Homework Assignment 3, Math 292, Spring 2014

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1. Let $\mathbf{v}(x,y)$ be the vector field defined on the right half-plane $U = \{(x,y) \ : \ x > 0\}$ by

$$\mathbf{v}(x,y) = \left(x \ , \ -\frac{1}{x^2} - 2y + x^2y^2\right) \ .$$

The system corresponding to this vector field is recursively coupled since the rate of change of x depends on x alone. This can be used to solve the system, but the system can also be completely decoupled by change of variables. There is a method for finding such a change of variables, but at this point in the course our goal is only to become familiar with how systems of differential equations transform under changes of variables. So we will start with the change of variables as a given.

(a) Define

$$u(x,y) = -\ln x \qquad \text{and} \qquad v(x,y) = x^2 y \ .$$

The transformation $(x,y) \to (u,v)$ invertible transforms the right half-plane onto all of $\mathbb{R}^2$. Compute the inverse transformation.

(b) Suppose that $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)$. Define $\mathbf{u}(t) = (u(\mathbf{x}(t)), v(\mathbf{x}(t))$. Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\mathbf{x}(t)) = \frac{\partial}{\partial x} u(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} u(\mathbf{x}(t)) y'(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} v(\mathbf{x}(t)) = \frac{\partial}{\partial x} v(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} v(\mathbf{x}(t)) y'(t) \ ,$$

find the vector field $\mathbf{w}(u,v)$ on the u,v plane such that

$$\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t)) \ .$$

You should find that this vector field describes a completely decoupled system.

(c) Solve the system $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ by separately solving the decoupled one dimensional equations. Show that the solution of this equation with $\mathbf{u}(0) = (u_o, v_0)$ exists for all t and is unique if and only if $|v_0| \leq 1$.

(d) Use the inverse transformation you found in part **(a)** to express the solution of $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{u}(0) = \mathbf{u}_0 = (u(x_0, y_0), v(x_0, y_0))$ in terms of x and y. Show that the resulting curve $\mathbf{x}(t)$ satisfies $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$.

(e) Show that the solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$ exists for all time and is unique if and only if $|x_0^2 y_0| \leq 1$, and give the solution for all such (x_0, y_0).

(f) Now go back to the original equation and use the fact that $x' = x$ is solved by $x(t) = x_0 e^t$ to convert the equation for y into a Ricatti equation, and solve this. Compare your two solutions.

SOLUTION: For **(a)**, we have $x = e^{-u}$, so that $v = e^{-2u} y$, and hence $y = e^{2u} v$. Thus,

$$x(u, v) = e^{-u} \qquad \text{and} \qquad y(u, v) = e^{2u} v\ .$$

For **(b)**, Since $\nabla u(x, y) = (\,-1/x, 0)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u(\mathbf{x}(t)) &= \frac{\partial}{\partial x} u(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} u(\mathbf{x}(t)) y'(t) \\
&= -1\ .
\end{aligned}$$

Since $\nabla v(x, y) = (2xy, x^2)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} v(\mathbf{x}(t)) &= \frac{\partial}{\partial x} u(\mathbf{x}(t)) x'(t) + \frac{\partial}{\partial y} u(\mathbf{x}(t)) y'(t) \\
&= 2x^x y - 1 - 2x^2 y + x^4 y^2 \\
&= v^2 - 1\ .
\end{aligned}$$

Therefore, defining

$$\mathbf{w}(u, v) = (\,-1, v^2 - 1)\ ,$$

we have that $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$.

(c) The system is completely decoupled, and is easily solved: $u' = 1$ is solved by $u(t) = u_0 - 1$, while to solve $v' = v^2 - 1$, we decompose

$$\frac{1}{v^2 - 1} = -\frac{1}{2}\left(\frac{1}{1+v} + \frac{1}{1-v}\right)\ ,$$

Then by Barrow's formula, taking $|v_0| < 1$, so that $1 + v_0$ and $1 - v_0$ are both positive,

$$t(v) = -\frac{1}{2}\int_{v_0}^{v} \left(\frac{1}{1+z} + \frac{1}{1-z}\right) \mathrm{d}z = \ln \sqrt{\frac{1-z}{1+z}}\Bigg|_{v_0}^{v}\ .$$

Solving for $v(t)$ we find

$$v(t) = \frac{(1 + v_0) - (1 - v_0)e^{2t}}{(1 + v_0) + (1 - v_0)e^{2t}}\ .$$

Clearly, with such initial data, the solution stays in $(-1, 1)$ for all t, as one can also see from the fact that v is Lipschitz on $(-1, 1)$.

On the other hand, for $v_0 > 1$,

$$\int_{v_0}^{\infty} \frac{1}{z^2 - 1} \mathrm{d}z < \infty$$

so the solution reaches ∞ in a finite time. The same reasoning shows that for $v_0 < -1$, the solution reaches $-\infty$ in a finite negative time. So a global solution exists only for $|v_0| < 1$, and is then given by the above formula.

(d) Now using

$$\mathbf{x}(t) = (x(u(t), v(t)), y(t), v(t)) = (e^{-u(t)}, e^{2u(t)} v(t))$$

and the above computation of $u(t)$ and $v(t)$, we get that

$$\mathbf{x}(t) = \left(e^{-u_0} e^t \ , \ e^{2u_0} \frac{(1+v_0)e^{-2t} - (1-v_0)}{(1+v_0) + (1-v_0)e^{2t}} \right)$$

Then since $e^{-u_0} = x_0$ and $v_0 = x_0^2 y_0$, we have, for $|x_0^2 y_0| = |v_0| < 1$,

$$\mathbf{x}(t) = \left(x_0 e^t \ , \ x_0^{-2} \frac{(1+x_0^2 y_0)e^{-2t} - (1-x_0^2 y_0)}{(1+x_0^2 y_0) + (1-x_0^2 y_0)e^{2t}} \right) \ . \tag{0.1}$$

(e) Direct differentiation verifies that $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$, but we also know this by the equivalence we established above. Since the system in the x, y variables is equivalent to the system in u, v variables, and since the latter has a global solution if and only if $|v_0| < 1$, the former has a global solution, given above, if and only if $|x_0^2 y_0| < 1$.

(f) Using $x(t) = x_0 e^t$, we find

$$y' = -\frac{1}{x_0^2} e^{-2t} - 2y + x_0^2 e^{2t} y^2 \ .$$

This is a Ricatti equation. Since the coefficients are multiples of powers of e^t, we try for a solution of the form

$$y_1(t) = C e^{\alpha t} \ .$$

Plugging this in, the powers of e^t match for $\alpha = -2$, and then we are left with

$$-2C = -\frac{1}{x_0^2} - 2C + x_0^2 C^2 \ .$$

From here we see that $C = x_0^{-2}$. Hence

$$y_1(t) = x_0^{-2} e^{-2t}$$

is a solution. Notice that this what one gets from (0.1) with $y_0 = x_0^{-2}$: Our solution must match one of the solutions found above, and it does.

Defining $u = y - y_1$, we see by the standard Ricatti change of variables that

$$u' = x_0^2 e^{2t} u^2 \ .$$

This is a Bernoulli equation, but since there is no linear term in u, it is even simpler than that; it is separable:

$$(u^{-1})' = -x_0^2 e^{2t}$$

So with $z = u^{-1}$,

$$z(t) = z_0 + \frac{1}{2} x_0^2 (1 - e^{2t}) \ .$$

Thus,

$$y(t) = \left(z_0 + \frac{1}{2}x_0^2(1 - e^{2t}) \right)^{-1} + x_0^{-2}e^{-2t} \ .$$

Evaluating at $t = 0$, we find

$$z_0 = \frac{x_0^2}{x_0^2 y_0 - 1}$$

so that finally,

$$y(t) = \left(\frac{x_0^2}{x_0^2 y_0 - 1} + \frac{1}{2}x_0^2(1 - e^{2t}) \right)^{-1} + x_0^{-2}e^{-2t} \ .$$

A bit of algebra shows that this is the same as what we found before, by the other method.

2. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix} .$$

Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}. (Note that this system is recursively coupled.)

SOLUTION Taking advantage of the recursive structure, we find the solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = (x_0, y_0, z_0)$ has

$$z(t) = e^{-2t}z_0 \ ,$$

and then

$$y(t) = e^{-2t}y_0 + 4te^{-2t}z_0 \ ,$$

and finally

$$x(t) = e^{-t}x_0 + (e^{-t} - e^{-2t})z_0 \ .$$

Altogether,

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 0 & e^{-t} - e^{-2t} \\ 0 & e^{-2t} & 4te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix} (x_0, y_0, z_0) \ ,$$

which gives us the matrix exponential.

3. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

SOLUTION The characteristic polynomial is $t^2 + 5t - 6 = (t - 1)(t + 6)$ so the eigenvalues are $\mu_1 = -6$ and $\mu_2 = 1$. The corresponding eigenvectors are $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (2, 5)$. Therefore

$$e^{At} = [e^{-6t}\mathbf{v}_1, e^t\mathbf{v}_2][\mathbf{v}_1, \mathbf{v}_2]^{-1} = \frac{1}{7}\begin{bmatrix} 2e^t + 5e6{-}6t & 2e^t - 2e^{-6t} \\ 5e^t - 5e^{-6t} & 5e^t + 2e^{-6t} \end{bmatrix} .$$

For **(b)** the initial data cannot include any component of the eigenvector with the positive eigenvalue. Thus, $\mathbf{x}_0$ must be a multiple of $\mathbf{v}_1$.

4. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

SOLUTION The characteristic polynomial is $t^2 - 6t + 9 = (t-3)^2$ so the only eigenvalue is $\mu_1 = 3$ and the only eigenvectors eigenvectors are the non-zero multiples of $\mathbf{v}_1 = (1, 2)$. Therefore

$$e^{At} = e^{3t}(I + t(A - 3I)) = e^{3t} \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix} .$$

For **(b)** since all of the eigenvalues are positive, the only such initial data is $\mathbf{x}_0 = \mathbf{0}$.