Solutions for Homework Assignment 6, Math 292, Spring 2014

Eric A. Carlen[1]
Rutgers University

April 27, 2014

1. This problem concerns d'Alembert's formula.

(a) Let $L > 0$. Let $g(x) = x(L - x)$ for $0 \leq x \leq L$. Let g be doubly antisymmetric about $x = 0$ and $x = L$. Show that on $nL \leq x \leq (n+1)L$,

$$g(x) = (-1)^n (x - nL)((n+1)L - x) \ .$$

(b) Let $h(x, t)$ be the solution of

$$\frac{\partial^2}{\partial t^2} h(x,t) = \frac{\partial^2}{\partial x^2} h(x,t)$$

for all $0 < x < 1$ and all $t > 0$ with $h(0, t) = h(1, t) = 0$ (so now we use $L = 1$) and

$$h(x, 0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h(x, 0) = 0$$

for all $0 < x < 1$, with g as in part **(a)**, with $L = 1$. Compute $h(1/4, 3/2)$ and $h(1/2, 3/2)$.

(c) Graph the function $h(x, 3/2)$ on $0 < x < 1$.

SOLUTION Since the extended function agrees with $x(L - x)$ on $[0, L]$, we need only check that the formula defines a doubly antisymmetric function, since every function on $[0, L]$ that is zero at the endpoints hs a unique doubly antisymmetric extension.

So we must check that for all x, $g(-x) = -g(x)$ and $g(L - x) = -g(L + x)$. We may suppose $x \in [nL, (n+1)L]$, in which case $-x \in [(-n-1)L, -nL]$. Using the formula, we find

$$g(-x) = (-1)^{-n-1}(-x-(-n-1)L)((-n)L+x) = -g(x) = (-1)^n(x-nL)((n+1)L-x) = -g(x) \ .$$

The fact that $g(L - x) = -g(L + x)$ is verifed in the same way.

(b) We use

$$h(x, t) = \frac{1}{2}[g(x + t) + g(x - t)] \ .$$

Hence

$$\begin{aligned} h(1/4, 3/2) &= \frac{1}{2}[g(7/4) + g(-5/4)] \\ &= \frac{1}{2}[-(3/4)(1/4) + (3/4)(1/4)] = 0 \end{aligned} \tag{0.1}$$

Likewise,

$$\begin{aligned} h(1/2,3/2) &= \frac{1}{2}[g(2)+g(-1)] \\ &= \frac{1}{2}[0+0]=0 \end{aligned} \tag{0.2}$$

(c) Generalizing from the above, it is easy to see that $h(x,3/2)=0$ for all $x \in [0,1]$, and hence the graph is flat.

2. This problem concerns solution of the wave equation by Fourier exansion.

(a) .Let $g(x)=\sin^3(x)$ and $v(x)=\sin^2(x)$. Find numbers a_1, a_2, a_3 and b_1, b_2 so that

$$g(x) = a_1\sin(x) + a_2\sin(2x) + a_2\sin(3x) \qquad \text{and} \qquad v(x) = \sum_{k=1}^{\infty} b_k \sin(kx)\ .$$

Hint: To expand $g(x)$, you can simply use andle additiona formulas to express $g(x)$ as a linear combinfation of the specified functions. In fact, one of the coefficients will even be zero. To expand $v(x)$, you need to compute the Fourier series by wring

$$v(x) = \sum_{k=1}^{\infty} \langle v, u_k\rangle u_k(x)$$

where $u_k(x) - \sqrt{2/\pi}\sin(kx)$ is orthonormal. Using the angle addition formula

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \text{and} \qquad \sin\theta\cos\phi = \frac{\sin(\theta+\phi)+\sin(\theta-\phi)}{2}$$

you can explicitly compute all of the intergals defining the inner products $\langle v, u_k\rangle$.

(b) Let $h(x,t)$ be the solution of

$$\frac{\partial^2}{\partial t^2}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t)$$

for all $0 < x < \pi$ and all $t > 0$ with $h(0,t) = h(\pi,t) = 0$ and

$$h(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(x,0) = v(x)$$

for all $0 < x < \pi$, with g and v as in part **(a)**. Find $h(x,t)$.

SOLUTION It is easy to expand $\sin^3 x$ as

$$\sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x\ .$$

One way is to use

$$\sin^3 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2$$

and expand. The same pocedure allows us to expeand any odd power of $\sin x$ as a linear combination of the functions $\sin kx$, $k \in \mathbb{N}$, using only odd values of ranging up to the power in question.

The same procedure applied to an even power would yield a linear combination of $\cos kx$, whihc is not what we want. So for even powers, even $\sin^2 x$, we must use a Fourier expansion.

We know we can write

$$\sin^2 x = \sum_{k=1}^{\infty} b_k \sin kx$$

where

$$b_k = \frac{2}{\pi} \int_0^{\pi} \sin^2 x \sin kx \mathrm{d}x \ .$$

Using the trigonometric identities suggested above,

$$\sin^2 x \sin kx = \frac{1}{2} \sin kx - \frac{1}{4} \sin(k+2)x - \frac{1}{4} \sin(k-2)x \ .$$

Integrating we find

$$b_k = \frac{1}{2\pi} \left[2\frac{1}{k} - \frac{1}{k+2} - \frac{1}{k-2} \right] (-2)$$

for k odd and 0 for k even. Simplifying,

$$b_k = \begin{cases} \frac{4}{\pi} \frac{1}{k(k^2-4)} & k \text{ odd} \\ 0 & k \text{ odd} \end{cases} .$$

For part **(b)** we use that fact that

$$h_k(x,t) = a_k \sin(kx) \cos(kt) + \frac{b_k}{k} \sin(kx) \sin(kt)$$

satisfes our wav equation and boundary conditions with

$$h(x,0) = a_k \sin(kx) \qquad \text{and} \qquad \frac{\partial}{\partial t} h(x,0) = b_k \sin(kx) \ .$$

Using our expansin, the solution we seek is

$$\begin{aligned} h(x,t) &= \frac{3}{4} \sin x \cos t - \frac{1}{4} \sin(3x) \cos(3t) \\ &+ \sum_{k \text{ odd}} \frac{4}{\pi} \frac{1}{k(k^2-4)} \frac{1}{k} \sin(kx) \sin(kt) \ . \end{aligned} \tag{0.3}$$

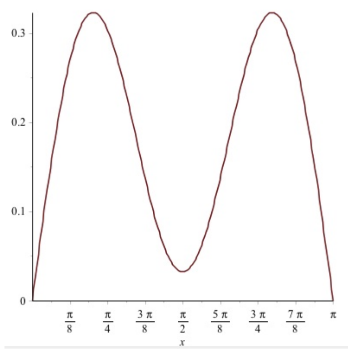
Though our solution is represented as an infinite series, the coefficents decay *very fast*. For example, the $k = 13$ term in the series is

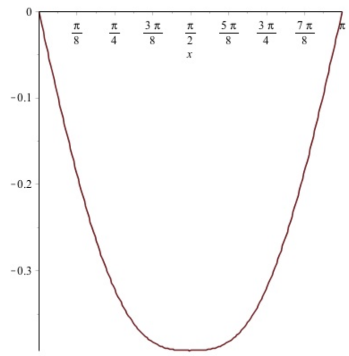
$$\frac{4}{27885} \frac{1}{\pi} \sin(13x) \sin(13t) \approx 4.57 \times 10^{-5} \sin(13x) \sin(13t) \ .$$

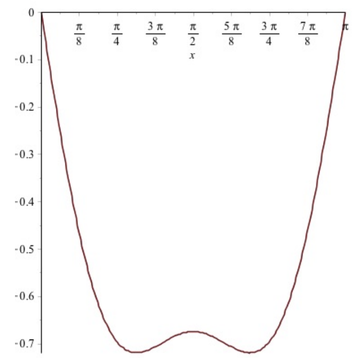
In fact, the kth term is a small multiple of k^{-4} for large k, and $\sum_{k=11}^{\infty} k^{-4} \leq \frac{1}{3} 10^{-3}$ by the integral test. Therefore, truncating after the $k = 10$ term, we would get about 3 decimal places of accuracy, at least. (In fact, this is a worst case analysis; there may be additional cancelation since not all of the rems in the sum will have the same sign.)

For example, suppose we approximate the solution $h(x,t)$ by summing only up to $k = 11$. The first term in the Fourier series that we leave out is

Based on what we have said above, we can expect the graph of the approximation to be indistinguishable from the graph of the actual function on plots of the size we provide beow. Here are plots of the approximate solutions at $t = \pi/4$, $t = \pi/2$ and $t = 3\pi/4$.







By what we have explained above, the eye could not distinguish these approximate plots from plots of the exact solution.

3. Let $k > 0$, and consider the equation

$$u''(x) + \frac{k}{x^2}u(x) = 0 \ .$$

Show that for $k > 1/4$, every non-trivial (i.e., not identically zero) solution has infinitely many zeros, but that for $k \le 1/4$, any such soluiton has only finitely many zeros.

SOLUTION We can solve this equation exactly. Since the coefficients are powers of x, we look for a solution of the form $u(x) = x^\alpha$. For this choice of u, we find

$$u''(x) + \frac{k}{x^2}u(x) = [\alpha(\alpha - 1) + k]x^{\alpha-2} \ .$$

Therefore, we get a solution provided

$$\alpha^2 - \alpha + k = 0 \ .$$

Completing the square, we find that the roots of this quadratic equation are

$$\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{4k-1} \ .$$

For $k \neq 1/4$, we get two indepedent solutions this way. For $k > 1/4$, the general solution is

$$ax^{\frac{1}{2}+\frac{1}{2}\sqrt{4k-1}} + bx^{\frac{1}{2}-\frac{1}{2}\sqrt{4k-1}} = x^{\frac{1}{2}+\frac{1}{2}\sqrt{4k-1}}[a + bx^{-\sqrt{4k-1}}] \ .$$

If either a or b is zero, there is no slution of $u(x) = 0$ (unless both are zer, but this would be the trivial solutions). If neither is zero, and both a and b have the same sign, again there is no solution of $u(x) = 0$. Finally, if they have opposite signs, the unique solution of $u(x) = 0$ is

$$x = -(|b|/|a|)^{1/\sqrt{4k-1}} \ .$$

If $\alpha = 1/4$, we get only one solution from out guess. but we know how to get a second one. Starting from the solution $u_1(x) = \sqrt{x}$, and using the usual formula, we see that

$$u_2(x) = \sqrt{x}\ln x$$

is a second solution. Thus, for $k = 1/4$, the general solution is

$$\sqrt{x}(a + b\ln x) \ .$$

Again, there is at most one solution.

If $k < 1/4$, then $\sqrt{4k-1} = i\sqrt{1-4k}$ so that

$$x^{\frac{1}{2}+\frac{1}{2}\sqrt{4k-1}} = \sqrt{x}e^{i\ln x\sqrt{1-4k}} = \sqrt{x}(\cos(\ln x\sqrt{1-4k}) + i\sin(\ln x\sqrt{1-4k})) \ .$$

The general solution has the form

$$A\sqrt{x}\sin(\ln x\sqrt{1-4k}) + \phi)$$

for constants A and ϕ. Clearly, in this case there are always infinitely many solutions of $u(x) = 0$.

4. Define $\mathcal{L}u(x)$ by

$$\mathcal{L}u(x) = (1+x)^3\left(\frac{u'(x)}{1+x}\right)' \ .$$

(a) Write the equation

$$\mathcal{L}u(x) = \lambda u(x)$$

in the form

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \ .$$

The function $Q(x)$ will depend on λ. Find the gneral solution of this equation for all λ.

(b) Compute the eigenvalues of $\mathcal{L}u(x)$ for Dircihlet boundary conditions on $[0, L]$. That is find all numbers λ so that there exists a solutions of $\mathcal{L}u(x) = \lambda u(x)$ such that $u(0) = u(L) = 0$.

SOLUTION Differentiating, we find

$$\mathcal{L}u(x) = (1+x)^2u''(x) - (1+x)u'(x) \ ,$$

and so the eigenvalue equation bcomes

$$(1+x)^2u''(x) - (1+x)u'(x) = \lambda u(x) \ .$$

We can write this in the form

$$u'' - (1+x)^{-1}u' - (1+x)^{-2}\lambda u = 0 \ ,$$

so that $P(x) = -(1+x)^{-1}$ and $Q(x) = -\lambda(1+x)^{-2}$.

We look for solutios of the form $u(x) = (1+x)^\alpha$. We find

$$(1+x)^2u''(x) - (1+x)u'(x) - \lambda u(x) = [\alpha(\alpha-1) - \alpha - \lambda]x^{\alpha-2} \ .$$

The roots of $\alpha^2 - 2\alpha - \lambda = 0$ are

$$\alpha = 1 \pm \sqrt{1+\lambda}\ .$$

The general solutionwill have more than one zero only in case $\lambda + 1 < 0$, in whihc case the general colution is

$$u(x) = x[a\sin(\sqrt{-\lambda - 1}\ln(1+x)) + b\cos(\sqrt{-\lambda-1}\ln(1+x))]\ .$$

The boundary condition $u(0) = 0$ forces $b = 0$. Then the condition $u(L) = 0$ forces $\sin(\sqrt{-\lambda-1}\ln(1+L)) = 0$ which means that $\sqrt{-\lambda-1}\ln L = k\pi$ for some $k \in \mathbb{N}$.Solving for λ, we find

$$\lambda_k = -\left(\frac{k^2\pi^2}{(\ln(1+L))^2} + 1\right)\ .$$

These are all of the eigenvalues.

5. Find upper and lower bounds on the kth eigenvalue of the problem

$$\frac{1}{(1+x^2)}[[(1+x^2)u'(x)]' - xu(x)] = \lambda u(x)$$

subject to $u(0) = u(1) = 0$ by comparing with two problems with constant coefficients.

SOLUTION Differentiating, we find that

$$u'' + Pu' + Qu = 0$$

where

$$P := \frac{2x}{1+x^2} \qquad \text{and} \qquad Q = -\lambda - \frac{x}{1+x^2}\ .$$

The zeros of every solution iof this equation mathch the zeros of a solution of

$$y'' + Vy = 0$$

where

$$V = Q - \frac{1}{4}P^2 - \frac{1}{2}P' = -\lambda - \frac{x^3+x+1}{(1+x^2)^2}\ .$$

We clearly have

$$-\lambda - 3 \le V(x) \le -\lambda - \frac{1}{4}$$

for all $x \in [0,1]$. (Note: I have simply made the numerator al large as possible, and the denominator as small as possible for the upper bound, and analogously for the lower bound. This is give a valid estimate, but you can compute the exact minimum and maximum in which case you will get slightly sharper bounds. So if you did something sharper at this point, your answer will be slightly different, but that is fine.)

If V is negative everywhere on $[0,1]$, there can be at most one zero, and we cannot satisfy the boundary conditions. So $-\lambda > 1/4$ for all eigenvlaues.

So we define

$$m_\lambda^2 = (-\lambda - 3)_+ \qquad \text{and} \qquad M_\lambda^2 = -\lambda - 1/4\ ,$$

where $(x)_+ = x$ for $x \ge 0$ and $(x)_+ = 0$ for $x \le 0$.

Then, for $\lambda = \lambda_n$, we have

$$-\lambda - 3 \leq n^2\pi^2 \leq -\lambda - 1/4 \ ,$$

so for $n \geq 0$,

$$n^2\pi^2 + \frac{1}{4} \leq \lambda_n \leq n^2\pi^2 + 3 \ .$$