \ .$$

Next, since $u_1(0) = u_2(1) = 0$,

$$D = u_1(0)u_2(1) - u_1(1)u_2(0) = -u_1(1)u_2(0) \ ,$$

and we do not need to compute this since it will cancel out below.

Then, since $u_1(0) = u_2(1) = 0$, we have

$$G(x,y) = \frac{1}{CD} u_2(0)u_1(1)u_1(x)u_2(y) = -\frac{1}{C} u_1(x)u_2(y) \qquad \text{for } y \geq x \ ,$$

and we interchange x and y for $y \leq x$.

Thus,

$$G(x,y) = \frac{1}{(1-2^{\sqrt{5}})\sqrt{5}} [(1+x)^{1/2+\sqrt{5}/2} - (1+x)^{1/2-\sqrt{5}/2}][(1+y)^{1/2+\sqrt{5}/2} - 2^{\sqrt{5}}(1+y)^{1/2-\sqrt{5}/2}]$$

for $y \geq x$, and

$$G(x,y) = \frac{1}{(1-2^{\sqrt{5}})\sqrt{5}} [(1+y)^{1/2+\sqrt{5}/2} - (1+y)^{1/2-\sqrt{5}/2}][(1+x)^{1/2+\sqrt{5}/2} - 2^{\sqrt{5}}(1+x)^{1/2-\sqrt{5}/2}]$$

for $x \geq y$.

Then the solution of the boundary value problem $\mathcal{L}u = e^x$ with $u(0) = u(1) = 0$ is

$$u(x) = \int_0^1 G(x,y)e^y$$

for the Green's function we have just computed. These integrals cannot be computed in terms of elementary functions, but we have reduces the solutions to explicit one dimensional integrals, and so it is solved for our purposes.

5. Let $L > 0$ and let

$$I[y] = \int_0^L [(y')^2 - y^2 - (\sin x)y]\mathrm{d}x \ .$$

Consider the problem of minimizing $I[y]$ subject to $y(0) = y(L) = 0$. Find the corresponding Euler-Lagrange equation. For which values of L does it have a solution subject to the boundary conditions? What is the greatest lower bound as a function of L, and for which values of L is it a minimum?

SOLUTION The Euler-Largange equations is

$$-y'' - y = \frac{1}{2}\sin x \ .$$

Lets us define $\mathcal{L}y = y'' + y$. Then our equation is

$$\mathcal{L}y = -\frac{1}{2}\sin x \ .$$

The homogenous equation $\mathcal{L}y = 0$ has the solutions

$$y(x) = a\sin x + b\cos x \ .$$

To satisfy $y(0) = y(L) = 0$, we must have $L = k\pi$, $h \in \mathbb{N}$. In particular, when $L = \pi$, there is a non-trivial solution of $\tilde{y}$ of $\mathcal{L}y = 0$, namely $\tilde{y}(x) = \sin x$. Since this is *not* orthogonal to the term of the right hand side of the inhomogeneous equation, we see that the Euler-Lagrange equation has no solution for $L = \pi$.

Since whenever a minimum exists, it must satisfy the Euler-Lagrange equation, we see there is no mimimum for $L = \pi$.

This can now easily be seen directly, now that we know what to look for: Define $y(x) = a\sin x$ for a an arbitrary constant. Then for $L = \pi$, y satisfies the boundary conditions, and a simple computation shows

$$I[y] = -a\int_0^\pi \sin^2 x\mathrm{d}x = -\frac{a\pi}{2} \ .$$

Taking a to infinity, this tends to minus infinity, and there is no minimum.

Next, for general L, the eigenvalues of $\mathcal{L}$ are

$$-\lambda_k = \frac{k^2\pi^2}{L^2} - 1 \ .$$

Let $\{u_k\}$ be the corresponding sequence of orthonormal eigenfunctions. Expanding a general y in its Fourier series, we have

$$y = \sum_{k=1}^\infty a_k u_k \ ,$$

and so

$$\int_0^L y(x)\mathcal{L}y(x)\mathrm{d}x = \langle y, \mathcal{L}y\rangle = \sum_{k=1}^\infty |a_k|^2\lambda_k \ .$$

As long as $L < \pi$, $\lambda_k < \lambda_1 < 0$ for all k, and in this case

$$\langle y, \mathcal{L}y\rangle < \lambda_1 \sum_{k=1}^{\infty} |a_k|^2 = \lambda_1 \|y\|_2^2 \ .$$

But, integrating by parts,

$$\langle y, \mathcal{L}y\rangle = -\int_0^L ((y')^2 - y^2)\mathrm{d}x \ .$$

Altogether, using our computed value for λ_1,

$$\int_0^L ((y')^2 - y^2)\mathrm{d}x \geq \left(1 - \frac{\pi^2}{L^2}\right) \|y\|_2^2$$

for $L < \pi$.

We have proved in class that whenever we have this strict positivity, there is exactly one solution of the Euler-Lagrange equation, and it is a minimum.

On the other hand, if $L > \pi$, the eigenfunction u_1 of $-\mathcal{L}$ is negative. In fact, it is $\pi^2/L^2 - 1$. Then a simple computation using $y = au_1$ for a constant a shows

$$I[au_1] = (\pi^2/L^2 - 1)a^2 + a\int_0^L u_1(x)\sin x \mathrm{d}x \ .$$

Taking $a \to \infty$, we see that there is no minimum.

Altogether, there is a minimum when $L < \pi$, but lot when $L \geq \pi$. The greatest lower bound is $-\infty$ for $L \geq \pi$, and it is given by $I[y_\star]$ for $0 < L < \pi$ where $y_\star$ is the unique solution to $y'' - y = \frac{1}{2}\sin x$.