Challenge Problem Set 5, Math 292 Spring 2014

Eric A. Carlen[1]
Rutgers University

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This challenge problem set concerns finding particular solutions of higher order –third order in this case – solutions of inhomogeneous linear equations.

Consider the third order linear equations

$$yx''(t) + P(t)x''(t) + Q(t)x'(t) + R(t)x(t) = f(t) \tag{0.1}$$

for given continuous functions $P(t)$, $Q(t)$, $R(t)$ and $f(t)$.

(1). Introduce the vector variable $\mathbf{x}(t) = (x(t), x'(t), x''(t))$. Find a 3×3 matrix $A(t)$ so that (0.1) is equivalent to the first order linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + (0, 0, f(t)) \ . \tag{0.2}$$

(2). Now suppose that you can find 3 solutions $x_1(t)$, $x_2(t)$ and $x_3(t)$ of (0.1). Define the matrix

$$M(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \\ x_1'(t) & x_2'(t) & x_3'(t) \\ x_1''(t) & x_2''(t) & x_3''(t) \end{bmatrix} \ .$$

Suppose that for some t, $M(t)$ is invertible. Show that then $M(t)$ is invertible for all t, and that with $[\Phi_{t,s}]$ is the 3×3 matrix defined by

$$[\Phi_{t,s}] = M(t)M^{-1}(s) \ ,$$

The solution of (0.2) with $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = [\Phi_{t,t_0}]\mathbf{x}_0 + \int_{t_0}^t [\Phi_{t,s}](0, 0, f(s))\mathrm{d}s \ .$$

(3). When we apply the result of Exercise **(2)** to solve (0.1), we only need the particular solution

$$\int_{t_0}^t [\Phi_{t,s}](0, 0, f(s))\mathrm{d}s$$

corresponding to $\mathbf{x}_0 = \mathbf{0}$ since we have the general solution to the homogeneous equation at hand already. (It is $ax_1(t) + bx_2(t) + cx_3(t)$ for arbitrary a, b and c.) Since we are only interested in the first component of this solution, and since only the third entry in the inhomogeous term is non-zero,, we only need concern ourselves with the upper-right entry of the matrix $[\Phi_{t,s}]$.

Show that if $K(t,s) = [\Phi_{t,s}]_{1,3}$, then

$$x(t) = \int_{t_0}^{t} K(t,s)f(s)$$

solves (0.1) with $x(t_0) = x'(t_0) = x''(t_0) = 0$.

(4). Consider the equation

$$t^2 x''' + 2tx'' - 4x'(t) + \frac{4}{t}x = 0 \ .$$

Look for solutions of the form $x(t) = t^\alpha$. You will find three of them.

Then use the results derived above to find a function $K(t,s)$ so that

$$x(t) = \int_{t_0}^{t} K(t,s)f(s)\mathrm{d}s$$

solves (0.1) with $x(t_0) = x'(t_0) = x'(t_0) = 0$.

Apply this to find the general solution of

$$t^2 x''' + 2tx'' - 4x'(t) + \frac{4}{t}x = t^4 \ .$$

Finally, find the solution of the boundary value problem

$$t^2 x''' + 2tx'' - 4x'(t) + \frac{4}{t}x = t^4 \quad \text{with} \quad x(1) = 1, x'(1) = 0, x(2) = 2 \ .$$