Practice Test 1B, Math 292 Spring 2013

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(1) Find the general solution of the equation

$$x'(t) = x^2(t) + tx(t) - (1+t) \ ,$$

and the particular solution with $x(0) = 1$.

SOLUTION Trying for a solution of the form $x_1 = Ct^\alpha$, we find $\alpha = 0$ and $c = 1$ works; i.e., $x_1(t) = 1$ is a solutions of the equations.

We now define $y(t) = x(t) - 1$. Then $y(t)$ satisfies

$$y' = (2+t)y + y^2 \ .$$

This is a Bernoulli equation with $n = 2$, so we put $z = y^{1-2} = 1/y$. Then

$$-\frac{1}{z^2}z' = (2+t)\frac{1}{z} + \frac{1}{z^2} \ ,$$

or

$$z' = -(2+t)z + 1 \ .$$

This is a linear first order equation. The integrating factor is $e^{2t+t^2/2}$, so

$$(ze^{2t+t^2/2})' = e^{2t+t^2/2} \ .$$

Thus,

$$z(t)e^{2t+t^2/2} = z(0) + \int_0^t e^{2s+t^s/2}\mathrm{d}s \ ,$$

so that

$$x(t) = 1 + \left(C + \int_0^t e^{2s+t^s/2}\mathrm{d}s\right)^{-1}$$

is the general solution, and the solution with $x(0) = 1$ is $x(t) = 1$; i.e., the particular solution x_1, which one obtains from the general solutions in the limit $C \to \infty$.

(2.) Let $v(x) = 4x - x^2$.

(a) Show that for each $x_0 \in (0,4)$ there exist a unique solution of

$$x'(t) = v(x) \qquad \text{with} \quad x(t) = x_0 \ . \qquad (*)$$

(b) How long does it take the solution of $(*)$ with $x_0 = 1$ to reach $x = 3$?

(c) Find an explicit formula for the flow transformation φ_t such that for each $x_0 \in (0,4)$, $x(t) = \varphi_t(x_0)$ is the solution to $(*)$. Compute $\varphi_t'(1)$.

SOLUTION Noe that $v > 0$ on $(0,4)$ and since v' is bounded on this interval, v is Lipschitz. Thus, the equation has a unique global (valid for all t) solution for each $x_0 \in (0,4)$. This answers **(a)**. Let us find the solutions: The solution is strictly monotone increasing, and its inverse function $t(x)$ is given by

$$t(x) = \int_{x_0}^{x} \frac{1}{z(4-z)} \mathrm{d}z = \frac{1}{4} \ln\left(\frac{x}{4-x}\right) - \frac{1}{4} \ln\left(\frac{x_0}{4-x_0}\right) .$$

Solving for x, we find

$$x(t) = \frac{4x_0 e^{4t}}{(4-x_0) + x_0 e^{4t}} .$$

We can now read off the answer to **(c)**: Replacing x_0 be x, we have

$$\varphi_t(x) = \frac{4x e^{4t}}{(4-x) + x e^{4t}} .$$

Finally, for **(b)**,

$$t(3) = \int_{1}^{3} \frac{1}{z(4-z)} \mathrm{d}z = \frac{\ln 3}{2} .$$

(3.) Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \qquad \text{and} \qquad B := \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix} .$$

(a) Compute e^{tA} and e^{tB}.

(b) Find all $\mathbf{x}_0$, if any, so that the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

(c) Find all $\mathbf{x}_0$, if any, so that the solution of $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

(d) Let $\mathbf{f}(t) = (1, t)$, and let $\mathbf{x}_0 = (2, 1)$. Find the solution of

$$\mathbf{x}'(t) = B\mathbf{x}(t) + \mathbf{f}(t)$$

with $\mathbf{x}(0) = \mathbf{x}_0$.

SOLUTION For A, the characteristic polynomial is $(t-3)^2$, so the only eigenvalue is 3. In this case $(A - 3I)^2 = 0$, so we easily compute

$$e^{tA} = e^{3t} \begin{bmatrix} 2t-1 & 2t \\ -2t & 2t+1 \end{bmatrix} .$$

For B, the eigenvalues are $-1 \pm i$, and from a single eigenvector we get two solutions and then

$$e^{tB} = e^{-t} \begin{bmatrix} \cos t + \sin t & -\sin t \\ -\sin t & \cos t - \sin t \end{bmatrix} .$$

This takes care of **(a)**. For **(b)** and **(c)**, since all eigenvalues of A are positive, the only such $\mathbf{x}_0$ is $\mathbf{x}_0 = \mathbf{0}$, while since all eigenvalues of B have a negative real part, every $\mathbf{x}_0 \in \mathbb{R}^2$ has this property.

Finally, for **(d)**, by the Duhamel formula,

$$\begin{aligned}\mathbf{x}(t) &= e^{tB}\mathbf{x}_0 + \int_0^t e^{(t-s)B}\mathbf{f}(s)\mathrm{d}s \\ &= e^{-t}(2\cos t, \cos t + \sin t) + (2 - t - 2e^{-t}\cos t, 1 - e^{-t}\cos t - e^{-t}\sin t) \\ &= (2-t, 1)\ ,\end{aligned}$$

which you can easily check.

(4.) Consider the *driving force* $\mathbf{f} = 3\cos(\omega t)(1, 2)$, where $\omega > 0$. Find the solution of

$$M\mathbf{x}'' = -A\mathbf{x} + \mathbf{f} \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{0}\ , \quad \mathbf{x}'(0) = \mathbf{0}\ . \tag{0.1}$$

where

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 12 & 2 \\ 2 & 3 \end{bmatrix}\ .$$

SOLUTION As in the online notes.