Challenge Problem Set 2, Math 292 Spring 2014

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February 26, 2014

Let $K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$. The object of this challenge problem set is to find and study the solution of

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t) \quad \text{with} \quad \mathbf{x}(0) = (1,2) \quad \text{and} \quad \mathbf{x}'(0) = (1,1) \ , \tag{0.1}$$

and

$$\mathbf{f}(t) = \sum_{j=1}^{4} \mathbf{f}_j(t) \tag{0.2}$$

where

$$\begin{aligned} \mathbf{f}_1(t) &= (1,3)\cos(\omega_1 t) \\ \mathbf{f}_2(t) &= (1,-1)\sin(\omega_2 t) \\ \mathbf{f}_3(t) &= (3,-1)\sin(\omega_3 t) \\ \mathbf{f}_4(t) &= (1,0)\cos(\omega_4 t) \ . \end{aligned}$$

The idea here is to take adavantage of the *superposition principle*. Let $\mathbf{x}_0(t)$ be the solution to the unforced equation

$$\mathbf{x}''(t) = -K\mathbf{x}(t) \quad \text{with} \quad \mathbf{x}(0) = (1,2) \quad \text{and} \quad \mathbf{x}'(0) = (1,1) \ .$$

Then, for $j = 1,2,3,4$, let $\mathbf{x}_j(t)$ be the solutions to

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}_j(t) \quad \text{with} \quad \mathbf{x}(0) = (0,0) \quad \text{and} \quad \mathbf{x}'(0) = (0,0) \ .$$

(Notice the zero initial data.)

Exercise 1: Show that the solution of (0.1) and (0.2) is given by

$$\mathbf{x}(t) = \sum_{j=0}^{4} \mathbf{x}_j(t) \ .$$

Exercise 2: Compute $\mathbf{x}_0(t)$.

Exercise 3: Compute $\mathbf{x}_1(t)$.

Exercise 4: Compute $\mathbf{x}_2(t)$.

Exercise 5: Compute $\mathbf{x}_3(t)$.

Exercise 6: Compute $\mathbf{x}_4(t)$.

Exercise 7: Put the results together, and write down the solutions to (0.1) and (0.2). For which values of ω_1, ω_2, ω_3 and ω_4 does the system exhibit resonance?

Exercise 8: Let $\omega_1 = 1$, $\omega_1 = 1.65$, $\omega_1 = 2.65$ and $\omega_1 = 3.75$. Then one of the five functions that were added up to obtain the solution is much larger (for most t) than the others, and the solution is well approximated by keeping only this main term. What is the main term that gives this approximate solution?