Solutions for Homework Assignment 2, Math 292, Spring 2014

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1. Let $v(x) = \sin(x)$. For all $0 \le x \le \pi$, Find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?
Hint: It will probably help to recall the identity

$$\frac{1 - \cos x}{\sin x} = \tan(x/2) \ .$$

SOLUTION By Barrow's formula,

$$t(x) - t_0 = \int_{x_0}^{x} \frac{1}{\sin z} \mathrm{d}z = \ln(\csc x - \cot x) - \ln(\csc x_0 - \cot x_0) \ .$$

Since

$$\csc x - \cot x = \frac{1 - \cos x}{\sin x} = \tan(x/2) \ ,$$

this means

$$t(x) = t_0 = \ln(\tan(x/2)) - \ln(\tan(x_0/2)) \ .$$

Therefore,

$$\tan(x/2) = e^{t-t_0} \tan(x_0/2) \ .$$

Finally,

$$x(t) = 2 \arctan(e^{t-t_0} \tan(x_0/2)) \ .$$

The solution is defined for all $T \in \mathbb{R}$, as one can see from the general form, or from the fact that $v(x)$ is Lipschitz on $[0, \pi]$.

2. Let $v(x) = \tan(x)$, which is continuous on $-\pi/2 < x < \pi/2$. For all x_0 in this interval, find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?

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SOLUTION By Barrow's formula,

$$t(x) - t_0 = \int_{x_0}^{x} \frac{\cos z}{\sin z} \mathrm{d}z = \ln(|\sin x|) - \ln(|\sin x_0|) \ .$$

Therefore

$$|\sin x| = |\sin x_0| e^{t-t_0} \ .$$

Since v is Lipschitz around $x = 0$, the solution never crosses $x = 0$, and so $x(t)$ has the same sign as x_0. Therefore,

$$\sin x(t) = \sin x_0 e^{t-t_0} \ .$$

as long as the right hand side stays in the interval $[-1, 1]$, the it remains in the domain of the arcsin function, and so for such t we have

$$x(t) = \arcsin(\sin x_0 e^{t-t_0}) \ .$$

Let T be the solution of $|\sin x_0| e^{t-t_0} = 1$, so that

$$T = t_0 - \ln(|\sin x_0)| \ .$$

This is the time at whihc the solution "blows up". Thus, the solution is defined on

$$(-\infty, t_0 - \ln(|\sin x_0)|) \ .$$

3. For $\alpha > 0$, let

$$v(x) = x|\ln|x||^{\alpha}$$

for $x \neq 0$, and $v(0) = 0$, so that v is continuous on $\mathbb{R}$. The interval $(0, 1)$ is a maximal interval for v since $v(0) = v(1) = 0$ and $v(x) > 0$ on $(0, 1)$.

(a) For all $\alpha > 0$, and all $x_0 \in (0, 1)$, and $t_0 \in \mathbb{R}$, find the solution of $x'(t) = v(t)$ for $x(t_0) = x_0$ for all t for which the solution stays in the interval $(0, 1)$. For which values of α does the solution remain in $(0, 1)$ for all $t > t_0$? For which values of α does the solution remain in $(0, 1)$ for all $t < t_0$?

(b) Note that $x = 0$ and $x = 1$ are both equilibrium points for v (as is $x = -1$). For which values of α is the steady state solution $x(t) = 0$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 0$? For which values of α is the steady state solution $x(t) = 1$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 1$?

(c) For which values of α is v Lipschitz on $(0, 1)$?

SOLUTION Note that for $x \in (0, 1)$, $x|\ln|x|| = x(-\ln x)$, and so $v(x) = x(-\ln x)^{\alpha}$. Therefore, by Barrow's formula,

$$t(x) - t_0 = \int_{x_0}^{x} \frac{1}{x(-\ln x)^{\alpha}} \mathrm{d}z \ .$$

We make the change of variables $u = -\ln x$ so that $\mathrm{d}u = -(1/x)\mathrm{d}x$. Thus,

$$t(x) - t_0 = -\int_{-\ln x_0}^{-\ln x} \frac{1}{u^{\alpha}} \mathrm{d}z \ .$$

In case $\alpha = 1$, this becomes

$$t(x) - t_0 = \ln(-\ln x_0) - \ln(-\ln x) \ .$$

Then

$$\ln(x) = (\ln x_0)e^{t_0 - t} \ ,$$

and so

$$x(t) = x_0^{e^{t_0-t}} \ .$$

Then,

$$\lim_{t\to\infty} x(t) = 1 \qquad \text{and} \qquad \lim_{t\to-\infty} x(t) = 0 \ .$$

This solutions exists for all $t \in \mathbb{R}$.

Next consider the case $\alpha > 0$, but $\alpha \neq 1$. In this case,

$$t(x) - t_0 = -\int_{-\ln x_0}^{-\ln x} \frac{1}{u^\alpha}\mathrm{d}z = -\frac{1}{1-\alpha}u^{1-\alpha}\Big|_{-\ln x_0}^{-\ln x} \ .$$

Therefore,

$$(-\ln x)^{1-\alpha} = (-\ln x_0)^{1-\alpha} + (1-\alpha)(t_0 - t) \ .$$

Thus,

$$x(t) = \exp\left[-[(-\ln x_0)^{1-\alpha} + (1-\alpha)(t_0 - t)]^{1/(1-\alpha)}\right] \ .$$

None of these solutions remains in $(0,1)$ for all t. If $\alpha > 1$, then

$$(-\ln x_0)^{1-\alpha} + (1-\alpha)(t_0 - t) = 0$$

for some $t < t_0$, at which time $x(t) = 0$. These solutions do remain in $(0,1)$ for all $t > t_0$.

However, if $\alpha < 1$, then

$$(-\ln x_0)^{1-\alpha} + (1-\alpha)(t_0 - t) = 0$$

for some $t > t_0$, at which time $x(t) = 1$. These solutions do remain in $(0,1)$ for all $t < t_0$.

In summary, for $\alpha = 1$ the solutions remain in $(0,1)$ for all t. For $\alpha > 1$ the solutions remain in $(0,1)$ for all $t > t_0$ but exits through $x = 0$ at some time $t < t_0$. For $\alpha < 1$ the solutions remain in $(0,1)$ for all $t < t_0$ but exits through $x = 1$ at some time $t > t_0$.

Next, the steady state solution $x = 0$ is unique if and only if it takes an infinite time to reach any $x > 0$ starting from $x = 0$. (Note the $v(x) \geq 0$, so the motion, if any, is to the right.)

By Barrow's formula the time to reach $x > 0$ from $x_0 = 0$ is

$$t = \int_0^x \frac{1}{z(-\ln z)^\alpha}\mathrm{d}z = \int_{-\ln x}^{\infty} \frac{1}{u^\alpha}\mathrm{d}u \ .$$

For $x \in (0,1)$, this is finite if and only if $\alpha > 1$. Thus, the steady state solution $x(t) = 0$ for all t is the unique solution of $x'(t) = v(x(t))$ with $x(0) = 0$ if and only if $\alpha > 0$.

Next, since the motion, if any, is to the right, there is a non-constant solution of $x'(t) = v(x(t))$ with $x(0) = 1$ if and only if it is possible to reach some $x > 1$ in a finite time starting from $x = 0$. For any such time x, Barrows formula gives the time to arrive as

$$t = \int_1^x \frac{1}{z(\ln z)^\alpha}\mathrm{d}z = \int_0^{\ln x} \frac{1}{u^\alpha}\mathrm{d}u \ .$$

For $x > 1$, this is finite if and only if $\alpha < 1$. Thus, the steady state solution $x(t) = 0$ for all t is the unique solution of $x'(t) = v(x(t))$ with $x(0) = 1$ if and only if $\alpha < 1$.

Finally, for $x \in (0,1)$,

$$v'(x) = (-\ln x)^\alpha + \alpha(-\ln x)^{\alpha-1} \ .$$

If $\alpha > 1$, then

$$\lim_{x\to 1} v'(t) = \infty$$

and so v is not Lipschitz on $(0,1)$. On the other hand, if $0 < \alpha \leq 1$,

$$\lim_{x\to 0} v'(t) = \infty$$

and so v is not Lipschitz on $(0,1)$.

Thus, despite the fact that v is not Lipschitz for $\alpha = 1$, solutions that start in $(0,1)$ exist and remain there fore all t. This vector field v, at $\alpha = 1$, despite being non-Lipschitz, is well behaved in some ways.

4. Consider the equation

$$x''(t) = F(x(t)) \qquad \text{where} \qquad F(x) = -\frac{\mathrm{d}}{\mathrm{d}x}V(x) \tag{0.1}$$

for some continuously differentiable function V.

(a) Define the function $H(x,y)$ by

$$H(x,y) = \frac{1}{2}y^2 + V(x) \ . \tag{0.2}$$

Show that if $x(t)$ is any solution of (0.1) defined on some open interval containing t_0, then

$$H(x(t), x'(t)) = H(x(t_0), x'(t_0))$$

for all t in the interval. Therefore, to solve (0.1) with $x(t_0) = x_0$ and $x'(t_0) = y_0$, we need only solve

$$x' = \pm\sqrt{2H(v_0, x_0)) - 2V(x)} \ . \tag{0.3}$$

(b) Let $V(x) = \frac{1}{2}x^2$, and take $x_0 = 1$ and $y_0 = 0$. There will be infinitely many solutions of (0.3). Describe all of them (The description will involve arbitrary "rest periods" at equilibrium points.). Of these solutions, how many are twice continuously differentiable?

(c) How many solutions of

$$(x'(t))^2 + (x(t))^4 = 1$$

are there with $x(0) = 1$? How many of these are twice continuously differentiable?

SOLUTION By the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H &= \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}x'' \\
&= \frac{\mathrm{d}}{\mathrm{d}x}V(x)x' + x'x'' \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}x}V(x) + x''\right]x' = 0 \ ,
\end{aligned}$$

where in the last line we used Newton's Second Law.

Next, for part **(b)**, $H(x_0, y_0) = 1/2$. Therefore, (0.3) becomes

$$x' = \pm\sqrt{1-x^2}\ . \tag{0.4}$$

We must have $|x| \le 1$ for all t, and so initially only $x' = -\sqrt{1-x^2}$ is relevant. There are two solutions of

$$x' = \sqrt{1-x62}\ , \quad x(0) = 1\ ,$$

namely the steady -state solution $x(t) = 1$, and $x(t) = \cos t$. There are thus infinitely many solutions $x(t)$. For all $T \ge 0$,

$$x(t) = \begin{cases} 1 & t \le t \\ \cos(t - T_1) & t \ge t \end{cases}$$

is a solution. But then at $t = T_1 + \pi$, $x(t) = -1$, and the solution has reached another equilibrium point. It can "rest" there an arbitrary amount of time T_2, including forever, or, after a finite rest, continue, and so forth. The only solution that is twice continuously differentiable is the solution that never rests. Indeed, during any "rest interval", $x''(t) = 0$. However, $F(\pm 1) \ne 0$ and so solutions with a rest have a jump in the value of the second derivative at the beginning and end of the rest.

Finally, for $(x'(t))^2 + (x(t))^4 = 1$, the situation is very much the same. The equilibrium points $x = \pm 1$ can be reached in a finite time since

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}} \mathrm{d}x < \infty\ .$$

To see this note that

$$1 - x^4 = (1-x^2)(1+x^2)\ .$$

so that for $-1 < x < 1$,

$$1 - x^2 \le 1 - x^4 \le 2(1-x^2)\ .$$

Therefore, for all $-1 < x < 1$,

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{1-x^4}} \le \frac{1}{\sqrt{1-x^2}}\ .$$

This shows that $\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\mathrm{d}x$ is a convergent improper integral if and only if $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ is a convergent improper integral. But by direct computation,

$$i\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \lim_{\epsilon\to 0} \arcsin(x)\Big|_{\epsilon}^{1-\epsilon} = \pi\ .$$

Thus both equilibrium points can be reached in a finite time (at most $\sqrt{2}\pi$), and can be left at any time. Hence there are infinitely many solutions. However, any solution which has a rest has a discontinuous second derivative. To see this, note that $x'' = 0$ during a rest. On the other hand if $x' = \pm\sqrt{1-x^4}$, then

$$x'' = \pm\frac{1}{2}\frac{-4x^3}{\sqrt{1-x^4}}x' = -2x^3\ .$$

so as $x(t)$ approaches $x = \pm 1$, $|x''(t)|$ approaches 2, not 0. Thus, if $x(t)$ takes a rest, the second derivative will be discontinuous.

Since the option to take a rest or not is the only source of non-uniqueness for the first order system, there is a unique twice continuously differentiable solution, the one that never rests.