Homework Assignment 4, Math 292, Spring 2014

Eric A. Carlen[1]
Rutgers University

February 22, 2014

1. (10 points) Let A be the matrix $A = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$.

(a) Compute A^2, A^3 and A^4 Observe the patterns, and deduce a formula for A^k for all positive integers k. (You will probably want to consider even and odd k separately.)

(b) Use the results of part **(a)** to compute e^{tA}.

2. (30 points) In this problem we consider driven oscillations with friction taken into account. We will consider a fricative force of the form $-ax'(t)$ where $a > 0$. That is the force is a negative multiple of the velocity. Combining this with the spring force, again assumed to be given by Hooke's Law, we have the Newton equation

$$mx''(t) = -kx(t) - ax'(t) + f(t) \tag{0.1}$$

where m is the mass, k is the spring constant, and $f(t)$ is the driving force.

(a) Introduce $y(t) = x'(t)$, and $\mathbf{x}(t) = \big(x(t), y(y)\big)$ and $\mathbf{g}(t) = \big(0, \frac{1}{m}f(t)\big)$. Find a 2×2 matrix B so that (0.1) is equivalent to

$$\mathbf{x}'(t) = B\mathbf{x}(t) + \mathbf{g}(t) \ .$$

(b) Compute e^{tB}. There will be three cases, according to whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ and $(a/m)^2 < 4(k/m)$.

(c) Using Duhamel's formula, find integral formulas for the solution of (0.1). You will need 3 formulas, depending on whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ or $(a/m)^2 < 4(k/m)$.

(d) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 5/4$.

(e) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 1/4$.

3. (20 points) Consider the vector field

$$\mathbf{v}(x, y) = \big((x + y)(x - y - 1), (x + y - 2)(x - y + 1)\big) \ .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x, y) = \big((x + y - 2)(x - y + 1) \ , (x + y)(x - y - 1), \big) \ .$$