Solutions for the Exercises from Chapter 1

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February 10, 2014

1.1 Find the general solution of the differential equation

$$tx' = 3x + t^4$$

for $t > 0$. Find the corresponding flow transformation, and the particular solution with $x(1) = 2$.

SOLUTION: We can put this in the form $x'(t) = p(t)x(t) + q(t)$ with $p(t) = 3/t$, which is the derivative of $P(t) = 3\ln t$. Therefore, we regroup and multiply through by $e^{-3\ln t} = t^{-3}$ to obtain

$$(t^{-3}x' - 3t^{-4}x) = 1$$

which is the same as $(t^{-3}x)' = 1$, and so

$$t^{-3}x = t + C \ ,$$

and thus the general solution is

$$x(t) = t^4 + Ct^3 \ .$$

(You can easily check that this is a solution for each C.)

To find the solution that passes through x_0 at time t_0 we solve $x_0 = x(t_0) = t_0^4 + Ct_0^3$ for C, finding

$$C = x_0 t_0^{-3} - t_0 \ .$$

Inserting this value of C into our general solution, we find that the solution that passes through x_0 at time t_0 is

$$x(t) = t^4 + [x_0 t_0^{-3} - t_0]t^3 \ .$$

Since by definition, $\Phi_{t_1,t_0}(x_0) = x(t_1)$ for this solution,

$$\Phi_{t_1,t_0}(x_0) = t_1^4 + [x_0 t_0^{-3} - t_0]t_1^3 \ .$$

Since this is true for every value of x_0, we can drop the subscript and write

$$\Phi_{t_1,t_0}(x) = t_1^4 + [x t_0^{-3} - t_0]t_1^3 \ ,$$

though this last step is merely cosmetic.

Finally, to get the solution with $x(1) = 2$, we only need to substitute $x_0 = 2$ and $t_0 = 1$ into $x(t)$. We find

$$x(t) = t^4 + t^3 \ .$$

1.2 Find the general solution of the differential equation

$$(1 + t^2)x' + 2tx = \cot t$$

for $0 < t < \pi$. Find the corresponding flow transformation, and the particular solution with $x(\pi/2) = 2$.

SOLUTION: The equation can be rewritten as

$$((1 + t^2)x)' = \cot t = (\ln(\sin t))' \ .$$

Integrating both sides,

$$x(t) = \frac{1}{1 + t^2}(\ln(\sin t) + C) \ .$$

This is the general solution. If $x(t_0) = x_0$, then

$$x_0 = \frac{1}{1 + t_0^2}(\ln(\sin t_0) + C) \ .$$

Solving for C we find

$$C = (1 + t_0^2)x_0 - \ln(\sin t_0 \ .$$

Therefore, the flow transformation $\Phi_{t_1,t_0}(x)$ is

$$\begin{aligned} \Phi_{t_1,t_0}(x) &= \frac{1}{1 + t_1^2}(\ln(\sin t_1) + (1 + t_0^2)x_0 - \ln(\sin t_0)) \\ &= \frac{1}{1 + t_1^2}\left[(1 + t_0^2)x_0 + \ln\left(\frac{\sin t_1}{\sin t_0}\right)\right] \ . \end{aligned}$$

The solutions with $x(\pi/2) = 2$ is

$$x(t) = \Phi_{t,\pi/2}(2) = \frac{1}{1 + t^2}\left[2 + \frac{\pi^2}{2} + \ln(\sin t)\right] \ .$$

1.3 The equation $(e^x - 2tx)x' = x^2$ is not linear, but think of t as a function of x, and recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}t(x) = \frac{1}{x'(t(x))} \ .$$

Use this to rewrite the equation as a linear first order equation for $t(x)$, and solve this.

SOLUTION: Substituting $x' = 1/t'$, our equation becomes

$$(e^x - 2tx)\frac{1}{t'} = x^2 \ .$$

Multiplying both sides by t'/x^2 we obtain

$$t' + \frac{2}{x}t = \frac{1}{x^2}e^x \ .$$

Multiplying both sides through by x^2, we obtain

$$(x^2 t)' = e^x \ ,$$

and so

$$x^2 t(x) = e^x + C \ .$$

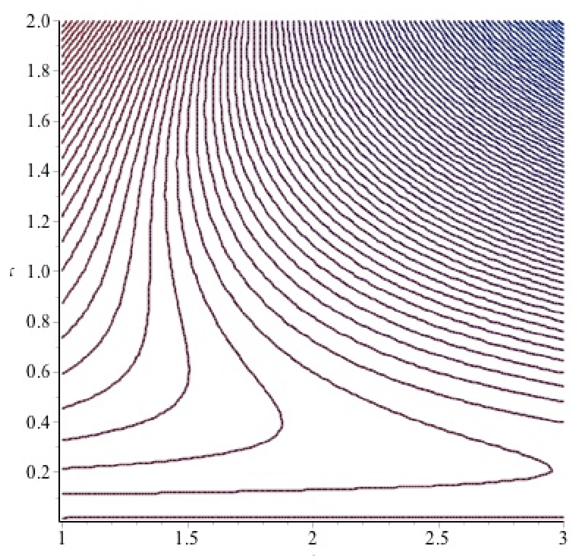
The general solution is

$$t(x) = \frac{e^x + C}{x^2} \ .$$

This function cannot be inverted globally to find $x(t)$ since it is not one to one, as we explain: Perhaps the best description of the solution curves is the implicit form

$$x^2 t - e^x = C \ .$$

Here is a contour plot showing the curves defined by this equation for various values of C in the region $(1,3) \times (0,2)$ in the t, x plane. You can see that the curves have vertical tangent at various points – those with $e^x = 2tx$ – and then the curve "double back" so that in the vicinity of such a point one cannot write x as a function of t. At all other points there is a function $x(t)$ passing through the point that solves the equation on some interval.



1.4 Use the method of the previous exercise to solve $x - tx' = x'x^2e^x$.

SOLUTION Using $x' = 1/t'$, we have

$$x - t\frac{1}{t'} = \frac{1}{t'}x^2 e^x \ .$$

multiplying through by t'/x, we have

$$t'(x) - \frac{1}{x}t(x) = xe^x \ .$$

Multiplying through by $1/x$, we obtain,

$$(t(x)/x))' = e^x \ .$$

Integrating,

$$t(x) = x(e^x + C) \ .$$

1.5 Find the general solution of $tx' + x = t^3x^3$.

SOLUTION This is a Bernoulli equation with $n = 3$. Therefore, the change of variable $z = x^{-2}$ will render it linear. Then with $x = z^{-1/2}$,

$$x' = -\frac{1}{2}z^{-3/2}z'$$

so our equation becomes $-\frac{1}{2}tz^{-3/2}z' + z^{-1/2} = t^3z^{3/2}$, so that

$$z' - 2z = 2t^3 \ .$$

Multiplying through by e^{-2t}, we find

$$(e^{-2t}z)' = 2e^{-2t}t^3 \ .$$

Integrating both sides,

$$e^{-2t}z(t) = -\frac{1}{4}(3 + 6t + 6t^2 + 4t^3)e^{-2t} + C \ .$$

Hence,

$$z(t) = -\frac{1}{4}(3 + 6t + 6t^2 + 4t^3) + Ce^{2t} \ .$$

(This is easily checked, and should be checked now.) Finally,

$$x(t) = \left(Ce^{2t} - \frac{1}{4}(3 + 6t + 6t^2 + 4t^3)\right)^{-1/2} \ .$$

1.6 Find the general solution of $x' = \frac{1}{3}x + e^{-2t}x^{-2}$. Also find the corresponding flow transformation, and the particular solution with $x(0) = 2$.

SOLUTION This is a Bernoulli equation with $n = -2$. Hence we introduce $z = x^3$ so that $x = z^{1/3}$. Then $x' = \frac{1}{3}z^{-2/3}z'$ and our equation becomes

$$\tfrac{1}{3}z^{-2/3}z' = \frac{1}{3}z^{1/3} + e^{-2t}z^{-2/3} \ .$$

Multiplying through by $3z^{2/3}$. we obtain

$$z' - z = 3e^{-2t} \ .$$

Multiplying through by e^{-t} we obtain

$$(e^{-t}z)' = 3e^{-3t} = -(e^{-3t})' \ .$$

Integrating, we find

$$e^{-t}z = C - e^{-3t}$$

so that

$$z(t) = e^t(C - e^{-3t}) = Ce^t - e^{-2t} \ .$$

Finally, $x(t) = z^3(t)$, so the general solution is

$$x(t) = \left(Ce^t - e^{-2t}\right)^{1/3} .$$

If $x(t_0) = x_0$, then

$$C = e^{-t_0}x_0^3 + e^{-3t_0} .$$

Thus, the solution with $x(t_0) = x_0$ is

$$x(t) = \left([e^{-t_0}x_0^3 + e^{-3t_0}]e^t - e^{-2t}\right)^{1/3} .$$

Therefore,

$$\Phi_{t_1,t_0}(x) = \left([e^{-t_0}x^3 + e^{-3t_0}]e^{t_1} - e^{-2t_1}\right)^{1/3} ,$$

and the solution with $x(0) = 2$ is

$$\Phi_{t,0}(2) = \left(9e^t - e^{-2t}\right)^{1/3} .$$

1.7 Find the general solution of $x' + \frac{4}{t}x = t^3x^2$, $t > 0$. Also find the corresponding flow transformation $\Phi_{t_1,t_0}(x)$ for those pairs of t_0 and t_1 for which it is defined, and the particular solution with $x(1) = 2$.

SOLUTION This is a Bernoulli equation with $n = 2$. Hence we introduce $z = x^{-1}$ so that $x = z^{-1}$. Then $x' = -z^{-2}z'$ and our equation becomes

$$z' - \frac{4}{t}z = -t^3 .$$

Multiplying both sides by t^{-4},

$$(t^{-4}z)' = -t^{-1} .$$

Thus, $t^{-4}z = -\ln t + C$, so that

$$z(t) = Ct^4 - t^4\ln t .$$

Finally, the general solution is

$$x(t) = \left(Ct^4 - t^4\ln t\right)^{-1} .$$

To find the solution passing through $x = x_0$ at $t = t_0$, we solve

$$x_0 = \left(Ct_0^4 - t_0^4\ln t_0\right)^{-1} ,$$

to find

$$C = \frac{1}{x_0t_0^4} + \ln(t_0) .$$

Of course this only makes sense if $x_0 \neq 0$. But if $x_0 = 0$, we have the steady state solution $x(t) = 0$ for all t. Otherwise, the solution is given by this formula. In what follows bellow, we suppose that $x_0 \neq 0$ Therefore, the solution passing through $x = x_0$ at $t = t_0$ is

$$x(t) = \left(\frac{t^4}{x_0t_0^4} + t^4(\ln(t_0) - \ln t)\right)^{-1} .$$

The particular solution with $x(1) = 2$ is obtained by setting $t_0 = 1$ and $x_0 = 2$, which gives

$$x(t) = \left(\frac{t^4}{2} - t^4 \ln t\right)^{-1} .$$

The flow transformation $\Phi_{t_1,t_0}(x)$ is the value at $t = t_1$ of the solution that passes through x at time t_0. By the above, this is

$$\Phi_{t_1,t_0}(x) = \left(\frac{t_1^4}{xt_0^4} + t_1^4(\ln(t_0) - \ln t_1)\right)^{-1} .$$

The solutions "blows up" (there is division by zero) at

$$t = t_0 e^{1/(xt_0^4)} .$$

Since we are considering $t > 0$, $t_0 > 0$, and so for $x_0 > 0$, the solution is defined for $t \in (-\infty, t_0 e^{1/(xt_0^4)})$ while for $x_0 < 0$, the solution is defined for $t \in (t_0 e^{1/(xt_0^4)}, \infty)$. Then $\Phi_{t_1,t_0}(x_0)$ is defined exactly when t_1 lies in one of these intervals (depending on the sign of x_0).

1.8 For $0 < c < 1/4$, and $x_0 > 0$, find the solution to

$$x' = x(1-x) - c \ , \quad x(0) = x_0 \ .$$

Show that for all $x_0 \geq \frac{1}{2} - \sqrt{\frac{1}{4} - c}$, the solution exists for all t, and compute $\lim_{t\to\infty} x(t)$ for such x_0. What happens for smaller (positive) values of x_0?

SOLUTION Let $v(x) = x(1-x) - c$. Then $v(x) = 0$ if and only if

$$x^2 - x = c \ ,$$

and the roots of this equation are

$$r_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c} \ .$$

Since $0 < c < 1/4$, both roots are real, and both lie in the interval $(0, 1)$. Then

$$v(x) = -(r_+ - x)(r_- - x)$$

and so

$$\begin{aligned} \frac{1}{v(x)} &= -\frac{1}{(r_+ - x)(r_- - x)} \\ &= \frac{1}{r_+ - r_-}\left(\frac{1}{r_+ - x} - \frac{1}{r_- - x}\right) \\ &= \frac{1}{\sqrt{1-4c}}\left(\frac{1}{r_+ - x} - \frac{1}{r_- - x}\right) . \end{aligned}$$

For $x_0, x < r_-$, both denominators are positive, so we may take their logarithms, and Barrow's formula give us

$$t(x) - t(x_0) = \frac{1}{\sqrt{1-4c}}\left[\ln\left(\frac{r_- - x}{r_+ - x}\right) - \ln\left(\frac{r_- - x_0}{r_+ - x_0}\right)\right] .$$

Therefore,

$$\frac{r_- - x}{r_+ - x} = e^{\sqrt{1-4c}(t-t_0)}\left(\frac{r_- - x_0}{r_+ - x_0}\right) .$$

This may be solved for $x < x_0$ (since $x(t)$ is decreasing for $x_0 < r_-$) only if the right hand side is less than t; i.e., if

$$e^{\sqrt{1-4c}(t-t_0)} < \left(\frac{r_+ - x_0}{r_- - x_0}\right) .$$

Solving for t under this condition, we find

$$x(t) = \frac{e^{\sqrt{1-4c}(t-t_0)}(r_- - x_0)r_+ - (r_+ - x_0)r_-}{e^{\sqrt{1-4c}(t-t_0)}(r_- - x_0) - (r_+ - x_0)} .$$

As t approaches the time defined above, at which time the denominator becomes 0, $x(t)$ approaches $-\infty$.

Next suppose $r_- < x_0 < r_+$. Then we write

$$\frac{1}{v(x)} = \frac{1}{\sqrt{1-4c}}\left(\frac{1}{r_+ - x} + \frac{1}{x - r_-}\right) .$$

This time Barrow's formula yields

$$t(x) - t(x_0) = \frac{1}{\sqrt{1-4c}}\left[\ln\left(\frac{x - r_-}{r_+ - x}\right) - \ln\left(\frac{x_0 - r_-}{r_+ - x_0}\right)\right] .$$

Solving for x we find

$$x(t) = \frac{e^{\sqrt{1-4c}(t-t_0)}(x_0 - r_-)r_+ + (r_+ - x_0)r_-}{e^{\sqrt{1-4c}(t-t_0)}(x_0 - r_-) + (r_+ - x_0)} .$$

In this case, the solution is defined for all t, and $\lim_{t\to\infty} x(t) = r_+$ and $\lim_{t\to\infty} x(t) = r_-$.

Finally, we consider $x > r_+$. Then we write

$$\frac{1}{v(x)} = \frac{1}{\sqrt{1-4c}}\left(-\frac{1}{x - r_+} + \frac{1}{x - r_-}\right) .$$

This time Barrow's formula yields

$$t(x) - t(x_0) = \frac{1}{\sqrt{1-4c}}\left[\ln\left(\frac{x - r_-}{x - r_+}\right) - \ln\left(\frac{x_0 - r_-}{x_0 - r_+}\right)\right] .$$

Solving for x we find

$$x(t) = \frac{e^{\sqrt{1-4c}(t-t_0)}(x_0 - r_-)r_+ - (x_0 - r_+)r_-}{e^{\sqrt{1-4c}(t-t_0)}(x_0 - r_-) - (x_0 - r_+)} .$$

In this case, there is a $t < t_0$ for which the denominator is zero, and $x(t)$ approaches $+\infty$ as t approaches this time, but the solution exists for all $t > t_0$, and $\lim_{t\to\infty} x(t) = r_+$.

1.9 Find the solution of

$$x'(t) = tx\frac{4 - x}{1 + t} \qquad x(0) = x_0 > 0 .$$

Also compute $\lim_{t\to\infty} x(t)$ for each x_0.

SOLUTION Note first the if $x_0 = 4$, then $x(t) = 4$ for all t is a solution, and $\lim_{t\to\infty} x(t) = 4$. In what follows, we assume that $x_0 \neq 4$.

The equation is separable; it can be written as

$$\frac{1}{x(4-x)}x' = \frac{t}{1+t} \ .$$

Integrating both sides,

$$\frac{1}{4}\left(\frac{x}{|4-x|}\right) = t - \ln(1+t) + C \ .$$

Exponentiating,

$$\frac{x}{|4-x|} = e^{4(t+C)}(1+t)^4 \ .$$

Setting $t = 0$, we conclude

$$\frac{x_0}{|4-x_0|} = e^{4C} \ .$$

Thus,

$$\frac{x}{|4-x|} = \frac{x_0}{|4-x_0|}e^{4t}(1+t)^4 \ .$$

We can already compute $\lim_{t\to\infty} x(t)$ without first solving for $x(t)$: Since the right side diverges as $t \to \infty$, so does the left. But this means that $x(t) \to 4$. Hence $\lim_{t\to\infty} x(t) = 4$ for all t.

Finally, some algebra leads to

$$x(t) = \frac{4x_0 e^{4t}(1+t)^4}{4 - x_0 + x_0 e^{4t}(1+t)^4} \ .$$

1.10 Find the general solution of the Ricatti equation

$$x' = -\frac{2}{t}x + t^3x^2 + t^{-5} \ .$$

SOLUTION We try for $y = Ct^\alpha$ since the coefficients are multiples of powers of t. Inserting this into the equation, we see that the powers of t are all equal in case

$$\alpha - 1 = 2\alpha + 3 = -5 \ ,$$

and this requires $\alpha = -4$. With this choice of α, the equation reduces to $-4C = -2C + C^2 + 1$, or $(C+1)^2 = 0$. Hence we must take $C = -1$. Thus we have one solution

$$x_1 = -t^{-4} \ .$$

1.11 Find the general solution of the Ricatti equation

$$x' = \frac{2\cos^2 t - \sin^2 t + x^2}{2\cos t} \ .$$

Then $y = x - x_1$ satisfies the Bernoulli equation

$$y' = -\frac{4}{t}y + t^3y^2 \ .$$

Introducing $z = y/y$, we find
$$z' - \frac{4}{t}z = t^3 \ ,$$
which reduces to
$$\left(\frac{z}{t^4}\right)' = -\frac{1}{t} \ .$$
Integrating,
$$z = t^4(C - \ln t) \ .$$
Finally,
$$x(t) = -t^{-4} + t^{-4}(C - \ln t)^{-1} \ .$$
Notice that the solution $x_1(t)$ is obtained from the general solution in the limit $C \to \infty$.

SOLUTION By inspection, trying powers of $\sin t$ and $\cos t$, we find that $x_1 = \sin t$ is a solution. Then the general solution is
$$x = \sin t + u$$
where u solves the Bernoulli equation
$$u' - \tan t u = \frac{1}{2}\sec t u^2 \ .$$
This gets us to a Bernoulli equation with $n = 2$. Making the change of variables $z = 1/u$, we convert to the linear equation
$$z' + \tan t z = \frac{1}{2}\sec t \ .$$
We multiply through by $e^{-\ln cos(t)} = \sec t$, and obtain
$$(\sec t z)' = \frac{1}{2}\sec^2 t \ .$$
Integrating both sides,
$$\sec t z = -\frac{1}{2}\tan t + C \ ,$$
so that
$$z = \frac{1}{2}(-\sin t + C\cos t) \ .$$
Then
$$u = z^{-1} = \frac{2}{C\cos t - \sin t} \ .$$
Therefore, the general solution of our Ricatti equation is
$$x(t) = \sin t + \frac{2}{C\cos t - \sin t} \ .$$

1.12 Find the general solution of the equation $xx'' + (x')^2 = 0$.

SOLUTION The independent variable t is not present, so we introduce $y = x$, and regard y as a function of x so that
$$x''(t) = \frac{\mathrm{d}y}{\mathrm{d}x}x' = y\frac{\mathrm{d}y}{\mathrm{d}x} \ .$$

Our equation becomes

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0$$

so that

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \ .$$

But this means that

$$\frac{\mathrm{d}y}{\mathrm{d}x}(xy(x)) = 0 \ ,$$

and so

$$xy(x) = c_1 \ .$$

Therefore, $x(t)x'(t) = c_1$ which is

$$\frac{1}{2}(x^2(t))' = c_1 \ .$$

Therefore

$$x^2(t) = 2(c_1 t + c_2) \ .$$

We may absorb the factor of 2 into the arbitrary constants, an find the general solution is

$$x(t) = \pm\sqrt{c_1 t + c_2} \ .$$

1.13 Find the general solution of the equation $x'' = 1 + (x')^2$.

SOLUTION We introduce $y = x'$, and then our equation becomes

$$\frac{1}{1+y^2}y' = 1 \ .$$

Integrating both sides, we have

$$\arctan(y) = t + C_1 \ .$$

Hence,

$$x'(t) = y(t) = \tan(t + C_1) \ .$$

Integrating once more,

$$x(t) = \frac{1}{2}\ln(1 + n\tan((t + C_1)^2) + C_2 \ .$$

1.14 Find the general solution of the equation $tx'' = x' + (x')^3$.

SOLUTION In this case the dependent variable x is not present. We introduce $y = x'$, still considered as a function of t, and then our equation becomes

$$ty' = y + y^3 \ .$$

This is a Bernoulli equation, but is simple enough to solve more directly: Dividing by t^2 and regrouping, we have

$$\left(\frac{y}{t}\right)' = \frac{y^3}{t^2} = t\left(\frac{y}{t}\right)^3 \ .$$

Introducing $z = y/t$, we have

$$z^{-3}z' = t \ ,$$

and so

$$-\frac{1}{2}z^{-2} = \frac{1}{2}t^2 + c_1 \ .$$

Therefore, with a new c_1,

$$z = \left(c_1 - t^2\right)^{-1/2} \ ,$$

and so

$$x' = y = t\left(c_1 - t^2\right)^{-1/2} \ .$$

Integrating

$$x(t) = -\left(c_1 - t^2\right)^{1/2} + c_2 \ .$$

1.15 Find the general solution of the equation $t^2x'' = 2tx' + (x')^2$.

SOLUTION We introduce $y = x'$, and then our equation becomes

$$t^2y' = 2ty + y^2 \ .$$

This is a Bernoulli equation with $n = 2$. We introduce $z = 1/y$ so $y = 1/z$ and $y' = -z^{-2}z'$. Our equation becomes

$$-t^2z^{-2}z' = 2tz^{-1} + z^{-2} \ ,$$

and hence

$$z' = -\frac{2}{t} - \frac{1}{t^2} \ ,$$

which is $(t^2z) = -1$. Therefore,

$$z = \frac{C_1 - t}{t^2} \quad \text{and hence} \quad x'(t) = y(t) = \frac{t^2}{C_1 - t} \ .$$

Integrating once more,

$$x(t) = -C_1t - \frac{1}{2}t^2 - C_1^2\ln(C_1 - t) + C_2 \ .$$