Homework Assignment 2, Math 292, Spring 2014

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1. Let $v(x) = \sin(x)$. For all $0 \le x \le \pi$, Fnd all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?
Hint: It will probably help to recall the identity

$$\frac{1 - \cos x}{\sin x} = \tan(x/2) \ .$$

2. Let $v(x) = \tan(x)$, which is contnuous on $-\pi/2 < x < \pi/2$. For all x_0 in this interval, find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?

3. For $\alpha > 0$, let

$$v(x) = x|\ln|x||^\alpha$$

for $x \neq 0$, and $v(0) = 0$, so that v is continuous on $\mathbb{R}$. The interval $(0, 1)$ is a maximal interval for v since $v(0) = v(1) = 0$ and $v(x) > 0$ on $(0, 1)$.

(a) For all $\alpha > 0$, and all $x_0 \in (0, 1)$, and $t_0 \in \mathbb{R}$, find the solution of $x'(t) = v(t)$ for $x(t_0) = x_0$ for all t for which the soltution stays in the interval $(0, 1)$. For which values of α does the solution remain in $(0, 1)$ for all $t > t_0$? For which values of α does the solution remain in $(0, 1)$ for all $t < t_0$?

(b) Note that $x = 0$ and $x = 1$ are both equilibrium points for v (as is $x = -1$). For which values of α is the stady state soltution $x(t) = 0$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 0$? For which values of α is the stady state soltution $x(t) = 1$ for all t the only solution of $x'(t) = v(x(t))$ with $x(0) = 1$?

(c) For which values of α is v Lipschitz on $(0, 1)$?

4. Consider the equation

$$x''(t) = F(x(t)) \qquad \text{where} \qquad F(x) = -\frac{\mathrm{d}}{\mathrm{d}x}V(x) \tag{0.1}$$

for some continously differentiable function V.

(a) Define the function $H(x,y)$ by

$$H(x,y) = \frac{1}{2}y^2 + V(x)\ . \tag{0.2}$$

Show that if $x(t)$ is any solution of (0.1) defined on some open interval containing t_0, then

$$H(x'(t), x(t)) = H(x'(t_0), x(t_0))$$

for all t in the interval. Therefore, to solve (0.1) with $x(t_0) = x_0$ and $x'(t_0) = v_0$, we need only solve

$$x' = \pm\sqrt{H(v_0, x_0)) - V(x)}\ . \tag{0.3}$$

(b) Let $V(x) = \frac{1}{2}x^2$, and take $x_0 = 1$ and $v_0 = 0$. There will be infinitely many solutions of (0.3. Describe all of them (The description will involve arbitrary "rest periods" at equilibrium points.). Of these solutions, how many are twice continuously differentiable?

(c) How many solutions of

$$(x'(t))^2 + (x(t))^4 = 1$$

are there with $x(0) = 1$? How many of these are twice continuously differentiable?