Homework Assignment 6, Math 292, Spring 2014

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1. This problem concerns d'Alembert's formula.

(a) Let $L > 0$. Let $g(x) = x(L-x)$ for $0 \leq x \leq L$. Let g be doubly antisymmetric about $x = 0$ abd $x = L$. Show that on $nL \leq x \leq (n+1)L$,

$$g(x) = (-1)^n (x - nL)((n+1)L - x) \ .$$

(b) Let $h(x,t)$ be the solution of

$$\frac{\partial^2}{\partial t^2} h(x,t) = \frac{\partial^2}{\partial x^2} h(x,t)$$

for all $0 < x < 1$ and all $t > 0$ with $h(0,t) = h(1,t) = 0$ and

$$h(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h(x,0) = 0$$

for all $0 < x < 1$, with g as in part **(a)**. Compute $h(1/4, 3/2)$ and $h(1/2, 3/2)$.

(c) Graph the function $h(x, 3/2)$ on $0 < x < 1$.

2. This problem concerns solution of the wave equation by Fourier exansion.

(a) .Let $g(x) = sin^3(x)$ and $v(x) = sin^2(x)$. Find numbers a_1, a_2, a_3 and b_1, b_2 so that

$$g(x) = a_1 \sin(x) + a_2 \sin(2x) + a_2 \sin(3x) \qquad \text{and} \qquad v(x) = \sum_{k=1}^{\infty} b_k \sin(kx) \ .$$

Hint: To expand $g(x)$, you can simply use andle additiona formulas to express $g(x)$ as a linear combinfation of the specified functions. In fact, one of the coefficients will even be zero. To expand $v(x)$, you need to compute the Fourrier series by wring

$$v(x) = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k(x)$$

where $u_k(x) - \sqrt{2/\pi} \sin(kx)$ is orthonormal. Using the angle addition formula

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \qquad \text{and} \qquad \sin\theta \cos\phi = \frac{\sin(\theta + \phi) - \sin(\theta - \phi)}{2}$$

you can explicitly compute all of the intergals defining the inner products $\langle v, u_k \rangle$.

(b) Let $h(x,t)$ be the solution of

$$\frac{\partial^2}{\partial t^2} h(x,t) = \frac{\partial^2}{\partial x^2} h(x,t)$$

for all $0 < x < \pi$ and all $t > 0$ with $h(0,t) = h(\pi,t) = 0$ and

$$h(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial}{\partial t} h(x,0) = v(x)$$

for all $0 < x < \pi$, with g and v as in part **(a)**.

3. Let $k > 0$, and consider the equation

$$u''(x) + \frac{k}{x^2} u(x) = 0 \ .$$

Show that for $k > 1/4$, every non-trivial (i.e., not identically zero) solution has infinitely many zeros, but that for $k \leq 1/4$, any such soluiton has only finitely many zeros.

4. Define $\mathcal{L}u(x)$ by

$$\mathcal{L}u(x) = (1+x)^3 \left(\frac{u'(x)}{1+x} \right)' \ .$$

(a) Write the equation

$$\mathcal{L}u(x) = \lambda u(x)$$

in the form

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \ .$$

The function $Q(x)$ will depend on λ. Find the gneral solution of this equation for al λ.

(b) Compute the eigenvalues of $\mathcal{L}u(x)$ for Dircihlet boundary conditions on $[0, L]$. That is find all number λ so that there exists a solutions of $\mathcal{L}u(x) = \lambda u(x)$ such that $u(0) = u(L) = 0$.

5. Find upper and lower bounds on the kth eigenvalue of the problem

$$\frac{1}{(1+x^2)} [[(1+x^2)u'(x)]' - xu(x)] = \lambda u(x)$$

subject to $u(0) = u(1) = 0$ by comparing with two problems with constant coefficients.