Homework Assignment 3, Math 292, Spring 2014

Eric A. Carlen[1]
Rutgers University

February 11, 2014

1. Let $\mathbf{v}(x, y)$ be the vector field defined on the right half-plane $U = \{(x, y) \ : \ x > 0\}$ by

$$\mathbf{v}(x,y) = \left(x \ , \ -\frac{1}{x^2} - 2y + x^2y^2\right) \ .$$

The system corresponding to this vector field is recursively coupled since the rate of change of x depends on x alone. This can be used to solve the system, but the system can also be completely decoupled by change of variables. There is a method for finding such a change of variables, but at this point in the course our goal is only to become familiar with how systems of differential equations transform under changes of variables. So we will start with the change of variables as a given.

(a) Define

$$u(x,y) = -\ln x \qquad \text{and} \qquad v(x,y) = x^2 y \ .$$

The transformation $(x, y) \to (u, v)$ invertible transforms the right half-plane onto all of $\mathbb{R}^2$. Compute the inverse transformation.

(b) Suppose that $\mathbf{x}(t)$ solves $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)$. Define $\mathbf{u}(t) = (u(\mathbf{x}(t)), v(\mathbf{x}(t))$. Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\mathbf{x}(t)) = \frac{\partial}{\partial x}u(\mathbf{x}(t))x'(t) + \frac{\partial}{\partial y}u(\mathbf{x}(t))y'(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}v(\mathbf{x}(t)) = \frac{\partial}{\partial x}v(\mathbf{x}(t))x'(t) + \frac{\partial}{\partial y}v(\mathbf{x}(t))y'(t) \ ,$$

find the vector field $\mathbf{w}(u, v)$ on the u, v plane such that

$$\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t)) \ .$$

You should find that this vector field describes a completely decoupled system.

(c) Solve the system $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ by separately solving the decoupled one dimensional equations. Show that the solution of this equation with $\mathbf{u}(0) = (u_o, v_0)$ exists for all t and is unique if and only if $|v_0| \leq 1$.

(d) Use the inverse transformation you found in part **(a)** to express the solution of $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{u}(0) = \mathbf{u}_0 = (u(x_0, y_0), v(x_0, y_0))$ in terms of x and y. Show that the resulting curve $\mathbf{x}(t)$ satisfies $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$.

(e) Show that the solution of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$ with $\mathbf{x}(0) = \mathbf{x}_0$ exists for all time and is unique if and only if $|x_0^2 y_0| \leq 1$, and give the solution for all such (x_0, y_0).

(f) Now go back to the original equation and use the fact that $x' = x$ is solved by $x(t) = x_0 e^t$ to convert the equation for y into a Ricatti equation, and solve this. Compare your two solutions.

2. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix} .$$

Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}. (Note that this system is recursively coupled.)

3. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

4. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.