Test 1, Math 292 Spring 2013

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(1) Find the general solution of the equation

$$x'(t) = -\cot(t)x(t) + 2t\csc(t) \ ,$$

and the particular solution with $x(\pi/2) = 1$.

(2.) Consider the two differential equations:

(i) $x'(t) = \sqrt{1 - x(t)} \qquad x(0) = 0$

and

(ii) $y'(t) = (1 - y(t))^2 \qquad y(0) = 0.$

One of the two solutions reaches 1 in a finite time; i.e., there is a finite T so that $x(T) = 1$ or $y(T) = 1$. Which one is it, and what is T.

(3.) Find the general solution of the differential equation

$$tx^2(t)x'(t) + x^3(t) = t\cos t \ .$$

(4.) Consider the vectorfield

$$\mathbf{v}(x,y) = (-y^3 - x, x^3 + y) \ .$$

(a) Find all equilibrium points, and determine which are Lyapunov stable, which are asymptotically stable, and which are unstable.

(b) Find a function $U(x,y)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}U(\mathbf{x}(t)) = 0$$

for all solutions of $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$. Does this equation have periodic solutions?

(5.) Consider the *driving force* $\mathbf{f} = 3\cos(\omega t)(1,2)$, where $\omega > 0$. Find the resonant frequencies of the system governed by

$$M\mathbf{x}'' = -A\mathbf{x} + \mathbf{f} \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{0} \ , \quad \mathbf{x}'(0) = \mathbf{0} \ . \tag{0.1}$$

where

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 12 & 2 \\ 2 & 3 \end{bmatrix} \ .$$

For which values of ω will resonance occur?