Note on the Brachistochrone Problem

Eric A. Carlen[1]
Rutgers University

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1 introduction

Consider two points $\mathbf{p}$ and $\mathbf{q}$ in the x, y plane.

We suppose that the height of $\mathbf{q}$ is lower than that of $\mathbf{p}$, so that the straight line segment from $\mathbf{p}$ to $\mathbf{q}$ "runs downhill" The *brachisochrone problem* is to find the curve C connecting $\mathbf{p}$ to $\mathbf{q}$ such that a bead sliding along the curve, started at rest and acted upon only by gravity and the forces that keep it on C, will arrive at $\mathbf{q}$ in the least time. Note that the forces which keep the bead on the curve C act orthogonally to the direction of motion, and thus do no work on the bead: Only gravity does work on the bead.

It will be convenient to choose our coordinates so that $\mathbf{p} = (0, 0)$, and so that y increases in the *downward* direction instead of the upward direction as usual. We take the x coordinate to be increasing to the right, and, without loss of generality we may assume that $\mathbf{q}$ lies to the right of $\mathbf{p}$. Thus $\mathbf{q} = (a, b)$ with $a, b > 0$.

As the bead slides down the wire, potential energy is converted into kinetic energy. As the curve passes through height y, the potential energy that is converted into kinetic energy is mgy where m is the mass of the bead, and g is the gravitational constant. The equivalent kinetic energy is $mv^2/2$, and so

$$v = \sqrt{2gy} \ .$$

It will be convenient to work in units such that $2g = 1$[1]

Now let suppose that the path is given by the graph of a continuously differentiable function $y(x)$. For small $h > 0$, the time it takes the bead to travel from $(x, y(x))$ to $(x + h, y(x + h))$ is approximately the arc length of this segment divided by the bead's speed at $y(x)$, namely $\sqrt{y(x)}$ (since h is small). This is

$$\frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y(x)}} \ .$$

Adding up increments, we find that the time it takes the bead to travel from $(x_0, y(x_0)$ to $(x_1, y(x_1))$ is

$$\int_{x_0}^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y(x)}} \mathrm{d}x \ . \tag{1.1}$$

[1]Since $g \approx 9.8$ meters per second squared, we can keen length in meters and define a new unit of time to be $\sqrt{4.9}$-seconds. Then our speeds and times refer to these units.

In particular, the time it takes the bead to travel from $\mathbf{p}$ to $\mathbf{q}$ is

$$I[y] := \int_0^a \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y(x)}} \mathrm{d}x \ .$$

2 Determination of what the minimal curve must be if there is one

Let K denote the set of continuously differentiable curves $y(x)$ defined on $[0, a]$ with $y(0) = 0$, $y(a) = b$ and $y(x) > 0$ for all $0 < x < a$. (The last condition is necessary since if $y(x) < 0$ for any $0 < x < a$, the bead will never reach $\mathbf{q}$.) K is the set of admissible curves. Notice that the final condition we have imposed leaves "wriggle room" about any curve in K at any point apart from the endpoints.

Thus, the Euler-Lagrange Theorem applies, and so if we assume that there exists a curve $y_0 \in K$ such that

$$I[y_0] \le I[y] \qquad \text{for all} \quad y \in K \ ,$$

then y_0 is a solution of the Euler-Lagrange equation. In this case, as we have seen in class, the Euler-Lagrange Equation reduces to

$$y[1 + (y')^2] = c \ . \tag{2.1}$$

Also as we have seen, the solution of this equation with $y(0) = 0$ is given in parametric form by

$$x(\theta) = r(\theta - \sin\theta) \qquad \text{and} \qquad y(\theta) = r(1 - \cos\theta) \ , \tag{2.2}$$

where$r = c/2$, with c given in (2.1).

This is the parametric representation of a *cycloid*: Consider a circle of radius $r > 0$, initially with its center at the point $(0, r)$. Mark the point on the cicle that is initially at the origin. Now roll the circle to the right along the x-axis, without any slippage. Then the marked point traces out the cycloid given by the parameterization (2.2). To see this, note that when the circle has rolled through an angle θ, the new point of tangency to the x-axis is $r\theta$ since the circle rolls without slipping. Hence the center of the circle is at $(r\theta, r)$ at this time. However, the marked point has rotated counterclockwise about the circle through an angle θ, so it is now at

$$r(\theta, 1) + r(-\sin\theta, -\cos\theta) \ .$$

That is, for given $r > 0$, the cycloid curve is parameterized by

$$\mathbf{x}(\theta) = (x(\theta), y(\theta)) = r(\theta - \sin\theta, 1 - \cos\theta) \ . \tag{2.3}$$

To match our cycloids, one for each $r > 0$, to our given data, we must choose r and θ so that

$$r(\theta - \sin\theta, 1 - \cos\theta) = \mathbf{q} = (a, b) \ . \tag{2.4}$$

It is geometrically clear that as one increases r from 0 to ∞ the resulting cycloids "sweep" across the positive quadrant, passing through each point exactly once, so that there is exactly one solution to (2.4) for each $a, b > 0$. We shall write $\theta(\mathbf{q})$ and $r(\mathbf{q})$ to denote the unique solution to (2.4) when the endpoint is $\mathbf{q}$.

Moreover, we see that the value of θ depends only on the ratio b/a:

$$\frac{1-\cos\theta}{\theta-\sin\theta} = \frac{b}{a} \ . \tag{2.5}$$

Let us now compute the time the bead takes to travel along the cycloid. Since on our cycloid (2.1) is satisfied, with $c = 2r$, (1.1) tells us that. with $T(x)$ denoting the time of travel to $(x, y(x))$,

$$\frac{\mathrm{d}}{\mathrm{d}x}T(x) = \frac{\sqrt{2r}}{y} \ ,$$

and then since

$$\frac{\mathrm{d}}{\mathrm{d}\theta}x(\theta) = r(1-\cos\theta) \ ,$$

we define $T(\theta) = T(x(\theta))$ and have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}T(\theta) = \sqrt{2r}\frac{r(1-\cos\theta)}{r(1-\cos\theta)} = \sqrt{2r} \ .$$

That is, as the bead slides don that cycloid θ increases at a constant rate, and the time it takes the bead to slide to the point $(x(\theta), y(\theta))$ is

$$T(\theta) = \sqrt{2r}\theta \ . \tag{2.6}$$

To see how well the cycloid does, let us compare it with the straight line segment from $\mathbf{p}$ to $\mathbf{q}$ which is given by

$$y(x) = \frac{b}{a}x \ .$$

Simlpe computations show that the time of travel is

$$2\sqrt{\frac{a^2}{b} + b} \ .$$

To facilitate comparison, we take $b = 1$ and a much larger than 1. In this case, it is clear that we must use a cycloid with $(2\pi r)/a \approx 1$, and $\theta \approx 2\pi$. Hence the time for the cycloid is approximately

$$4\pi^{3/2}\sqrt{a} \ .$$

For large a, where out approximations are valid, this is *much smaller* than $2\sqrt{a^2+1}$, the result with a straight line slope. In fact, notice that the cycloid achieves this short time by dipping *far below* the straight line segment to pick up considerable speed for the long horizontal trip.

3 Proof that the cycloid does actually minimize travel time

We shall now prove a remarkably strong theorem that completely solves the brachistochrone problem. First, we establish some notation.

Let $\mathcal{C}_{\mathbf{p},\mathbf{q}}$ denote the set of piecewise continuously differentiable curves in the upper half plane starting at $\mathbf{p} = (0,0)$ and ending at $\mathbf{q}$ in the open upper half plane. Let $C_\star$ be the curve in $\mathcal{C}_{\mathbf{p},\mathbf{q}}$ that has the parameterization (2.2) with r such that (2.4) is satisfied for some $\theta < 2\pi$

For any curve $C \in \mathcal{C}_{\mathbf{p},\mathbf{q}}$, let $\mathbf{x}(s) = (x(s), y(s))$ be an arc length parameterization of it so that $\mathbf{x}(s_1) = \mathbf{q}$. Then, by what we have explained in the first section,

$$T(C) = \int_0^{s_1} \frac{1}{\sqrt{y(s)}} \mathrm{d}s$$

is the travel time along the curve on this route from $\mathbf{p}$ to $\mathbf{q}$. By (2.6),

$$T(C_\star) = \sqrt{2r(\mathbf{q})}\theta(\mathbf{q}) \ . \tag{3.1}$$

3.1 THEOREM. *With $\mathbf{p}$, $\mathbf{q}$, $\mathcal{C}_{\mathbf{p},\mathbf{q}}$ and $C_\star$ as above,*

$$T(C) \geq T(C_\star)$$

for all $C \in \mathcal{C}_{\mathbf{p},\mathbf{q}}$, and there is equality if and only if $C = C_\star$.

Notice that in Theorem 3.1, we no longer assume that that the curve is the graph of a function $y(x)$. Instead, we consider all piecewise continuously differentiable parameterized curves $\mathbf{x}(s)$ that joined $\mathbf{p}$ to $\mathbf{q}$.

The proof of the theorem makes use of a set of coordinates for the half-plane $\{(x, y) \ : \ y > 0\}$ that is based on the cycloid. Without the formal deduction of the cycloid as the only candidate for a time-minimizing curve, it is hard to see how one might discover the relevance of this coordinate system.

In what follows, fix $r = r(\mathbf{q})$, so that we are building our coordinates using the cycloid that the Euler-Lagrange equation suggests is optimal. Let $\mathbf{x}(\theta)$ be given by (2.3). Let

$$\mathbf{z}(\theta) = r(\theta, 0) \ .$$

be the point of tangency of the rolling circle on the x-axis.

Consider the line in the plane through $\mathbf{x}(\theta)$ and $\mathbf{z}(\theta)$ that is parameterized by

$$(1 - u)\mathbf{x}(\theta) + u\mathbf{z}(\theta) \ .$$

For each θ, this line crosses the x-axis at $u = 1$, and it is easy to see that the lines "fan out" across the upper half plane, never intersecting there.

Thus we may use this one parameter family of lines to introduce a coordinate system on the upper half plane: Define

$$\begin{aligned} (x(\theta, u), y(\theta, u)) &= (1 - u)\mathbf{x}(\theta) + u\mathbf{z}(\theta) \\ &= r(\theta + (u - 1)\sin\theta, (1 - u)(1 - \cos\theta)) \ . \end{aligned} \tag{3.2}$$

Because of what we have explained above, as θ ranges over $(0, 2\pi)$ and u ranges over $(-\infty, 1)$, $(x(\theta, u), y(\theta, u))$ ranges over the upper half plane in a one-to-one manner.

That is

$$(\theta, u) \mapsto r(\theta + (u - 1)\sin\theta, 1 - u + (u - 1)\cos\theta)$$

is a one-to-one continuously differentiable transformation of $(0, 2\pi) \times (-\infty, 1)$ onto the upper half plane. Let $(\theta(x, y), u(x, y))$ be the inverse transformation.

Now, note that since $\mathbf{q}$ is on $C_\star$, the $\theta(\mathbf{q})$ in (3.1) is exactly the θ coordinate of $\mathbf{q}$, and of course $\theta(\mathbf{p}) = 0$.

Therefore

$$T(C_\star) = \sqrt{2r}(\theta(\mathbf{q}) - \theta(\mathbf{p})) \ . \tag{3.3}$$

To compare this to any other curve $C \in \mathcal{C}_{\mathbf{p},\mathbf{q}}$, we write this difference as a line integral:

$$\theta(\mathbf{q}) - \theta(\mathbf{p}) = \int_{C_\star} \nabla\theta(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} \ . \tag{3.4}$$

However, since our line integral is of a gradient vector field, the same identity holds for any other $C \in \mathcal{C}_{\mathbf{p},bq}$:

$$\theta(\mathbf{q}) - \theta(\mathbf{p}) = \int_{C} \nabla\theta(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} \ . \tag{3.5}$$

Next, by the Cauchy-Schwarz inequality, if $\mathbf{x}(s)$, $0 \le s \le s_1$ is the arc length parameterization of C,

$$\int_C \nabla\theta(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_0^{s_1} \nabla\theta(\mathbf{x}(s)) \cdot \mathbf{T}(s)\mathrm{d}s \le \int_0^{s_1} \|\nabla\theta(\mathbf{x}(s))\|\mathrm{d}s \ . \tag{3.6}$$

Combing (3.3) through (3.6), we obtain

$$T(C_\star) \le \sqrt{2r}\int_0^{s_1} \|\nabla\theta(\mathbf{x}(s))\|\mathrm{d}s \ . \tag{3.7}$$

The key to the proof of Theorem 3.1 is the following lemma:

3.2 LEMMA. *For all (x, y) in the upper half plane,*

$$\|\nabla\theta(x, y)\| \le \frac{1}{\sqrt{2ry}} \ ,$$

and there is equality if and only if $u(x, y) = 0$, which is the case if and only if (x, y) lies on the curve $C_\star$.

Let us assume the lemma is true for the moment, and use it to complete the proof of Theorem 3.1:

Proof of Theorem 3.1. From (3.7) we have

$$T(C_\star) \le \int_0^{s_1} \frac{1}{\sqrt{y}(s)}\mathrm{d}s = T(C) \ ,$$

and there is equality if and only if

$$\|\nabla\theta(x(s), y(s))\| = \frac{1}{\sqrt{2ry(s)}}$$

for each $0 < s < s_1$, and by the lemma, this is the case if and only if $u(x(s), y(s)) = 0$ for each such s, meaning that $\mathbf{x}(s)$ is on $C_\star$ for all s, in which case $C = C_\star$. □

We now turn to the proof of Lemma 3.2.

Proof of Lemma 3.2. This would be easy if we could compute an explicit formula for $\theta(x, y)$, but this is not possible in a form that is sufficiently explicit to be useful. However, a good way to compute $\nabla\theta(x, y)$ is to note that the Jacobian matrix $\left[\frac{\partial(\theta, u)}{\partial(x, y)}\right]$ is given by

$$\left[\frac{\partial(\theta, u)}{\partial(x, y)}\right] = \begin{bmatrix} \partial\theta/\partial x & \partial\theta/\partial y \\ \partial u/\partial x & \partial u/\partial y \end{bmatrix} ,$$

so that $\nabla\theta(x, y)$ is the first row of this matrix. Then since

$$\left[\frac{\partial(\theta, u)}{\partial(x, y)}\right] = \left[\frac{\partial(x, y)}{\partial(\theta, u)}\right]^{-1} = \begin{bmatrix} \partial x/\partial\theta & \partial x/\partial u \\ \partial y/\partial\theta & \partial y/\partial u \end{bmatrix}^{-1} .$$

From the definition (3.2), we readily compute

$$\left[\frac{\partial(x, y)}{\partial(\theta, u)}\right]^{-1} = \begin{bmatrix} \partial x/\partial\theta & \partial x/\partial u \\ \partial y/\partial\theta & \partial y/\partial u \end{bmatrix} = r \begin{bmatrix} 1 + (u-1)\cos\theta & \sin\theta \\ (1-u)\sin\theta & 1 - \cos\theta \end{bmatrix} .$$

Computing the first row of the inverse, we find

$$\nabla\theta(x(\theta, u), y(\theta, u)) = \frac{1}{r(2-u)(1-\cos\theta)}(1-\cos\theta, -\sin\theta) .$$

Then since

$$\|(1-\cos\theta, -\sin\theta)\| = \sqrt{2}\sqrt{(1-\cos\theta)} ,$$

$$\|\nabla\theta(x(\theta, u), y(\theta, u))\| = \frac{\sqrt{2}}{\sqrt{r}(2-s)} \frac{1}{\sqrt{r(1-\cos\theta)}} .$$

But from (3.2), we have $y(\theta, u) = r(1-u)(1-\cos\theta)$. Thus,

$$\frac{1}{\sqrt{r(1-\cos\theta)}} = \sqrt{\frac{1-u}{y(\theta, u)}} .$$

Therefore,

$$\|\nabla\theta(x, y)\| = \frac{2\sqrt{1-u}}{2-u} \frac{1}{\sqrt{2ry}} .$$

Define

$$\phi(u) := \frac{2\sqrt{1-u}}{2-u} .$$

A simple computation yileds

$$\phi'(u) = -\frac{u}{\sqrt{1-u}(2-u)^2} .$$

This shows that $\phi(u)$ has a strict global maximum value of 1 at $u = 0$, which proves the lemma. $\square$

Theorem 3.1 shows that the cycloid does give the minimal time, and thus deserves the name "brachistochrone", meaning "shortest time" in greek.

3.1 The method behind the proof

The proof may have seemed a bit miraculous. There is a method behind it – the *method of calibrations*. See [1] and [2].

References

[1] G. Lawlor *A new minimization proof for the brachistochrone*, Amer. Math. Monthly, **103**, March 1996, 242-249.

[2] F. Morgan *Area-minimizing surfaces, faces of Grassmannians, and calibrations* Amer. Math. Monthly **95** September 1988, 813822.