Challenge Problem Set 1, Math 292 Spring 2014

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This challenge problem set concerns the matrix exponential function. For any $n \times n$ matrix A this is defined by

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

where $A^0 = I$ by definition. We have seen that this power series converges, entry by entry, with an infinite radius of convergence.

We have seen that for any $\mathbf{x}_0 \in \mathbb{R}^n$, if we define $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, then $\mathbf{x}(t)$ is the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$.

Conversely, if we can find n solutions $\mathbf{z}_j(t)$ with

$$\mathbf{z}_j'(t) = A\mathbf{z}_j(t) \ , \mathbf{z}_j(0) = \mathbf{v}_j \ ,$$

and if $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ is invertible, then

$$e^{tA} = [\mathbf{x}_1(t), \dots, \mathbf{x}_j(t)][\mathbf{v}_1, \dots, \mathbf{v}_n]^{-1} \ ,$$

even if the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and the solutions $\mathbf{z}_1(t), \dots, \mathbf{z}_n(t)$ are complex.

The point of working with complex solutions, is that there always is suitable set of n complex solutions for which the initial data vectors $\mathbf{v}_j$ are generalized eigenvectors of A. Working with these provides the easiest general way to compute e^{tA}, and to understand its properties. The exercises here are designed to illustrate this.

1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} -2 & 4 & 2 \\ -1 & -1 & 1 \\ -3 & 3 & 3 \end{bmatrix} \ .$$

(a) Show that the only eigenvector of A is $\mu = 0$.

(b) Show that $A^3 = 0$, and therefore that

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 \ .$$

That is the entries of A are quadratic polynomials. Show also that the solution of $\mathbf{x}(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ has a quadratic (at most) polynomial in each entry no mater what $\mathbf{x}_0 \in \mathbb{R}^3$ is.

(c) Let B be any 3×3 matrix. Show that every solution of $\mathbf{x}'(t) = B\mathbf{x}(t)$ is a vector whose entries are quadratic polynomials if and only if 0 is the only eigenvalue of B. Hint: If there is a non-zero eigenvalue, there is a corresponding eigenvector and solution.

2. Let B be the 3×3 matrix

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{bmatrix} .$$

(a) Find all of eigenvalues and eigenvectors of B. Hint: $L\mathbf{e}_2 = -2\mathbf{e}_2$.

(b) Show that there does not exist a set of three linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ each of which is an eigenvector of B.

(c) Find a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of B, and $\mathbf{v}_3$ is a generalized eigenvector of B. Find the solutions $\mathbf{x}_j'(t) = B\mathbf{x}_j(t)$, $\mathbf{x}_j(0) = \mathbf{v}_j$, and use these to explicitly compute the matrix exponential e^{tB}.

3. Let B be the 3×3 matrix from Exercise 2. Consider the unit vector

$$\mathbf{u} = \frac{1}{9}(8, 1, 4) .$$

(a) Compute $\|e^{2B}\mathbf{u}\|$, and show that this is larger than $\|\mathbf{u}\| = 1$. That is, the solution of $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{u}$ satisfies $\|\mathbf{x}(2)\| > \|\mathbf{x}(0)\|$.

(b) Show that for all $\mathbf{x}_0 \in \mathbb{R}^3$, the solution of $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies

$$\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0 .$$

4. Let C be the 3×3 matrix

$$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix} .$$

(a) Find all of eigenvalues and eigenvectors of B. Two of the eigenvalues are complex (in fact, pure imaginary), and so the corresponding eigenvectors will necessarily be complex.

(b) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of eigenvectors corresponding to the three distinct eigenvalues $\{\mu_1, \mu_2, \mu_3\}$ you found in part **(a)**. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Show that V is invertible, and compute the inverse.

(c) Use the formula

$$e^{tC} = [e^{t\mu_1}\mathbf{v}_1, e^{t\mu_2}\mathbf{v}_2, e^{t\mu_3}\mathbf{v}_3]V^{-1}$$

to compute e^{tC}. Use Euler's identity $e^{i\theta} = \cos\theta + i \sin\theta$ to express your answer in *purely real terms.*

(d) Show that for all $\mathbf{x}_0 \in \mathbb{R}^3$,

$$\|e^{tC}\mathbf{x}_0\| = \|\mathbf{x}_0\| .$$

Hint: Let $\mathbf{x}(t) = e^{tC}\mathbf{x}_0$ and compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}(t)\|^2 .$$

The fact that C is antisymmetric makes this easy.