Practice Test for Test 2, Math 292, April 25, 2013

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1. The differential equation

$$t^2x''(t) - 3tx'(t) + 4x(t) = 0$$

has polynomial coefficients.

(a) Find one polynomial solution to this equation.

(b) Find the general solution of this equation.

(c) Find the general solution of

$$t^2x''(t) - 3tx'(t) + 4x(t) = t^2 \ln t \ .$$

SOLUTION: (a) We look for a solution of the form $y = t^\alpha$. Putting this into the equation, we find it is satisfied in case

$$\alpha(\alpha - 1) - 3\alpha + 4 = 0 \ ,$$

which is $(\alpha - 2)^2$. There is thus only one solution of this form, namely

$$y_1(t) = t^2 \ .$$

(b) We get a second solution

$$y_2(t) = v(t)y_1(t) \qquad \text{where} \qquad v(t) = \int \frac{1}{y_1^2} e^{\int 3/t} \mathrm{d}t \ .$$

This works out to

$$v(t) = \int \frac{1}{t^4} e^{3\ln t} \mathrm{d}t \ = \int \frac{1}{t} \mathrm{d}t \ = \ln(t) \ .$$

Hence our second solution is

$$y_2(t) = t^2 \ln(t)$$

and the general solution of the homogeneous equation is

$$c_1 t^2 + c_2 t^2 \ln(t) \ .$$

(c) We use the variation of constants formula

$$y_p = -y_1 \int \frac{y_2 r}{W(y_1, 2_2)} \mathrm{d}t + y_2 \int \frac{y_1 r}{W(y_1, 2_2)} \mathrm{d}t \ ,$$

where

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = t^3$$

in this case. Thus,

$$y_p = -t^2 \int \frac{(\ln(t))^2}{t} \mathrm{d}t + t^2 \ln(t) \int \frac{(ln(t)}{t} \mathrm{d}t = \frac{t^2}{6} (\ln(t))^3 \ .$$

Thus the general solution is

$$c_1 t^2 + c_2 t^2 \ln(t) \ + \frac{t^2}{6} (\ln(t))^3 \ .$$

2. Consider the differential equation

$$y'(x) = f(x, y) \qquad \text{where} \qquad f(x, y) = -\frac{1}{x^3} - \frac{2}{x} y + x y^2 \ . \tag{0.1}$$

Consider also the change of variables

$$h(x, y) = (u(x, y), v(x, y)) = (-\ln x, x^2 y). \tag{0.2}$$

(a) Compute the transformed slope field $h_*(1, f)(u, v)$, and find the general solution of the transformed equation.

(b) Find the general solution of the equation (0.1).

SOLUTION (a) We compute

$$(1.f) \circ h^{-1}(u, v) = -e^{-3u}(1 + 2v - v^2) \ ,$$

and

$$[J_h(x, y)]^{-1} = \begin{bmatrix} -e^u & 0 \\ 2e^u v & e^{-2u} \end{bmatrix} \ .$$

Multiplying, we get $-e^{-u}(1, 1-v^2)$, hence the transformed slope field is $(1, 1-v^2)$, and the equivalent equation is

$$v' = 1 - v^2 \ .$$

Separating variables

$$\int \frac{\mathrm{d}v}{1 - v^2} = u + c \ ,$$

and so

$$v(u) = \tanh(u + c) \ .$$

(b) Transforming back to x and y, we find

$$y(x) = \frac{1}{x^2} \tanh(-\ln x + c) \ .$$

3. Consider the equation

$$y''(x) - x y'(x) + \frac{x^2}{2} y(x) = 0 \ . \tag{0.3}$$

(a) Find a function $q(x)$ so that whenever $y(x)$ is a solution of (0.3), there is a solution $z(x)$ of

$$z''(x) + q(x)z(x) = 0 \tag{0.4}$$

that has the same set of zeros as $y(x)$.

(b) Find a number $L > 0$ so that if $y(x)$ solves (0.3) and satisfies $y(0) = 0$ and $y'(0) = 1$, then for some x_1 with $0 < x_1 < L$, $y(x_1) = 0$. Justify your answer.

SOLUTION(a) To transform the equation

$$y'' + Py' + Qy = 0$$

into its normal form

$$u'' + qu = 0 \ ,$$

we define

$$v = \exp(\left(-\frac{1}{2}\int P\mathrm{d}x\right) \ ,$$

and put $y = uv$. Then u satisfies $u'' + qu = 0$ with

$$q = Q - \frac{1}{4}P^2 - \frac{1}{2}P' \ .$$

Since the exponential function is never zero, $v(x) \neq 0$ for any x, and so $y(x) = 0$ if and only if $u(x) = 0$. In this case, we find $v(x) = e^{x^2/2}$ and

$$q(x) = \frac{1}{2} + \frac{1}{4}x^2 \ .$$

(b) Since $q(x) \geq 1/2$ everywhere on $[0, 1]$, any solution to $u'' + qu = 0$ with $u(0) = 0$, $u'(0) \neq 0$, is oscilating at least as fast as the solutions of

$$z'' + \frac{1}{2}z = 0 \ .$$

The solutions of this equation are multiples of

$$z(x) = \sin(x/\sqrt{2}) \ .$$

The first zero of this equation is at $x = \sqrt{2}\pi$. Then by the Sturm Comparison Theorem, $u(x)$, and hence $y(x)$, has a zero in $(0, L)$ where $L\sqrt{2}\pi$.

4. Find the continuously differentiable curve $y(x)$ such that $y(0) = 1$ and $y(1) = 0$ that minimizes the functional

$$I[y] = \int_0^1 [|y'(x)|^2 + |y(x)|^2]\mathrm{d}x \ .$$

Justify your answer.

SOLUTION The Euler-Lagange equation works out to be

$$y'' - y = 0 \ .$$

This has the general solution

$$y(x) = c_1 e^x + c_2 e^{-x} \ .$$

From the boundary conditions

$$\begin{aligned} 1 &= c_1 + c_2 \\ 0 &= c_1 e + c_2 e^{-1} \end{aligned}$$

Solving, we find

$$c_2 = \frac{e^2}{e^2 - 1} \qquad \text{and} \qquad c_1 = \frac{1}{1 - e^2} \ .$$

Thus,

$$y(x) = \frac{1}{1 - e^2} e^x + \frac{e^2}{e^2 - 1} e^{-x}$$

is a solution of the Euler-Lagrange equation that satisfies the boundary conditions. Since the functional is quadratic with strictly positive coefficients, and such solution is the unique minimizer.