Practice Final Exam, Math 292, Spring 2013

Eric A. Carlen[1]
Rutgers University

May 5, 2013

The question on this practice exam cover *only material from the first half of the semester.* This is what I will review in the final class, Monday May 6. The actual final exam ail be **twice as long**, covering material from the second half as well, which we have recently reviewed.

1. Find the general solution of the differential equation

$$x^3y' + x^2y - y^2 = 2x^4 \ .$$

2. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}y(t) = -y(t) + \sin(t) \ .$$

(a) Find the general solution of this equation.

(b) There is a unique periodic solution of this equation, $y_p(t)$. Find $y_p(t)$.

(c) Show that for all initial data y_0, the solution $y(t)$ with $y(0) = y_0$ satisfies

$$\lim_{t\to\infty}(y(t) - y_p(t)) = 0 \ ,$$

where y_p is the unique periodic solution.

3. Consider the two equations

$$\textbf{I.} \quad (y')^2 + y^2 = 1 \qquad\qquad \textbf{II.} \quad (y')^2 - y^2 = 1 \ .$$

Let $-1 < y_0 < 1$.

(a) One of these equations has a unique solution $y(t)$ with $y(0) = y_0$, and the other has infinitely many such solutions. Which one has unique solutions? Justify your answer.

(b) For each $-1 < y_0 < 1$, find infinitely many solutions of the equations for which there is no uniqueness.

4.

Let

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \ .$$

(a) Compute e^{tA}, and find the solution of

$$\begin{aligned} x' &= 3x + y \\ y' &= -x + 5y \end{aligned}$$

with $x(0) = 1$ and $y(0) = 2$.

(b) Compute e^{tB}, and find the solution of

$$\begin{aligned} x' &= -2y \\ y' &= x + 2y \end{aligned}$$

with $x(0) = 1$ and $y(0) = 2$.

(c) Solve

$$\mathbf{x}'(t) = A\mathbf{x}(t) + (t, t) \qquad \text{with} \qquad \mathbf{x}(0) = (1, 2) \ .$$

5. Let

$$\mathbf{v}(x, y) = (xy + 12, x^2 + y^2 - 25) \ .$$

(a) Find all equilibrium points of $\mathbf{v}$.

(b) Which, if any, of these points are Lyapunov stable, asymptotically stable, or unstable? Justify your answer.