Homework Assignment 4, Math 292, Spring 2014

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1. (10 points) Let A be the matrix $A = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$.

(a) Compute A^2, A^3 and A^4 Observe the patterns, and deduce a formula for A^k for all positive integers k. (You will probably want to consider even and odd k separately.)

(b) Use the results of part **(a)** to compute e^{tA}.

SOLUTION We compute $A^2 = -\kappa I$. Therefore, $A^3 = -\kappa A$ and $A^4 = \kappa^2 I$. From this it follows that for all integrs $m \geq 0$,

$$A^{2m+1} = (-\kappa)^m A = (-1)^m(\sqrt{\kappa})^{2m} A \qquad \text{and} \qquad A^{2m} = (-\kappa)^m I = (-1)^m(\sqrt{\kappa})^{2m} I \ .$$

Then from $e^{tA} = \sum_{j=0}^{\infty}(t^j/j!)A^j$, we have

$$e^{tA} = \begin{bmatrix} \cos(\sqrt{\kappa}t) & \sqrt{\kappa}^{-1}\sin(\sqrt{\kappa}t) \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}t) & \cos(\sqrt{\kappa}t) \end{bmatrix} .$$

2. (30 points) In this problem we consider driven oscillations with friction taken into account. We will consider a fricative force of the form $-ax'(t)$ where $a > 0$. That is the force is a negative multiple of the velocity. Combining this with the spring force, again assumed to be given by Hooke's Law, we have the Newton equation

$$mx''(t) = -kx(t) - ax'(t) + f(t) \tag{0.1}$$

where m is the mass, k is the spring constant, and $f(t)$ is the driving force.

(a) Introduce $y(t) = x'(t)$, and $\mathbf{x}(t) = (x(t), y(y))$ and $\mathbf{g}(t) = (0, \frac{1}{m}f(t))$. Find a 2×2 matrix B so that (0.1) is equivalent to

$$\mathbf{x}'(t) = B\mathbf{x}(t) + \mathbf{g}(t) \ .$$

(b) Compute e^{tB}. There will be three cases, according to whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ and $(a/m)^2 < 4(k/m)$.

(c) Using Duhamel's formula, find integral formulas for the solution of (0.1). You will need 3 formulas, depending on whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ or $(a/m)^2 < 4(k/m)$.

(d) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 5/4$.

(e) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 1/4$.

SOLUTION Let $\kappa = k/m$ and $\alpha = a/m$. Then the matrix B is Let A be the matrix $B = \begin{bmatrix} 0 & 1 \\ -\kappa & -\alpha \end{bmatrix}$.

The characteristic polynomial of B is $t^2 + t\alpha + \kappa$, and so the eigenvalues are

$$\mu_\pm = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\kappa}}{2} \ .$$

To simplify what follows, introduce

$$\mu = -\frac{\alpha}{2} \qquad \text{and} \qquad \nu = \frac{\sqrt{\alpha^2 - 4\kappa}}{2} \ ,$$

so that the eigenvalues are

$$\mu \pm \nu.$$

One then finds the corresponding eigenvectors to be

$$\mathbf{v}_+ = (1, \mu + \nu) \qquad \text{and} \qquad \mathbf{v}_+ = (1, \mu - \nu) \ .$$

Therefore,

$$\begin{aligned} e^{tB} &= e^{t\mu} \begin{bmatrix} e^{t\nu} & e^{-t\nu} \\ e^{t\nu}(\mu + \nu) & e^{-t\nu}(\mu - \nu) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \mu + \nu & \mu - \nu \end{bmatrix}^{-1} \\ &= \frac{1}{2\nu} e^{t\mu} \begin{bmatrix} e^{t\nu} & e^{-t\nu} \\ e^{t\nu}(\mu + \nu) & e^{-t\nu}(\mu - \nu) \end{bmatrix} \begin{bmatrix} \nu - \mu & 1 \\ \nu + \mu & -1 \end{bmatrix} \\ &= e^{t\mu} \begin{bmatrix} \cosh(t\nu) - \mu \frac{\sinh(t\nu)}{\nu} & \frac{\sinh(t\nu)}{\nu} \\ (\nu^2 - \mu^2)\frac{\sinh(t\nu)}{\nu} & \cosh(t\nu) + \mu \frac{\sinh(t\nu)}{\nu} \end{bmatrix} \ . \end{aligned} \tag{0.2}$$

For $(a/m)^2 > 4(k/m)$, the formula is in final form, since then ν is real. For $(a/m)^2 = 4(k/m)$, $\nu = 0$, and the formula reduces to

$$e^{tB} = e^{t\mu} \begin{bmatrix} \cosh(t\nu) - \mu t & t \\ (\nu^2 - \mu^2)t & \nu \cosh(t\nu) + \mu t \end{bmatrix} \tag{0.3}$$

where we have used

$$\lim_{\nu \to 0} \frac{\sinh(t\nu)}{\nu} = t \ .$$

For $(a/m)^2 < 4(k/m)$, ν is imaginary so we have $\nu = i|\nu|$ in this case, and then since

$$\cosh(i|\nu|t) = \cos(t|\nu|) \qquad \text{and} \qquad \sinh(i|\nu|t) = i \sinh(t|\nu|) \ ,$$

we have

$$e^{tB} = e^{t\mu} \begin{bmatrix} \cos(t\nu) - \mu \frac{\sin(t|\nu|)}{|\nu|} & \frac{\sin(t|\nu|)}{|\nu|} \\ -(\nu^2 + \mu^2)\frac{\sin(t|\nu|)}{|\nu|} & \cos(t|\nu|) + \mu \frac{\sin(t|\nu|)}{|\nu|} \end{bmatrix} \ . \tag{0.4}$$

For part **(c)**, we use

$$\mathbf{x}(t) = e^{tB}\big((x(0), x'(0) + \int_0^t e^{(t-s)B}\mathbf{g}(t)\ ,$$

and so, using (0.2),

$$\begin{aligned} x(t) &= e^{t\mu}\left[\left(\cosh(t\nu) - \mu\frac{\sinh(t\nu)}{\nu}\right)x(0) + \frac{\sinh(t\nu)}{\nu}x'(0)\right] \\ &= \frac{1}{m}\int_0^t e^{\mu(t-s)}\frac{\sinh(\nu(t-s))}{\nu}f(s)\mathrm{d}s\ . \end{aligned}$$

This formula is also valid for $\nu = 0$ and for imaginary ν by taking the limits and using the substitutions described above.

For part **(c)**, when $m = 1$, $a = 1$ and $k = 5/4$, we have $\mu = -1/2$ and $\nu = i$, and then we have

$$\begin{aligned} x(t) &= \int_0^t e^{(s-t)/2}\sin(t-s)\cos(s)\mathrm{d}s \\ &= \frac{1}{17}\left([4 - 4e^{-t/2}]\cos t + [16 - 18e^{-t/2}]\sin t\right)\ . \end{aligned}$$

For part **(c)**, when $m = 1$, $a = 1$ and $k = 1/4$, we have $\mu = -1/2$ and $\nu = 0$, and then we have

$$\begin{aligned} x(t) &= \int_0^t e^{(s-t)/2}(t-s)\cos(s)\mathrm{d}s \\ &= \frac{1}{25}\left([12 - 10t]e^{-t/2} + 16\sin t - 12\cos t\right)\ . \end{aligned}$$

3. (20 points) Consider the vector field

$$\mathbf{v}(x,y) = \big((x+y)(x-y-1), (x+y-2)(x-y+1)\big)\ .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = \big((x+y-2)(x-y+1)\ , (x+y)(x-y-1),\big)\ .$$

SOLUTION For **(a)**, we solve to find two equilibrium points

$$\mathbf{x}_1 = \big(-\tfrac{1}{2}, \tfrac{1}{2}\big) \qquad \text{and} \qquad \mathbf{x}_2 = \big(\tfrac{3}{2}, \tfrac{1}{2}\big)\ .$$

The Jacobian matrix is

$$[D_{\mathbf{v}}(x,y)] = \begin{bmatrix} 2x-1 & -2y-1 \\ 2x-1 & -2y+3 \end{bmatrix}\ .$$

At $\mathbf{x}_1$ we have

$$[D_{\mathbf{v}}(\mathbf{x}_1)] = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}\ .$$

The characteristic polynomial is $t^2 - 8$, so the eigenvalues $\pm 2\sqrt{2}$. There is a positive eigenvalue; this equilibrium point is unstable.

At $\mathbf{x}_2$ we have

$$[D\mathbf{v}(\mathbf{x}_2)] = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

The characteristic polynomial is $t^2 - -4t + 8$, so the eigenvalues $2 \pm 2i$. Both have positive real parts so this equilibrium point is unstable.

For **(b)**, the entries of the vector field have simply been swapped, so the equilibrium points are the same, and the Jacobian matrix is the same as above except the rows are swapped. Thus:

$$[D\mathbf{v}(x, y)] = \begin{bmatrix} 2x - 1 & -2y - 1 \\ 2x - 3 & -2t + 1 \end{bmatrix}.$$

Proceeding in the same way:

At $\mathbf{x}_1$ we have

$$[D\mathbf{v}(\mathbf{x}_1)] = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}.$$

The characteristic polynomial is $t^2 + 4t + 8 = (t + 2)(t + 4)$, so the eigenvalues $-2 \pm 2i$. Both real parts are negative. This equilibrium point is therefore asymptotically stable.

At $\mathbf{x}_2$ we have

$$[D\mathbf{v}(\mathbf{x}_1)] = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}.$$

The characteristic polynomial is $t^2 - 8$, so the eigenvalues $\pm 2\sqrt{2}$. There is a positive eigenvalue; this equilibrium point is unstable.