Challenge Problem Set 4, Math 292 Spring 2014

Eric A. Carlen[1]
Rutgers University

April 23, 2014

This challenge problem set concerns the construction of Green's functions for $\mathcal{L}u = f$ subject to boundary conditions other than $u(a) = u(b) = 0$. As always, we take

$$\mathcal{L}u(x) = (p(x))u'(x))' + q(x)u(x) \ ,$$

and we assume that $p(x)$ and $q(x)$ are continuous on $[a, b]$ and that $p(x)$ is strictly positive on $[a, b]$.

We know that under these conditions, there exist two linearly independent solutions $u_1(x)$ and $u_2(x)$ of $\mathcal{L}u(x) = 0$, and that the general solution is a linear combination of u_1 and u_2. That is, the general solution has the form

$$u(x) = \alpha u_1(x) + \beta u_2(x)$$

for some $\alpha, \beta \in \mathbb{R}$.

Now fix two unit vectors (σ_j, τ_j), $j = 1, 2$, and let us consider the boundary conditions

$$(\sigma_1, \tau_1) \cdot (u(a), u'(a)) = 0 \qquad \text{and} \qquad (\sigma_2, \tau_2) \cdot (u(b), u'(b)) = 0 \ .$$

The boundary conditions we have considered so far were $u(a) = u(b) = 0$ which correspond to the choice

$$(\sigma_1, \tau_1) = (\sigma_2, \tau_2) = (1, 0) \ .$$

To simplify the notation for general boundary conditions, let us define the unit vectors

$$\boldsymbol{\sigma}_a = (\sigma_1, \tau_1) \qquad \text{and} \qquad \boldsymbol{\sigma}_b = (\sigma_2, \tau_2) \ ,$$

and the vector

$$\mathbf{u}(x) = (u(x), u'(x)) \ .$$

Then our boundary conditions can be expressed as

$$\boldsymbol{\sigma}_a \cdot \mathbf{u}(a) = 0 \qquad \text{and} \qquad \boldsymbol{\sigma}_b \cdot \mathbf{u}(b) = 0 \ .$$

1. Let u_1 and u_2 be two linearly independent solutions of $\mathcal{L}u = 0$ on (a, b). Show that there exists a non-trivial solution of $\mathcal{L}u = 0$ such that

$$\boldsymbol{\sigma}_a \cdot \mathbf{u}(a) = 0 \qquad \text{and} \qquad \boldsymbol{\sigma}_b \cdot \mathbf{u}(b) = 0$$

if and only if

$$\begin{bmatrix} \boldsymbol{\sigma}_a \cdot \mathbf{u}_1(a) & \boldsymbol{\sigma}_a \cdot \mathbf{u}_2(a) \\ \boldsymbol{\sigma}_b \cdot \mathbf{u}_1(b) & \boldsymbol{\sigma}_b \cdot \mathbf{u}_2(b) \end{bmatrix}$$

is invertible; i.e., has non-zero determinant.

2. We know from our study of Duhamel's formula that the general solution of $\mathcal{L}u = f$ on (a, b) is

$$u(x) = \alpha u_1(x) + \beta u_2(x) + \int_a^x K(x, y) f(y) \mathrm{d}y$$

where

$$K(x, y) = \frac{1}{C}[u_2(x)u_1(y) - u_1(x)u_2(y)]$$

and

$$C = [u_1(a)u_2'(a) - u_2(a)u_1'(a)]p(a) \ .$$

Then since $K(x, x) = 0$, it follows that

$$\begin{aligned} u'(x) &= \alpha u_1'(x) + \beta u_2'(x) + \int_a^x \frac{\partial}{\partial x} K(x, y) f(y) \mathrm{d}y + K(x, x) f(x) \\ &= \alpha u_1'(x) + \beta u_2'(x) + \int_a^x \frac{\partial}{\partial x} K(x, y) f(y) \mathrm{d}y \end{aligned}$$

Show that there exists a unique solution of $\mathcal{L}u = f$ subject to

$$\boldsymbol{\sigma}_a \cdot \mathbf{u}(a) = 0 \qquad \text{and} \qquad \boldsymbol{\sigma}_b \cdot \mathbf{u}(b) = 0 \tag{0.1}$$

if and only if the matrix

$$\begin{bmatrix} \boldsymbol{\sigma}_a \cdot \mathbf{u}_1(a) & \boldsymbol{\sigma}_a \cdot \mathbf{u}_2(a) \\ \boldsymbol{\sigma}_b \cdot \mathbf{u}_1(b) & \boldsymbol{\sigma}_b \cdot \mathbf{u}_2(b) \end{bmatrix}$$

is invertible; i.e., has non-zero determinant. Supposing this is the case, find the values of α and β

$$u(x) = \alpha u_1(x) + \beta u_2(x) + \int_a^x K(x, y) f(y) \mathrm{d}y \tag{0.2}$$

satisfies (0.1).

3. Find a function $G(x, y)$ such that the (0.2) can be written as

$$u(x) = \int_a^b G(x, y) f(y) \mathrm{d}y \ .$$

This is the *Green's function* for these boundary conditions.

4. Let $a = 0$ and $b = 1$ and let

$$\boldsymbol{\sigma}_0 = (1, 0) \qquad \text{and} \qquad \boldsymbol{\sigma}_1 = (1, 1) \ .$$

Find the Green's function for $\mathcal{L}u = u''$ subject to

$$\boldsymbol{\sigma}_0 \cdot \mathbf{u}(0) = 0 \qquad \text{and} \qquad \boldsymbol{\sigma}_1 \cdot \mathbf{u}(1) = 0 \ .$$

Solve this equation, subject to these boundary conditions for $f(x) = x^2$.